CATEGORICAL CLOSURE OPERATORS VIA GALOIS CONNECTIONS

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0. INTRODUCTION

Closure operators are well-known in topology and order theory. In the setting of an $\langle \boldsymbol{E}, \mathscr{M} \rangle$-category for sinks we show that the categorical abstraction of the notion of closure operator is such that closure operators appear as essentially the fixed points (i.e., as the Galois-closed members) of a natural Galois connection. We identify a common principle that underlies the construction of various types of closure operators, from regular closure operators induced by some class of objects by means of the Salbany construction, to idempotent modal closure operators induced by some Grothendieck topology. Our analysis also sheds additional light on the Pumplün–Röhrl connection and its factorizations, previously dealt with in [6] and [8]. Proofs and a more detailed treatment can be found in [4] and [7].

In Section 1 we present preliminary definitions and constructions, and in Section 2 we show how certain natural Galois connections yield closure operators (of various types) as precisely their Galois fixed points.

In Section 3 we introduce the notion of regular closure operator relative to Galois connections of the form $\boldsymbol{P}(\mathscr{M} \diamond \mathscr{M}) \xrightarrow{\rho} \bullet \mathscr{D}$ and $\boldsymbol{P}(\mathscr{M}) \xrightarrow{\tau} \bullet \mathscr{D}$ (where $\mathscr{M} \diamond \mathscr{M}$ denotes the family of composable pairs of members of $\mathscr{M}$), and we analyze under what conditions ρ and τ factor through the Galois connections that are fundamentally related to closure operators.

Section 4 is devoted to applications where the Galois connections ρ and τ are polarities induced by relations.

1. PRELIMINARIES

Our main tool will be a notion of orthogonality that generalizes the one introduced by Tholen (cf. [14]), and encompasses part of the defining properties of factorization structures for sinks and for sources as well as one of the essential features of closure operators (cf. Definition 1.01).

1.00 Definition. (cf. [12]) A pair $\langle \boldsymbol{a}, a' \rangle$ consisting of a sink $\boldsymbol{a} = \langle A_i \xrightarrow{a_i} A' \rangle_I$ and a morphism $A' \xrightarrow{a'} A''$ in a category $\mathscr{X}$ is called **left orthogonal to** a pair $\langle b, \boldsymbol{b}' \rangle$ consisting of a morphism $B \xrightarrow{b} B'$ and a source $\boldsymbol{b}' = \langle B' \xrightarrow{b'_j} B''_j \rangle_J$, written as $\langle \boldsymbol{a}, a' \rangle \perp \langle b, \boldsymbol{b}' \rangle$, iff for any

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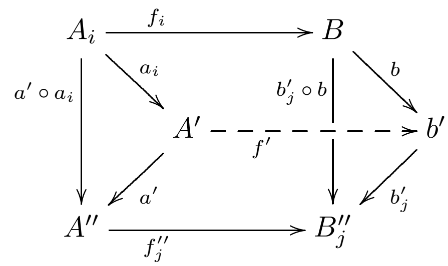
$i \in I$ and each $j \in J$ the rear square of following diagram commutes



(1-00)

there exists a unique $\mathscr{X}$-morphism $A' \xrightarrow{f'} B'$ such that all other squares commute. In this case the pair $\langle b, \boldsymbol{b'} \rangle$ is called **right orthogonal to** $\langle \boldsymbol{a}, a' \rangle$.

The topologically-motivated notion of a closure operator for a category $\mathscr{X}$ is defined relative to a class $\mathscr{M} \subseteq \boldsymbol{Mor}(\mathscr{X})$ (corresponding to the embeddings in $\boldsymbol{Top}$). We regard $\mathscr{M}$ as a full subcategory of the arrow category of $\mathscr{X}$. An $\mathscr{M}$**-morphism** $\langle f, g \rangle$ from m to n is a pair of $\mathscr{X}$-morphisms that satisfy $g \circ m = n \circ f$. The **domain functor** $\mathscr{M} \xrightarrow{U} \mathscr{X}$ maps $\langle f, g \rangle$ to f, and the **codomain functor** V maps $\langle f, g \rangle$ to g. We call $m \xrightarrow{\langle f, g \rangle} n$ **cartesian** iff it is V**-initial** (cf. [1]). This is equivalent to $m \xrightarrow{\langle f, g \rangle} n$ constituting a pullback square in $\mathscr{X}$.

1.01 Definition. A **closure operator** $F = \langle (\,)_F, (\,)^F \rangle$ on $\mathscr{M}$ maps each $m \in \mathscr{M}$ to a pair $\langle m_F, m^F \rangle$ with $m = m^F \circ m_F$ and $m^F \in \mathscr{M}$,[1] such that $F(m) \perp F(n)$ for all $n \in \mathscr{M}$.

$m \in \mathscr{M}$ is called F**-closed** (resp. F**-dense**) if m_F (resp. m^F) is an isomorphism. $\nabla_*(F)$ and $\Delta^*(F)$ denote the classes of F-closed and F-dense members of $\mathscr{M}$, respectively.

1.02 Remarks. (0) A succinct categorical formulation of the concept of closure operator, first proposed by Dikranjan and Giuli [9], views $(\,)^F$ as an endofunctor $\mathscr{M} \xrightarrow{(\,)^F} \mathscr{M}$ that satisfies $V(\,)^F = V$, and views $(\,)_F$ as the domain-part of a natural transformation $id_{\mathscr{M}} \xrightarrow{\delta} (\,)^F$ that satisfies $id_V \delta = id_V$. The uniqueness part of the orthogonality condition then says that $\langle \delta, (\,)^F \rangle$ is a pre-reflection in the sense of Börger [2], cf. also [15].

(1) If $\mathscr{M}$ has the following cancellation property

$$p \circ n \in \mathscr{M} \quad \text{and} \quad p \in \mathscr{M} \quad \text{implies} \quad n \in \mathscr{M} \tag{1-01}$$

then every closure operator on $\mathscr{M}$ is also a density operator on $\mathscr{M}$. Furthermore, if F is a closure operator on a class $\mathscr{M}$ of monos, as we will assume below, the uniqueness part of the orthogonality condition is automatically satisfied.

In order to have analogues to the complete subspace lattices of topological spaces, from now on we assume that $\mathscr{X}$ is an $\langle \boldsymbol{E}, \mathscr{M} \rangle$-category for sinks. This insures that $\mathscr{X}$ is sufficiently nice to support certain constructions (cf. [4]). In particular, $\mathscr{M}$ then consists of monos, is closed under composition, and satisfies the cancellation property (1-01). We write $\mathscr{M} \diamond \mathscr{M}$ for the category whose objects are all composable pairs of members of $\mathscr{M}$, i.e., the pullback of U and V.

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[1] If we require $m_F \in \mathscr{M}$ rather than $m^F \in \mathscr{M}$, then we obtain the notion of **density operator**.

the $\mathscr{M}$-morphism $p\circ n \xrightarrow{\langle d,c\rangle} r\circ q$. For each $m\in\mathscr{M}$ its W**-fiber,** i.e., the comma category W/m, is pre-ordered by $\langle n,p\rangle \ll \langle q,r\rangle$ iff there exists a (necessarily unique) $\mathscr{X}$-morphism b such that $\langle n,p\rangle \xrightarrow{\langle id,b,id\rangle} \langle q,r\rangle$ is an $\mathscr{M}\diamond\mathscr{M}$-morphism.

The W-fibers form (possibly large) complete lattices under $\ll$. Intersections and $\langle \boldsymbol{E},\mathscr{M}\rangle$-factorizations of the (collections of) second components yield infima and suprema, respectively.

The codomain functor V and the composition functor W both are bi-fibrations. For an $\mathscr{X}$-morphism $X \xrightarrow{g} Y$ the V-**inverse image functor** $V/Y \xrightarrow{g^{\leftarrow}} V/X$ maps an $\mathscr{M}$-subobject of Y to its pullback along g, while an $\mathscr{M}$-morphism $m \xrightarrow{\langle f,g\rangle} n$ induces a W-inverse image functor $W/m' \xrightarrow{\langle f,g\rangle^{\leftarrow}} W/m$ that maps $\langle s,t\rangle\in W/n$ to the unique $\langle q,r\rangle\in W/m$, whose second component is the chosen pullback of t along g. The corresponding adjoints are the V-direct image functor $V/X \xrightarrow{g_{\exists}} V/Y$ (that maps an $\mathscr{M}$-subobject m of X to the $\mathscr{M}$-component of the chosen $\langle \boldsymbol{E},\mathscr{M}\rangle$-factorization of $g\circ m$), and the W-direct image functor $W/m \xrightarrow{\langle f,g\rangle_{\exists}} W/n$ (that maps $\langle q,r\rangle\in W/m$ to the unique $\langle s,t\rangle\in W/n$ for which there exists an $\mathscr{X}$-morphism d such that $\langle\langle d,s\rangle,t\rangle$ is the chosen $\langle \boldsymbol{E},\mathscr{M}\rangle$-factorization of the 2-sink $\langle g\circ r,n\rangle$).

1.04 Definition. A closure operator F is called

(0) **idempotent** iff m^F is F-closed for every $m\in\mathscr{M}$, i.e., iff $(\,)^F(\,)^F\cong(\,)^F$;
(1) **weakly hereditary** iff m_F is F-dense for every $m\in\mathscr{M}$, i.e., iff $(\,)_F(\,)_F\cong(\,)_F$.
(2) **hereditary** iff n^F is a pullback of m^F along p whenever $\langle n,p\rangle\in W/m$;
(3) **modal** iff n^F is a pullback of m^F along g whenever $n \xrightarrow{\langle f,g\rangle} m$ is cartesian, i.e., iff F preserves cartesian $\mathscr{M}$-morphisms.

$\boldsymbol{CL}(\mathscr{M})$ denotes the collection of all closure operators on $\mathscr{M}$, pre-ordered by $F\sqsubseteq G$ iff $F(m)\ll G(m)$ for all $m\in\mathscr{M}$, while $\boldsymbol{iCL}(\mathscr{M})$, $\boldsymbol{wCL}(\mathscr{M})$, $\boldsymbol{hCL}(\mathscr{M})$, $\boldsymbol{mCL}(\mathscr{M})$, and $\boldsymbol{iwCL}(\mathscr{M})$ stand for the subcollections of idempotent, weakly hereditary, hereditary, modal, and idempotent weakly hereditary closure operators, respectively.

1.05 Definition. (0) A closure operator F is called $\mathscr{Z}$**-modal**, if F commutes with all W-inverse images along members of $\mathscr{Z}$, i.e., $F(m)\cong\langle f,g\rangle^{\leftarrow}(F(n))$ if $m \xrightarrow{\langle f,g\rangle} n$ belongs to $\mathscr{Z}$. We write $\boldsymbol{CL}\langle\mathscr{M},\mathscr{Z}\rangle$ for the collection of all $\mathscr{Z}$-modal closure operators on $\mathscr{M}$.

(1) $C\subseteq\mathscr{M}\diamond\mathscr{M}$ is called $\mathscr{Z}$**-stable**, if whenever $m \xrightarrow{\langle f,g\rangle} n$ belongs to $\mathscr{Z}$, and $\langle s,t\rangle\in C\cap W/n$, then the W-inverse image of $\langle s,t\rangle$ along $\langle f,g\rangle$ belongs to C. We call $A\subseteq\mathscr{M}$ $\mathscr{Z}$**-stable**, if $\gamma_*(A)$ has this property.

(2) For any class Y a relation $R\subseteq(\mathscr{M}\diamond\mathscr{M})\times Y$ (resp. $T\subseteq\mathscr{M}\times Y$) is called $\mathscr{Z}$**-stable** if for each $y\in Y$ the class $\{\langle n,p\rangle\in\mathscr{M}\diamond\mathscr{M}\mid\langle\langle n,p\rangle,y\rangle\in R\}$ (resp. $\{m\in\mathscr{M}\mid\langle m,y\rangle\in T\}$) has this property.

1.06 Remark. The following choices for $\mathscr{Z}$ deserve particular attention:

$$\boldsymbol{hCL}(\mathscr{M})=\boldsymbol{CL}\langle\mathscr{M},\mathscr{Z}\rangle \quad\text{for}\quad \mathscr{Z}=\{\langle f,g\rangle\in\boldsymbol{Mor}(\mathscr{M})\mid f\text{ iso and }g\in\mathscr{M}\}$$
$$\boldsymbol{mCL}(\mathscr{M})=\boldsymbol{CL}\langle\mathscr{M},\mathscr{Z}\rangle \quad\text{for}\quad \mathscr{Z}=\{\langle f,g\rangle\in\boldsymbol{Mor}(\mathscr{M})\mid\langle f,g\rangle\text{ cartesian}\}$$
$$\boldsymbol{CL}(\mathscr{M})=\boldsymbol{CL}\langle\mathscr{M},\mathscr{Z}\rangle \quad\text{for}\quad \mathscr{Z}=\{\langle f,g\rangle\in\boldsymbol{Mor}(\mathscr{M})\mid\langle f,g\rangle\text{ iso}\}$$

Every closure operator F has an **idempotent hull** (i.e., reflection) $F^{\mathrm{i}}\in\boldsymbol{iCL}(\mathscr{M})$ as well as a **weakly hereditary core** (i.e., coreflection) $F_{\mathrm{w}}\in\boldsymbol{wCL}(\mathscr{M})$. Each $\boldsymbol{CL}\langle\mathscr{M},\mathscr{Z}\rangle$ is closed under the formation of suprema in $\boldsymbol{CL}(\mathscr{M})$. If $\boldsymbol{E}$ is stable under $V_{\exists}(\mathscr{Z})$-pullbacks, $\boldsymbol{CL}\langle\mathscr{M},\mathscr{Z}\rangle$ is closed under the formation of infima as well, cf. [5]. The $\mathscr{Z}$-modal hull and core (if it exists) see, e.g., [3], [9], [10], and [5].

2. CLOSURE OPERATORS ARE GALOIS FIXED POINTS

$\mathscr{A} \xrightarrow{\pi}\!\bullet \mathscr{B}$ will be used to denote a Galois connection from the preordered class $\mathscr{A}$ to the preordered class $\mathscr{B}$, i.e., a pair of order-preserving functions $\mathscr{A} \xrightarrow{\pi_*} \mathscr{B}$ and $\mathscr{B} \xrightarrow{\pi^*} \mathscr{A}$ such that $\pi_* \dashv \pi^*$. We call π a **reflection** (resp. **coreflection**) provided that $\pi_* \circ \pi^* \cong id$ (resp. $\pi^* \circ \pi_* \cong id$). Recall that for every relation $R \subseteq X \times Y$ there are two naturally induced Galois connections $\boldsymbol{P}(X) \xrightarrow{R_+{}^+}\!\bullet \boldsymbol{P}(Y)^{\mathrm{op}}$, called a **polarity**, and $\boldsymbol{P}(X) \xrightarrow{R_\exists{}^\forall}\!\bullet \boldsymbol{P}(Y)$, called an **axiality** (cf. [11]).

Let $\boldsymbol{P}(\mathscr{M} \diamond \mathscr{M}) \xrightarrow{\omega}\!\bullet \boldsymbol{P}(\mathscr{M} \diamond \mathscr{M})^{\mathrm{op}}$ be the polarity induced by the restriction of the orthogonality relation $\perp$ to $\mathscr{M} \diamond \mathscr{M}$. The following theorem shows that the closure operators on $\mathscr{M}$ are essentially the Galois fixed points of ω.

2.00 Theorem. *There exists a reflection* $\boldsymbol{P}(\mathscr{M} \diamond \mathscr{M}) \xrightarrow{\dot\omega}\!\bullet \boldsymbol{CL}(\mathscr{M})$ *as well as a coreflection* $\boldsymbol{CL}(\mathscr{M}) \xrightarrow{\ddot\omega}\!\bullet \boldsymbol{P}(\mathscr{M} \diamond \mathscr{M})^{\mathrm{op}}$ *such that* $\omega = \ddot\omega \circ \dot\omega$. □

In particular, every subclass $C \subseteq \mathscr{M} \diamond \mathscr{M}$ yields two closure operators, $\dot\omega_*(C)$ and $\ddot\omega^*(C)$. Moreover, every closure operator F up to isomorphism can be recovered from $\dot\omega^*(F)$, the class of its relatively dense pairs, and from $\ddot\omega_*(F)$, the class of its relatively closed pairs.

Let $\boldsymbol{P}(\mathscr{M}) \xrightarrow{\gamma}\!\bullet \boldsymbol{P}(\mathscr{M} \diamond \mathscr{M})$ be the natural axiality that arises from the inverse of the first projection function $\mathscr{M} \diamond \mathscr{M} \longrightarrow \mathscr{M}$, considered as a relation, and for the second projection function let $\boldsymbol{P}(\mathscr{M} \diamond \mathscr{M})^{\mathrm{op}} \xrightarrow{\delta}\!\bullet \boldsymbol{P}(\mathscr{M})^{\mathrm{op}}$ be the corresponding axiality. The following theorem shows that the closure operators on $\mathscr{M}$ that are simultaneously idempotent and weakly hereditary are essentially the Galois fixed points of the composite Galois connection $\delta \circ \omega \circ \gamma$.

2.01 Theorem. *There exists a reflection* $\boldsymbol{P}(\mathscr{M}) \xrightarrow{\dot\upsilon}\!\bullet \boldsymbol{iwCL}(\mathscr{M})$ *as well as a coreflection* $\boldsymbol{iwCL}(\mathscr{M}) \xrightarrow{\ddot\upsilon}\!\bullet \boldsymbol{P}(\mathscr{M})^{\mathrm{op}}$ *such that* $\delta \circ \omega \circ \gamma = \ddot\upsilon \circ \dot\upsilon$. □

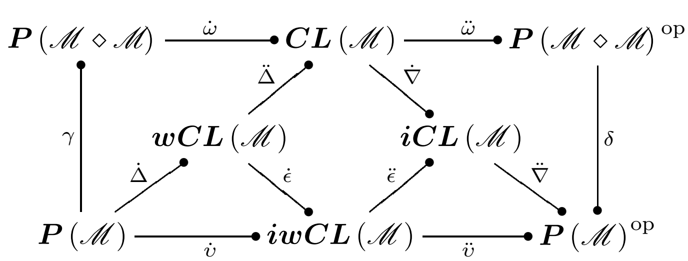
Let $\boldsymbol{wCL}(\mathscr{M}) \xrightarrow{\ddot\Delta}\!\bullet \boldsymbol{CL}(\mathscr{M})$ and $\boldsymbol{CL}(\mathscr{M}) \xrightarrow{\dot\nabla}\!\bullet \boldsymbol{iCL}(\mathscr{M})$ be the coreflection the reflection induced by $(\ )_{\mathrm{w}}$ and $(\ )^{\mathrm{i}}$, respectively. Next we see that both the weakly hereditary closure operators on $\mathscr{M}$ and the idempotent closure operators on $\mathscr{M}$ are also essentially the Galois fixed points of natural Galois connections.

2.02 Theorem. *There exists a reflection* $\boldsymbol{P}(\mathscr{M}) \xrightarrow{\dot\Delta}\!\bullet \boldsymbol{wCL}(\mathscr{M})$ *with* $\dot\omega \circ \gamma = \ddot\Delta \circ \dot\Delta$, *and there exists a coreflection* $\boldsymbol{iCL}(\mathscr{M}) \xrightarrow{\ddot\nabla}\!\bullet \boldsymbol{P}(\mathscr{M})^{\mathrm{op}}$ *such that* $\delta \circ \ddot\omega = \ddot\nabla \circ \dot\nabla$. □

The idempotent weakly hereditary closure operators on $\mathscr{M}$ appear (in a natural way) as a Galois fixed point lattice for a second time.

2.03 Theorem. *There exists a reflection* $\boldsymbol{P}(\mathscr{M}) \xrightarrow{\dot\epsilon}\!\bullet \boldsymbol{iwCL}(\mathscr{M})$ *as well as a coreflection* $\boldsymbol{iwCL}(\mathscr{M}) \xrightarrow{\ddot\epsilon}\!\bullet \boldsymbol{iCL}(\mathscr{M})$ *such that* $\dot\nabla \circ \ddot\Delta = \ddot\epsilon \circ \dot\epsilon$. □

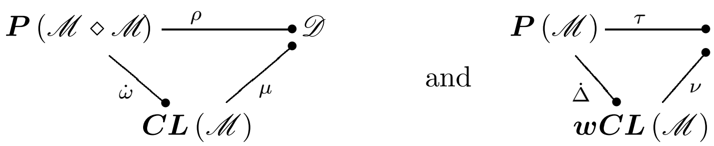
We summarize the results of this section with the following commutative diagram of Galois connections between complete lattices.



(2-00)

Consider an arbitrary pre-ordered class $\mathscr{D}$, and arbitrary Galois connections $\boldsymbol{P}(\mathscr{M} \diamond \mathscr{M}) \xrightarrow{\rho} \!\!\bullet \mathscr{D}$ and $\boldsymbol{P}(\mathscr{M}) \xrightarrow{\tau} \!\!\bullet \mathscr{D}$. In the applications these will in fact be polarities induced by suitable relations.

3.00 DEFINITION. Suppose that ρ and τ factor through $\dot{\omega}$ and $\dot{\Delta}$ via Galois connections $\boldsymbol{CL}(\mathscr{M}) \xrightarrow{\mu} \!\!\bullet \mathscr{D}$ and $\boldsymbol{wCL}(\mathscr{M}) \xrightarrow{\nu} \!\!\bullet \mathscr{D}$, respectively, i.e.,



(3-00)

are commutative diagrams of Galois connections. $F \in \boldsymbol{CL}(\mathscr{M})$ (resp. $F \in \boldsymbol{wCL}(\mathscr{M})$) is called ρ-**regular** (resp. τ-**regular**), if F is a fixed point of $\mu^* \circ \mu_*$ (resp. of $\nu^* \circ \nu_*$).

Clearly, since $\dot{\omega} \circ \gamma$ factors through $\dot{\Delta}$, the right triangle can be constructed from the left one whenever τ is taken to be $\rho \circ \gamma$.

Let us first consider the question of how many Galois connections μ and ν can exist that make the diagrams in (3-00) commute. To find suitable candidates for such Galois connections, we use the fact that $\dot{\omega}$ and $\dot{\Delta}$ are well-behaved in the sense of the following lemma.

3.01 LEMMA. *If Galois connections α, ϕ and ψ satisfy $\phi \circ \alpha = \psi$, and if α is a reflection, then $\phi_* \cong \psi_* \circ \alpha^*$ and $\phi^* \cong \alpha_* \circ \psi^*$.* □

Since $\dot{\omega}$ and $\dot{\Delta}$ are both reflections, the only possible candidates for μ and ν (up to isomorphism) are

$$\mu := \langle \rho_* \circ \dot{\omega}^*, \dot{\omega}_* \circ \rho^* \rangle \qquad \text{and} \qquad \nu := \langle \tau_* \circ \dot{\Delta}^*, \dot{\Delta}_* \circ \tau^* \rangle$$

3.02 PROPOSITION. *For every element D of $\mathscr{D}$*

(0) *if $\rho^*(D)$ is a fixed point of $\dot{\omega}^* \circ \dot{\omega}_*$, then μ is a Galois connection;*

(1) *if $\tau^*(D)$ is a fixed point of $\dot{\Delta}^* \circ \dot{\Delta}_*$, then ν is a Galois connection.* □

It can be shown that $\dot{\omega}^* \circ \dot{\omega}_*$ and $\omega^* \circ \omega_*$ have the same fixed points; indeed $C \subseteq \mathscr{M} \diamond \mathscr{M}$ is a fixed point of $\omega^* \circ \omega_*$ (and hence of $\dot{\omega}^* \circ \dot{\omega}_*$) iff C satisfies the conditions

(C0) C is closed under the formation of W-direct images.

(C1) $\sup_{\ll}\big(C \cap W/m\big) \in C$ for every $m \in \mathscr{M}$.

(C2) C is downward closed with respect to $\ll$.

We use this characterization when applying part (0) of Proposition 3.02. Next we characterize the fixed points of $\dot{\Delta}^* \circ \dot{\Delta}_*$.

3.03 THEOREM. *$A \subseteq \mathscr{M}$ is a fixed point of $\dot{\Delta}^* \circ \dot{\Delta}_*$ iff A satisfies the following conditions*

($\bar{\text{C}}$0) *$\gamma_*(A)$ is closed under the formation of W-direct images;*

($\bar{\text{C}}$1) *$\sup_{\ll}(\gamma_*(A) \cap W/m) \in \gamma_*(A)$ for every $m \in \mathscr{M}$.* □

It is important to notice that ($\bar{\text{C}}$0) is not equivalent to A being closed under the formation of V-direct images. (If $m = p \circ n$, then $\langle p, id_{V(p)} \rangle$ always is a W-direct image of $\langle m, id_{V(m)} \rangle$ along $\langle n, id_{V(p)} \rangle$, but p need not be an V-direct image of m along $id_{V(m)}$.) Translating (C2) for $\gamma_*(A)$ in terms of A yields the cancellation property ($\bar{\text{C}}$2): $p \circ n \in A$ and $p \in \mathscr{M}$ implies $n \in A$. By Corollary 2.03(0) of [5] ($\bar{\text{C}}$2) characterizes hereditary closure operators. The fact that this condition is not needed to characterize the fixed points of $\dot{\Delta}^* \circ \dot{\Delta}_*$ indicates that A may be a fixed point of $\dot{\Delta}^* \circ \dot{\Delta}_*$ without $\gamma_*(A)$ being a fixed point of $\omega^* \circ \omega_*$.

and $\boldsymbol{wCL}(\mathscr{M})$, respectively. Proposition 3.02 makes it clear that we need to characterize the classes $\dot{\omega}^*(F)$ for idempotent closure operators F, as well as the classes $\dot{\Delta}^*(F)$ for idempotent weakly hereditary closure operators F. From Proposition 2.02 of [5] it follows that the classes $\dot{\omega}^*(F)$ and $\dot{\Delta}^*(F)$ for $\mathscr{L}$-modal closure operators F are characterized by $\mathscr{L}$-stability.

3.04 Proposition. *$C \subseteq \mathscr{M} \diamond \mathscr{M}$ is of the form $\dot{\omega}^*(F)$ for some idempotent closure operator F iff in addition to conditions* (C0) – (C2) *above we have*

(C3) *C is stable under* **left-shifting**, *i.e., $\langle l, p \circ n \rangle \in C$ and $\langle n, p \rangle \in C \Rightarrow \langle n \circ l, p \rangle \in C$.*

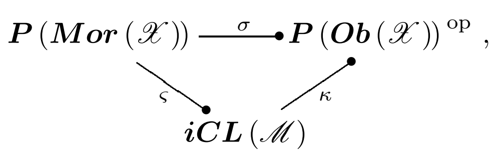
Similarly, $A \subseteq \mathscr{M}$ is of the form $\dot{\Delta}^*(F)$ for some idempotent weakly hereditary closure operator F iff in addition to conditions ($\bar{\text{C}}$0) *and* ($\bar{\text{C}}$1) *of Theorem 3.03 we have*

($\bar{\text{C}}$3) *A is closed under composition.* □

4. APPLICATIONS

We now consider the case that relations induce the Galois connections ρ and τ as polarities. Our main interest concerns suitable restrictions of the orthogonality relation $\perp$ and modifications thereof. According to Definition 1.00 $\perp$ relates the collection $\boldsymbol{L}$ of all pairs $\langle \boldsymbol{l}, l \rangle$ consisting of an $\mathscr{X}$-sink $\boldsymbol{l}$ and an $\mathscr{X}$-morphism l with matching codomain and domain, respectively, with the collection $\boldsymbol{R}$ of all pairs $\langle r, \boldsymbol{r} \rangle$ consisting of an $\mathscr{X}$-morphism r and an $\mathscr{X}$-source $\boldsymbol{r}$, also with matching codomain and domain, respectively. Let $\boldsymbol{R}^{\text{iso}}$ (resp. $\boldsymbol{R}_{\text{mono}}$ denote the subcollection of $\boldsymbol{R}$ that consists of those pairs that have an isomorphism in the first component (resp. a monosource in the second component). We want to restrict $\boldsymbol{L}$ to $\mathscr{M} \diamond \mathscr{M}$, or to $\mathscr{M} \diamond \boldsymbol{Id}(\mathscr{X}) \cong \mathscr{M}$, or in case of the Pumplün-Röhrl connection to $\boldsymbol{Mor}(\mathscr{X}) \diamond \boldsymbol{Id}(\mathscr{X}) \cong \boldsymbol{Mor}(\mathscr{X})$. On the other side we consider a subcollection $\boldsymbol{H}$ of $\boldsymbol{R}$. We will see that depending on whether or not we require $\boldsymbol{R}^{\text{iso}}$ to contain $\boldsymbol{H}$, the polarities in question will even factor through classes of idempotent closure operators.

Two of the motivating examples we want to be able to explain are the factorization of the Pumplün–Röhrl polarity established in [6], and the Salbany construction of a closure operator induced by a class of objects, cf. [13]. In both cases one considers the class consisting of all identity 2-sources $\boldsymbol{H} = \{ \langle X, \langle id_X, id_X \rangle \rangle \mid X \in \boldsymbol{Ob}(\mathscr{X}) \}$, which we identify with $\boldsymbol{Ob}(\mathscr{X})$. In [6] we established the commutativity of the diagram



(4-00)

provided that $\mathscr{X}$ has equalizers and $\mathscr{M}$ contains all regular monomorphisms or, equivalently, provided that $\boldsymbol{E}$ consists of epi-sinks. σ ($= (\alpha, \beta)$ in [6]) is the Pumplün–Röhrl connection, i.e., the polarity induced by the "separating relation" $S \subseteq \boldsymbol{Mor}(\mathscr{X}) \times \boldsymbol{Ob}(\mathscr{X})$ that consists of all $\langle f, X \rangle$ that satisfy $\langle f, id \rangle \perp \langle X, \langle id_X, id_X \rangle \rangle$. The Galois connection ς is determined by ς^* which maps $F \in \boldsymbol{iCL}(\mathscr{M})$ to the class of F-dense $\mathscr{X}$-morphisms. Finally, κ is determined by κ_* which maps each $F \in \boldsymbol{iCL}(\mathscr{M})$ to the class of F-**separated** objects, i.e., those objects X that satisfy $F(m) \perp \langle X, \langle id_X, id_X \rangle \rangle$ for every $m \in \mathscr{M}$. The adjoint κ^* maps each $\mathscr{Y} \subseteq \boldsymbol{Ob}(\mathscr{X})$ to the Salbany (or regular) closure operator induced by $\mathscr{Y}$. The Salbany closure $m^{\kappa^*(\mathscr{Y})}$ of $m \in \mathscr{M}$ is defined as the intersection of all equalizers of parallel pairs $\langle r, s \rangle$ of $\mathscr{X}$-morphisms with codomain in $\mathscr{Y}$ and the property that $r \circ m = s \circ m$. In particular, $\kappa^*(\mathscr{Y})(m)$ is a supremum in W/m of all those pairs $\langle n, p \rangle$ with the property that if m equalizes any parallel of the class $\rho^*(\mathscr{Y}) \cap W/m$.

A special subclass of sinks in $\boldsymbol{E}$ described below plays a crucial role in most of our results. In particular, it defines the interesting subclasses $\boldsymbol{H}$ of $\boldsymbol{R}$ to which one may want to restrict the orthogonality relation $\perp$.

4.00 Definition. (0) $\boldsymbol{E}' \subseteq \boldsymbol{E}$ denotes the collection of all sinks in $\boldsymbol{E}$ that either consist entirely of members of $\mathscr{M}$, or else have exactly two components at least one of which belongs to $\mathscr{M}$.

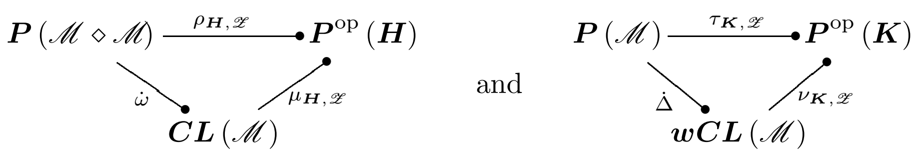
(1) $\boldsymbol{J}$ denotes the class of all $\langle h, \boldsymbol{h}\rangle \in \boldsymbol{R}_{\text{mono}}$ that satisfy $\boldsymbol{e} \perp \boldsymbol{h}$ for each $\boldsymbol{e} \in \boldsymbol{E}'$, and $\boldsymbol{J}_0 := \boldsymbol{J} \cap \boldsymbol{R}^{\text{iso}}$.

4.01 Definition. Let $\boldsymbol{H}$ be a subcollection of $\boldsymbol{R}$, and let $\mathscr{X}$ be a pullback-stable class of $\mathscr{M}$-morphisms. We denote the restrictions of $\perp$ to $(\mathscr{M} \diamond \mathscr{M}) \times \boldsymbol{H}$ and $\mathscr{M} \times \boldsymbol{H}$, respectively, by $R_{\boldsymbol{H}}$ and $T_{\boldsymbol{H}}$, and write $R_{\boldsymbol{H},\mathscr{X}}$ and $T_{\boldsymbol{H},\mathscr{X}}$ for the largest $\mathscr{X}$-stable relations contained in $R_{\boldsymbol{H}}$ and $T_{\boldsymbol{H}}$, respectively. $\rho_{\boldsymbol{H},\mathscr{X}}$ and $\tau_{\boldsymbol{H},\mathscr{X}}$ stand for the polarites induced by them.

We first address the question as to whether $\rho_{\boldsymbol{H},\mathscr{X}}$ and $\tau_{\boldsymbol{H},\mathscr{X}}$ factor through appropriate classes of closure operators.

4.02 Theorem. (0) *For any* $\boldsymbol{H} \subseteq \boldsymbol{R}$ *we have* $\rho_{\boldsymbol{H},\mathscr{X}} \circ \gamma = \tau_{\boldsymbol{H},\mathscr{X}}$.

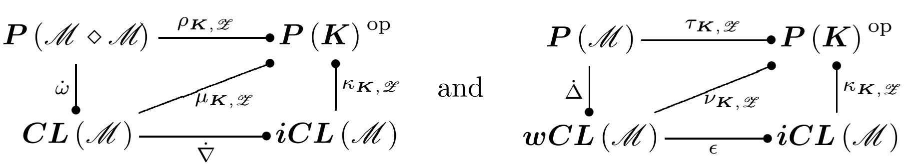
(1) *If* $\boldsymbol{E}'$ *is stable under* $V_\exists(\mathscr{X})$*-pullbacks, then for every* $\boldsymbol{H} \subseteq \boldsymbol{J}$ *and every* $\boldsymbol{K} \subseteq \boldsymbol{J}_0$ *there exist Galois connections* $\mu_{\boldsymbol{H},\mathscr{X}}$ *and* $\nu_{\boldsymbol{K},\mathscr{X}}$ *such that*



(4-01)

commute.

(2) *If* $\boldsymbol{K} \subseteq \boldsymbol{J}_0$, *there also exists a Galois connection* $\kappa_{\boldsymbol{K},\mathscr{X}}$ *such that both*



(4-02)

commute. In particular, in this case $\tau_{\boldsymbol{K},\mathscr{X}}$ *factors through* $\boldsymbol{iwCL}(\mathscr{M})$. □

In general, we may choose any subclass of $\boldsymbol{J}$ for $\boldsymbol{H}$. Singleton sources $\boldsymbol{h}$ that consist of an $\mathscr{M}$-element are always admissible. To include other types of sources, we may need additional information about $\boldsymbol{E}'$. For example, we have the following:

4.03 Corollary. *Let* $\boldsymbol{K} = \{\langle X, \langle id_X, id_X\rangle\rangle \mid X \in \boldsymbol{Ob}(\mathscr{X})\} \cup \{\langle id, m\rangle \mid m \in \mathscr{M}\}$ *and let* $\mathscr{X} = \boldsymbol{Iso}(\mathscr{M})$. *If* $\boldsymbol{E}'$ *consists of epi-sinks, then for all* $\mathscr{Y} \subseteq \boldsymbol{Ob}(\mathscr{X})$ *and all* $\mathscr{A} \subseteq \mathscr{M}$ *the class*

$$(\tau_{\boldsymbol{K},\mathscr{X}})^*\big(\{\langle X, \langle id_X, id_X\rangle\rangle \mid X \in \mathscr{Y}\} \cup \{\langle id, m\rangle \mid m \in \mathscr{A}\}\big)$$

is the class of dense $\mathscr{M}$*-elements for the largest closure operator* F *with respect to which all objects in* $\mathscr{Y}$ *are separated, and all elements of* $\mathscr{A}$ *are closed. Moreover,* F *is both idempotent and weakly hereditary.* □

In order to derive the factorization of the Pumplün-Röhrl connection displayed in Diagram (4-00) from Theorem 4.02, we need to link σ with $\tau_{\boldsymbol{H},\mathscr{X}}$. Recall that any relation $Q \subseteq X \times Y$ induces a Galois connection $\boldsymbol{P}(X) \xrightarrow{\langle Q_\exists, Q^\forall\rangle} \boldsymbol{P}(Y)$ where $Q_\exists(A) = \{y \in Y \mid \exists_{x\in A}\ \langle x, y\rangle \in Q\}$

relation $K \subseteq \boldsymbol{Mor}(\mathscr{X}) \times \mathscr{M}$ defined by

$$\langle f, m\rangle \in K \iff \exists_{e \in \boldsymbol{E}}\ f = m \circ e$$

induces a Galois connection $\boldsymbol{P}(\boldsymbol{Mor}(\mathscr{X})) \xrightarrow{K_{\exists}{}^{\forall}} \bullet \boldsymbol{P}(\mathscr{M})$.

4.04 Corollary. *Let* $\boldsymbol{H} = \{\langle X, \langle id_X, id_X\rangle\rangle \mid X \in \boldsymbol{Ob}(\mathscr{X})\}$ *, and* $\mathscr{Z} = \boldsymbol{Iso}(\mathscr{M})$ *.*

(0) *If* $\boldsymbol{E} \cap \boldsymbol{Mor}(\mathscr{X})$ *consists of epimorphisms, then* $\sigma = \tau_{\boldsymbol{H},\mathscr{Z}} \circ K_{\exists}{}^{\forall}$ *;*

(1) *If* $\boldsymbol{E}$ *consists of epi-sinks, then* $(\kappa_{\boldsymbol{K},\mathscr{Z}})^*$ *(up to isomorphism) agrees with the Salbany operator* κ^* *, and* σ *factors as* $\sigma = \kappa_{\boldsymbol{K},\mathscr{Z}} \circ \epsilon \circ \dot{\Delta} \circ K_{\exists}{}^{\forall}$ *.* □

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