Closure Operators with Respect to a Functor

Dedicated to Lamar Bentley on his 60th birthday

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Abstract. A notion of closure operator with respect to a functor U is introduced. This allows us to describe a number of mathematical constructions that could not be described by means of the already existing notion of closure operator. Some basic results and examples are provided.

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Introduction

The notion of closure operator in an arbitrary category $\mathcal{A}$ introduced in [7] depends on one parameter: subobjects. Precisely, for every object $A \in \mathcal{A}$, a class of subobjects sub$A$ of A is given and then a closure operator is defined on these classes. This notion has lead to a useful theory. For more details on closure operators see, e.g., [2], [6-7], [8-9], [15] and [16]. However, many very natural and useful closures cannot be described by means of the previous notion, since they do not act on the given subobjects of the category $\mathcal{A}$, but rather on subobjects of another category $\mathcal{X}$ related to $\mathcal{A}$ via a functor $\mathcal{A} \xrightarrow{U} \mathcal{X}$. More precisely, for a given class of subobjects in the category $\mathcal{X}$ and for a given functor $\mathcal{A} \xrightarrow{U} \mathcal{X}$, for each A in the category $\mathcal{A}$, we close the subobjects of UA in such a way that, for each $\mathcal{A}$-morphism $A \xrightarrow{f} B$, Uf preserves the closure. The main motivating example is the Kuratowski

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closure in the category of topological groups: with the previous notion of closure operator we can only close subgroups while the Kuratowski closure acts on every subset of a given group. Other examples of this phenomenon are: the subgroup generated by a subset of a given group, the convex hull and the radial hull of a subset of a convex space, the down closure and the Scott closure of a subset of a directed-complete partially ordered set.

It is worth to observe that the move from (ordinary) closure operators to the types of operators considered in this paper is analogous to the move from factorization structures to factorizations along a functor, as considered by Herrlich ([13]) and Tholen ([21]), which (in Proposition 12) also provides an analogue of Proposition 1.5.

The aim of this paper is first of all to introduce the notion of closure operator with respect to a functor U (also called *U-closure operator*) that, besides containing the previous notion as a special case, is general enough to include all the above mentioned examples and more. Then some basic results that were widely used with the previous notion of closure operator can be extended to this setting as well (e.g., every idempotent and weakly hereditary U-closure operator induces, under suitable conditions, a factorization structure on $\mathcal{A}$).

The U-closure operator induced by a suitable subclass of the class $\mathcal{M}$ of all subobjects of all objects of the category $\mathcal{X}$ is defined, and a notion of *regular* U-closure operator is derived as a special case.

We provide conditions under which a class of $\mathcal{A}$-morphisms $\mathcal{N}$ can be used to construct a U-closure operator (called the *hull operator* induced by $\mathcal{N}$). In the case when $\mathcal{N}$ is the second factor of a proper factorization structure for morphisms of the category $\mathcal{A}$ then, under suitable conditions, the hull operator induced by $\mathcal{N}$ composed with every closure operator of $\mathcal{A}$ with respect to $\mathcal{N}$ gives a U-closure operator. U-closure operators that can be obtained this way are called *standard*. It turns out that for these U-closure operators compactness (cf. [3], [4]), connectedness in the sense of [23] and [5] and Hausdorffness, are only depending on their second component (which is a usual closure operator of $\mathcal{A}$) so that the investigation goes back to the classical case. For non-standard U-closure operators (e.g., the Kuratowski closure in topological groups) the situation is different. For instance for the Kuratowski U-closure in topological groups it is not known whether the corresponding class of compact objects (which are the usual compact topological groups by an adaptation of the Kuratowski-Mrowka theorem given for topological spaces) coincides or is properly contained in the class of compact objects with respect to the restriction of the above closure to subgroups (see [10] and [4]).

Many examples that support the theory are included at the end of the paper.

We use the terminology of [1] throughout. We also acknowledge that Paul Taylor's commutative diagrams macro package was used to typeset most of the diagrams in this paper.

1. U-closure operators

Throughout we consider a finitely complete category $\mathcal{X}$ with a proper $(\mathcal{E},\mathcal{M})$-factorization structure for morphisms.

For every object $X \in \mathcal{X}$, the class $\mathcal{M}_X$ of morphisms of $\mathcal{M}$ with codomain X is preordered by $m \leq n$ if there is a morphism t such that $n \circ t = m$. We define the following equivalence relation in $\mathcal{M}_X$: $m \simeq m'$ if $m \leq m'$ and $m' \leq m$. The quotient class will be denoted by $subX$ and called the subobject class of X with respect to $(\mathcal{E},\mathcal{M})$. Since $subX$ may be large, we assume that $\mathcal{X}$ has multiple pullbacks of arbitrary large families of $\mathcal{M}$-morphisms with common codomain, with the pullback in $\mathcal{M}$. Consequently, $subX$ forms a (possibly large) complete lattice.

In what follows we will not distinguish between $m \in \mathcal{M}_X$ and the $\mathcal{M}$-subobject that it defines (i.e., $m = n$ means $m \simeq n$ when m and n are taken as subobjects).

If $X \xrightarrow{f} Y$ is an $\mathcal{X}$-morphism, $M \xrightarrow{m} X$ is a subobject of X and $M \xrightarrow{e} f(M) \xrightarrow{f(m)} Y$ is the $(\mathcal{E},\mathcal{M})$-factorization of $f \circ m$, then $f(M) \xrightarrow{f(m)} Y$ is the *image* of m along f. Moreover, if $N \xrightarrow{n} Y$ is a subobject of Y, then the pullback $f^{-1}(N) \xrightarrow{f^{-1}(n)} X$ of n along f is the *inverse image* of n along f. Clearly $f(-)$ and $f^{-1}(-)$ yield an adjoint situation between $subY$ and $subX$.

For the general theory of factorization structures for morphisms we refer to [11], [22] and [1].

Let $\mathcal{A}$ be any category, let $\mathcal{X}$ be as above and let $\mathcal{A} \xrightarrow{U} \mathcal{X}$ be a given functor. A pair (A, m) with A object of $\mathcal{A}$ and $M \xrightarrow{m} UA$ in $\mathcal{M}$ will be called *U-subobject* of A. We will write $sub_U A$ for the class of all U-subobjects of A (i.e., $sub_U A = \{A\} \times sub\, UA$) and we will simply refer to m for a U-subobject (A, m) of A when no confusion is possible.

1.1. DEFINITION. A *U-closure operator* c of $\mathcal{A}$ (with respect to $(\mathcal{E}$,$\mathcal{M})$) is a family of functions

$$(sub_U A \xrightarrow{c_A} sub_U A)_{A \in \mathcal{A}}$$

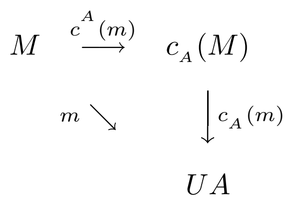
with the following properties that hold for each $A \in \mathcal{A}$ and U-subobjects m and n of A:

(a) $m \le c_A(m)$;

(b) $m \le n \Rightarrow c_A(m) \le c_A(n)$;

(c) For each $\mathcal{A}$-morphism $A \xrightarrow{f} B$, $Uf(c_A(m)) \le c_B(Uf(m))$.

Condition (a) implies that for every U-closure operator c on $\mathcal{A}$, every U-subobject $M \xrightarrow{m} UA$ of any object A has a canonical factorization

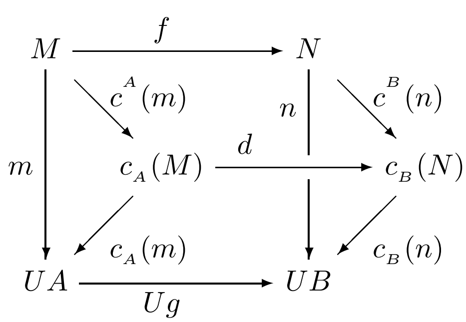


where $c_A(m)$ is called the U-*closure* of the U-subobject m of A.

1.2. REMARK.

(1) Notice that under condition (b) of the above definition, condition (c) is equivalent to the following statement concerning inverse images: if $A \xrightarrow{f} B$ is an $\mathcal{A}$-morphism and m is a U-subobject of B, then $c_A((Uf)^{-1}(m)) \le (Uf)^{-1}(c_B(m))$, i.e., the U-closure of the inverse image of m is less than or equal to the inverse image of the U-closure of m.

(2) Under condition (a), both conditions (b) and (c) together are equivalent to the following: given m and n U-subobjects of A and B, respectively, if f is an $\mathcal{X}$-morphism and g is an $\mathcal{A}$-morphism such that $n \circ f = Ug \circ m$, then there exists a unique morphism d such that the following diagram



commutes.

(3) The notion of closure operator of a category $\mathcal{A}$ with respect to a class

of monomorphisms $\mathcal{M}$ introduced in [7] can be easily seen as a special case of U-closure operator by simply taking $U = id_{\mathcal{A}}$.

We recall that an $\mathcal{A}$-morphism $A \xrightarrow{f} B$ is U-initial if for every $\mathcal{A}$-morphism $D \xrightarrow{h} B$ and $\mathcal{X}$-morphism $UD \xrightarrow{g} UA$ such that $(Uf) \circ g = Uh$, there exists a unique $\mathcal{A}$-morphism $D \xrightarrow{g'} A$ such that $Ug' = g$ and $f \circ g' = h$. (cf. [14, Definition 2.1(1)]).

1.3. DEFINITION. Given a U-closure operator c of $\mathcal{A}$, we say that a U-subobject m of an object A is *c-closed* (*c-dense*) if $m = c_A(m)$ ($m = c^A(m)$, equivalently, $c_A(m) = id_{UA}$). An $\mathcal{A}$-morphism $A \xrightarrow{f} B$ is called *c-dense* if $(Uf)(id_{UA})$ is c-dense. We call c *idempotent* provided that $c_A(m)$ is c-closed for every $A \in \mathcal{A}$ and U-subobject m of A. c is said to be *weakly hereditary* if for every $A \in \mathcal{A}$ and U-subobject m of A, whenever there is a U-initial $\mathcal{A}$-morphism $P \xrightarrow{p} A$ such that $c_A(m) = U(p)$, we have that $c_P(c^A(m)) = id_{c_A(M)}$, that is $c^A(m)$ is c-dense in $P = c_A(M)$. We call c *additive* if for every $A \in \mathcal{A}$ and U-subobjects m and n of A, we have that $c_A(m \vee n) = c_A(m) \vee c_A(n)$. We say that c is *grounded* if $c_A(0_A) = 0_A$ for every $A \in \mathcal{A}$, where 0_A is the smallest element of the lattice $sub_U A$

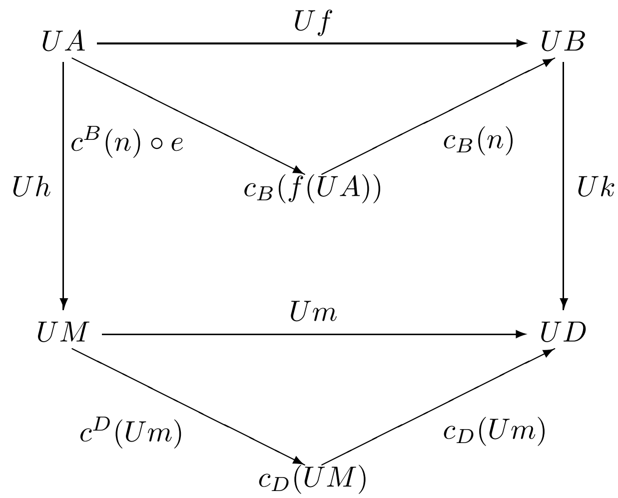
In what follows we extend to U-closure operators two classical results (Propositions 3.1 and 3.2 of [7]).

For any U-closure operator c on $\mathcal{A}$, $\mathcal{E}_c$ will denote the class of all c-dense $\mathcal{A}$-morphisms and $\mathcal{M}_c$ will denote the class of all U-initial $\mathcal{A}$-morphisms m such that $Um \in \mathcal{M}$ is c-closed. Notice that every morphism in $\mathcal{M}_c$ is a monomorphism.

1.4. PROPOSITION. (Diagonalization Property) *Let c be a U-closure operator and let $m \in \mathcal{M}_c$, $f \in \mathcal{E}_c$ and h and k $\mathcal{A}$-morphisms such that $k \circ f = m \circ h$; then there exists a unique $\mathcal{A}$-morphism $B \xrightarrow{d} M$ such that the following diagram commutes*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ h\big\downarrow & d\swarrow & \big\downarrow k \\ M & \xrightarrow[m]{} & D \end{array}$$

Proof. Let us consider the following commutative diagram



where (e, n) is the $(\mathcal{E}, \mathcal{M})$-factorization of Uf. Since $e \in \mathcal{E}$ and $Um \in \mathcal{M}$, the $(\mathcal{E}, \mathcal{M})$-diagonalization property yields a morphism $f(UA) \xrightarrow{t} UM$ such that $t \circ e = Uh$ and $(Um) \circ t = (Uk) \circ n$. From Remark 1.2(2) we obtain a morphism $c_B(f(UA)) \xrightarrow{d'} c_D(UM)$ such that $d' \circ c^B(n) = c^D(Um) \circ t$ and $c_D(Um) \circ d' = Uk \circ c_B(n)$. Since f is c-dense and Um is c-closed, $c_B(n)$ and $c^D(Um)$ are isomorphisms. Therefore, the composite of their inverses with d' yields a morphism $UB \xrightarrow{d''} UM$ such that $Um \circ d'' = Uk$ and $d'' \circ Uf = Uh$. Initiality of m implies the existence and uniqueness of the wanted $\mathcal{A}$-morphism d. □

1.5. PROPOSITION. *Let c be an idempotent and weakly hereditary U-closure operator. If, for every $\mathcal{A}$-morphism $A \xrightarrow{f} B$, the c-closure of the image of id_{UA} under Uf has an initial lift, then the pair $(\mathcal{E}_c, \mathcal{M}_c)$ is a factorization structure on $\mathcal{A}$.*

Proof. Let $A \xrightarrow{f} B$ be an $\mathcal{A}$-morphism and let $UA \xrightarrow{e} P \xrightarrow{m} UB$ be the $(\mathcal{E}, \mathcal{M})$-factorization of Uf. Since U initially lifts the c-closure of the image of id_{UA} under Uf, we obtain a factorization $A \xrightarrow{f} B = A \xrightarrow{e'} D \xrightarrow{m'} B$, where m' is the U-initial lift of $c_B(m)$ and e' is such that $U(e') = c^B(m) \circ e$ We shall show that (e', m') is an $(\mathcal{E}_c, \mathcal{M}_c)$-factorization of f. Notice that since c is idempotent, $c_B(m)$ is c-closed, and so $m' \in \mathcal{M}_c$. To see that $e' \in \mathcal{E}_c$, let us consider the commutative diagram

$$\begin{array}{ccc} UA & \stackrel{e}{\longrightarrow} & P \\ {\scriptstyle e_1}\big\downarrow & & \big\downarrow{\scriptstyle c^B(m)} \\ M_1 & \underset{m_1}{\longrightarrow} & c_B(P) \end{array}$$

where (e_1, m_1) is the $(\mathcal{E}, \mathcal{M})$-factorization of $U(e')$. The $(\mathcal{E}, \mathcal{M})$-diagonalization property yields a morphism d such that $m_1 \circ d = c^{B}(m)$ and $d \circ e = e_1$. From the monotonicity of c we have that $c_D(c^{B}(m)) \leq c_D(m_1) \leq id_{c_B(P)}$. Since c is weakly hereditary, we have that $c_D(c^{B}(m)) = id_{c_B(P)}$, and consequently, $c_D(m_1) = id_{c_B(P)}$. Therefore, $e' \in \mathcal{E}_c$.

The diagonalization property is Proposition 1.4. $\square$

2. Hull Operators and standard closure operators

A familiar way to produce idempotent closure operators of a category $\mathcal{X}$ is by means of a subclass $\mathcal{M}'$ of $\mathcal{M}$ which is stable under pullback. The same procedure works for U-closure operators if we assume that $\mathcal{M}'$ is stable under U-pullback. That is: for every $\mathcal{A}$-morphism $A \xrightarrow{f} B$ and $M' \xrightarrow{m'} UB$ in $\mathcal{M}'$ the pullback $(Uf)^{-1}(m')$ lies in $\mathcal{M}'$.

Let , in fact, $\mathcal{M}' \subset \mathcal{M}$ be stable under U-pullback. For every U-subobject m of A set

$$c_A^{\mathcal{M}'}(m) = \wedge\{M' \xrightarrow{m'} UA : m' \in \mathcal{M}' \text{ and } m \leq m'\}.$$

Then $c^{\mathcal{M}'}$ is an idempotent U-closure operator of $\mathcal{A}$ (the proof is provided by a slight adaptation of the proof in the classical case).

A special case of the above construction is obtained, for a given full subcategory $\mathcal{B}$ of $\mathcal{A}$, by taking $\mathcal{M}'$ to be the class of all existing equalizers of pairs of $\mathcal{X}$-morphisms of the form Uf, Ug with $f, g \in \mathcal{A}(A, B)$, $B \in \mathcal{B}$. The corresponding U-closure operator $c^{\mathcal{B}}$ is called (as in the classical case) the *regular* U-closure operator of $\mathcal{A}$ induced by $\mathcal{B}$ and is as useful as in the classical case. Precisely, assuming U to be faithful, a morphism in the category $\mathcal{B}$ is an epimorphism if and only if it is $c^{\mathcal{B}}$-dense.

Another special case is obtained by choosing a suitable class of $\mathcal{A}$-morphisms.

2.1. PROPOSITION. *Let $\mathcal{N}$ be a class of $\mathcal{A}$-morphisms that is stable under pullback and that satisfies the following conditions:*

(a) $U(\mathcal{N}) \subseteq \mathcal{M}$;

(b) U *preserves pullbacks of morphisms of* $\mathcal{N}$ *along arbitrary morphisms.*

Then the operator h *defined by setting, for every object* A *of* $\mathcal{A}$ *and* U*-subobject* m *of* A,

$$h_A^{\mathcal{N}}(m) = \wedge\{U(n) : n \in \mathcal{N} \text{ and } m \leq U(n)\}$$

is an idempotent U*-closure operator of* $\mathcal{A}$.

Proof. It is enough to show that $U(\mathcal{N})$ is stable under U-pullback. So, let $A \xrightarrow{f} B$ be an $\mathcal{A}$-morphism and let $n' = U(n)$ with $n \in \mathcal{N}$. Let us consider the following pullback diagrams:

$$\begin{array}{ccc} UA & \xrightarrow{Uf} & UB \\ {\scriptstyle (Uf)^{-1}(n')}\big\uparrow & & \big\uparrow{\scriptstyle n'} \\ (Uf)^{-1}(N') & \xrightarrow[t]{} & N' \end{array} \quad \text{and} \quad \begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle f^{-1}(n)}\big\uparrow & & \big\uparrow{\scriptstyle n} \\ f^{-1}(N) & \xrightarrow[d]{} & N \end{array}$$

By applying U to the diagram on the right, we obtain that $Uf \circ U(f^{-1}(n)) = n' \circ U(d)$. Condition (b) implies that $U(f^{-1}(n)) = (Uf)^{-1}(n')$. Therefore, since $\mathcal{N}$ is stable under pullback, we obtain that $(Uf)^{-1}(n') \in U(\mathcal{N})$. ☐

The U-closure operator $h_A^{\mathcal{N}}(m)$ is called the *hull operator* defined by $\mathcal{N}$. If U preserves meets of elements of $\mathcal{N}$ then the hull operator $h_A^{\mathcal{N}}(m)$ can be obtained by first producing, for every U-subobject of A, the $\mathcal{A}$-morphism

$$\bar{h}_A^{\mathcal{N}}(m) = \wedge\{n \in \mathcal{N} : m \leq U(n)\},$$

and then taking $U(\bar{h}_A^{\mathcal{N}}(m))$.

If now $\mathcal{N}$ is the second factor of a proper factorization structure of $\mathcal{A}$ (with $U(\mathcal{N}) \subset \mathcal{M}$) then we define, for every closure operator c of $\mathcal{A}$ with respect to $\mathcal{N}$, a closure ch of U-subobjects m of any object A in $\mathcal{A}$, as follows

$$ch_A^{\mathcal{N}}(m) = U(c_A(\bar{h}_A^{\mathcal{N}}(m)))$$

2.2. PROPOSITION. *Let* U *preserve images, inverse images and meets of elements of* $\mathcal{N}$. *Then the operator* ch *is a* U*-closure operator of* $\mathcal{A}$ *and it is idempotent whenever* c *is idempotent.*

Proof. Since U preserves meets of elements of $\mathcal{N}$ clearly $m \leq ch_A(m)$. and the monotonicity follows from the fact that both c and $\bar{h}$ are monotone.

Let $A \xrightarrow{f} B$ be an $\mathcal{A}$-morphism, we need to show that for every U-subobject m of A, we have that $Uf(ch_A(m)) \leq ch_B(Uf(m)) = U(c_B(\bar{h}_B(Uf(m))))$. Now, we have that $Uf(ch_A(m)) = UfU(c_A(\bar{h}_A(m))) = U(f(c_A(\bar{h}_A(m)))) \leq U(c_B(f(\bar{h}_A(m))))$. We also have that $f(\bar{h}_A(m)) = f(\wedge\{p \in \mathcal{N} : m \leq U(p)\}) \leq \wedge\{f(p) : p \in \mathcal{N}$ and $m \leq U(p)\}$ and $\bar{h}_B(Uf(m)) = \wedge\{p' \in \mathcal{N} : Uf(m) \leq U(p')\}$. Now let p' be such that $Uf(m) \leq Up'$ and let $p = f^{-1}(p')$. Since U preserves inverse images, i.e., $U(f^{-1}(p')) = (Uf)^{-1}(Up')$, we have that $U(p) = (Uf)^{-1}(Up') \geq (Uf)^{-1}((Uf)(m)) \geq m$. Finally, this together with $f(p) = f(f^{-1}(p')) \leq p'$ implies that $\wedge\{f(p) : p \in \mathcal{N}$ and $m \leq U(p)\} \leq \wedge\{p' \in \mathcal{M}' : U(f(m)) \leq U(p')\}$. Consequently we can conclude that $f(\bar{h}_A(m)) \leq \bar{h}_B(Uf(m))$. Thus, by applying c_B and the functor U to this last inequality we obtain the wanted result.

To show idempotency let $A \in \mathcal{A}$ and let m be a U-subobject of A. Consider the set $S = \{n \in \mathcal{N} : m \leq U(n)\}$. Its meet, $\bar{h}_A(m)$ belongs to $\mathcal{N}$ and so does $c_A(\bar{h}_A(m))$. Now, let us consider the set $S' = \{n \in \mathcal{N} : ch_A(m) \leq U(n)\}$. Since $m \leq ch_A(m) = U(c_A(\bar{h}_A(m)))$, we have that $c_A(\bar{h}_A(m))) \in S'$. Consequently, we obtain that $\wedge S' \leq c_A(\bar{h}_A(m)))$ and $c_A(\wedge S') \leq c_A(c_A(\bar{h}_A(m)))) = c_A(\bar{h}_A(m)))$. Notice that in this last equality we have used the idempotency of c. Thus, we have that $ch_A(ch_A(m)) = U(c_A(\wedge S')) \leq U(c_A(\bar{h}_A(m))) = ch_A(m)$. This together with $ch_A(m) \leq ch_A(ch_A(m))$ yields the idempotency of ch. □

2.3. DEFINITION. A U-closure operator a is called *standard* if there is a factorization structure $(\mathcal{F},\mathcal{N})$ of $\mathcal{A}$ and a closure operator c on $\mathcal{A}$ (with respect to $\mathcal{N}$) such that $a = ch$.

It is clear that a U-closure operator a will be *non-standard* if there is an a-closed U-subobject m of $\mathcal{A}$ for which there is no monomorphism n in $\mathcal{A}$ with $U(n) = m$.

Of course every hull operator is standard (e.g., the algebra generated by a subset of a given algebra, the convex hull of a subset of a (totally) convex space (see Example 3.5 below), the direct closure of a subset of a directed-complete partially ordered set (see Example 2.5 below)). Here are less trivial examples of standard operators.

2.4. EXAMPLE. A projection space is a set X equipped with a sequence of functions $(X \xrightarrow{\alpha_n} X)_{n\in\mathbf{N}}$ such that $\alpha_n \circ \alpha_m = \alpha_{min(n,m)}$. A morphism $(X, (\alpha_n)_{n\in\mathbf{N}}) \xrightarrow{f} (X, (\beta_n)_{n\in\mathbf{N}})$, that is a function $X \xrightarrow{f} Y$ such that for every $n \in \mathbf{N}$, $\beta_n \circ f = f \circ \alpha_n$ will be called a projection function. **Pro** will denote the category of projection spaces with projection functions. Let $\mathbf{Pro} \xrightarrow{\mathbf{U}} \mathbf{Set}$ be the obvious forgetful functor and let us consider the (surjective,injective)-

factorization structure in **Set**.

A U-closure operator of **Pro** is defined as follows:

$$s_X(M) = \{x \in X : \forall n \in \mathbf{N}, \exists p \in \mathbf{N} \text{ and } a \in M \text{ with } \alpha_n(x) = \alpha_p(a)\}.$$

The U-closure operator s is standard. Let, in fact, $\mathcal{F}$ and $\mathcal{N}$ denote the class of all surjective projection functions and the class of all injective projection functions, respectively. Then $(\mathcal{F}, \mathcal{N})$ is a proper factorization structure of the category **Pro** and $U(\mathcal{N}) \subset \mathcal{M}$. An explicit description of the hull operator h defined by $\mathcal{N}$ is

$$h_X(M) = \{x \in X : x \in M \text{ or } x = \alpha_n(a) \text{ for some } n \in \mathbf{N} \text{ and } a \in M\}$$

Let now c_∞ be the closure operator of **Pro** with respect to $\mathcal{N}$ studied in [12, Section 4]:

$$c_\infty(M) = \{x \in X : \alpha_n(x) \in M, \forall n \in \mathbf{N}\}$$

Then, it is easy to verify that $s_X(M) = c_\infty(h_X M)$.

2.5. EXAMPLE. A partially ordered set is directed-complete if every directed subset has a supremum. **DCPO** will denote the category of all directed-complete partially ordered sets and functions preserving directed joins. Let **DCPO** $\xrightarrow{\mathbf{U}}$ **Set** be the obvious forgetful functor and let us consider the (surjective,injective)-factorization structure in **Set**, say $(\mathcal{E}, \mathcal{M})$, and the (surjective,injective)-factorization structure, $(\mathcal{F}, \mathcal{N})$, in **DCPO**. The operator s which associates to every subset of any directed-complete partially ordered set the smallest Scott closed subset containing it (a subset is said to be Scott-closed if it is closed under directed joins and it is down closed (cf. [9, p. 63], [20] and Example 3.6 below)) is a standard U-closure operator of **DCPO** with respect to $\mathcal{M}$. In fact it can be considered as the composition of the hull operator induced by $\mathcal{N}$, which associates to every subset the smallest directed-complete subset containing it, and the closure operator of **DCPO** with respect to $(\mathcal{E}, \mathcal{M})$ (denoted by *scott* in [9]), which associates to every directed-complete subset the smallest Scott-closed subset containing it.

The next section is dedicated to examples of non-standard U-closure operators.

3. Examples of non-standard U-closure operators

3.1. EXAMPLE. In the category **TopGrp** of topological groups let $(\mathcal{F}, \mathcal{N})$ be the usual (surjective,embedding)-factorization structure, let **TopGrp** $\xrightarrow{\mathbf{U}}$

Top the functor that forgets the algebraic structure and let $(\mathcal{E}, \mathcal{M})$ be the usual (surjective,embedding)-factorization structure of **Top**. The Kuratowski closure induced by the topology is clearly a non-standard U-closure operator of **TopGrp** with respect to $(\mathcal{E}, \mathcal{M})$. The hull operator h defined by $\mathcal{N}$ gives the topological subgroup generated by a subspace (= subset). On the other hand, the Kuratowski closure k induced by the topology, when restricted to topological subgroups, is a closure operator of **TopGrp** with respect to $(\mathcal{F}, \mathcal{N})$. The combination of h followed by k gives the standard U-closure operator kh. This U-closure strongly differs from the Kuratowski closure induced by the topology, which is clearly non-standard.

3.2. EXAMPLE. Let **Rng** be the category of unitary rings and ring-homomorphisms, let $\mathbf{Rng} \xrightarrow{\mathbf{U}} \mathbf{Set}$ be the usual forgetful functor and let $(\mathcal{E}, \mathcal{M})$ be the (surjective, injective)-factorization structure of **Set**. The class $\mathcal{M}'$ of all inclusions of ideals is closed under U-pullbacks, then for every ring A and for every subset $M \xrightarrow{m} UA$, the operator which associates to every subset M of a ring A the (underlying set) of the ideal generated by M is an idempotent U-closure operator and it is non-standard since no proper ideal is a subring.

3.3. EXAMPLE. Let **Vec** be the category of real vector spaces and linear transformations, let $\mathbf{Vec} \xrightarrow{\mathbf{U}} \mathbf{Set}$ be the usual forgetful functor and let $(\mathcal{E}, \mathcal{M})$ be the (surjective,injective)-factorization structure of **Set**. If M is a subset of a vector space V, let us define:

$$r_V(M) = \{x \in V : \alpha x \in M \text{ for some } \alpha \in (0, 1]\}.$$

Since $1v \in M$ for every $v \in M$, we have that $M \subseteq r_V(M)$. Also, it is straightforward that if $M \subseteq N$, then $r_V(M) \subseteq r_V(N)$. Now, let $V \xrightarrow{f} W$ be a linear transformation and let M be a subset of V. If $\alpha x \in M$, then $\alpha(f(x)) = f(\alpha x) \in f(M)$. Therefore, $f(r_V(M)) \subseteq r_W(f(M))$. The U-closure operator r is the identity on linear subspaces and it is clearly non-standard. As a matter of fact, if $0 \notin M$, then we also have that $0 \notin r_V(M)$. Therefore, in this case $r_V(M)$ is not the underlying set of a linear subspace. Another non-standard U-closure operator can be defined by:

$$c_V(M) = \{x \in V : x = \alpha a + (1-\alpha)b \text{ for some } 0 \leq \alpha \leq 1 \text{ and } a, b \in M\}$$

This U-closure operator is the identity on linear subspaces. It is neither idempotent nor additive. As a matter of fact, the c-closure of three linearly independent vectors is the border of the triangle having the three vectors as vertices but the c-closure of this border includes also the inside of the triangle. Moreover, the c-closure of a point is the point itself but the c-closure of two points is the segment that joins them. The idempotent hull $\hat{c}$ of c is clearly the convex hull construction, that is, $\hat{c}_V(M)$ is the smallest convex subset of V containing M.

3.4. EXAMPLE. For each subset M of a projection space $(X, (\alpha_n)_{n\in\mathbf{N}})$, (cf. Example 2.4) let us define

$$t_X(M) = \{x \in X : x \in M, \text{ or } \alpha_n(x) \in M, \forall n \in \mathbf{N}\}$$

Trivially t_X is extensive and monotone. For the continuity property, assume that $y \in f(t_X(M))$, i.e., assume that $y = f(x)$ for some $x \in t_X(M)$. If $x \in M$, then $y = f(x) \in f(M) \subseteq f(t_X(M))$. If $x \in t_X(M) - M$, then for every $n \in \mathbf{N}$, $\alpha_n(x) \in M$. This implies that $\beta_n(y) = \beta_n(f(x)) = f(\alpha_n(x)) \in f(M)$, for every $n \in \mathbf{N}$. Thus, we have that $y \in t_Y(f(M))$. Notice that the above closure coincides with s and consequently with c_∞ on subprojection spaces. Moreover, $t \leq s$ and they differ on subsets that are not subprojection spaces. In fact, $x \in t_X(M) - M$, i.e., $\alpha_n(x) \in M$ for every $n \in \mathbf{N}$, implies that for every $n \in \mathbf{N}$, $\alpha_n(x) = \alpha_n(\alpha_n(x)) = \alpha_n(y)$, with $y \in M$. So, $x \in s_X(M)$. If M is not a subprojection space and $y \in M$ is such that $z = \alpha_m(y)$ is not in M for some $m \in \mathbf{N}$, then $z \notin t_X(M)$ since $\alpha_m(z) = \alpha_m(\alpha_m(y)) = \alpha_m(y) = z \notin M$. However, $z \in s_X(M)$, since for every $n \in \mathbf{N}$, $\alpha_n(z) = \alpha_n(\alpha_m(y)) = \alpha_{min(n,m)}(y)$ and $y \in M$. The U-closure operator t is clearly non-standard since closures of non-subprojection spaces are not subprojection spaces.

The last two examples show that non-standard U-closure operators may be useful to describe, with the help of hull operators, classical closure operators.

3.5. EXAMPLE. Consider the set $\Omega = \{(\alpha_n)_{n\in\mathbf{N}} \in \Re^{\mathbf{N}} : \sum_n |\alpha_n| \leq 1\}$. A totally convex space in the sense of [18] is a nonempty set X with an "operation" induced by Ω. The result of this "operation" is written as a formal sum $\sum_n \alpha_n x_n$ with $x_n \in X$ and the operations are required to satisfy the axioms:

(TC1) $\sum_n \delta_n^m x_n = x_m$

(TC2) $\sum_n \alpha_n (\sum_m \beta_m^n x_m) = \sum_m (\sum_n \alpha_n \beta_m^n) x_m$.

The morphisms in the category **TC** of totally convex spaces are functions $X \xrightarrow{f} Y$ that satisfy $f(\sum_n \alpha_n x_n) = \sum_n \alpha_n (f(x_n))$.

The pair $(\mathcal{F}, \mathcal{N})$ =(surjective,injective) is a factorization structure of **TC**. A subspace M of a totally convex space X is called radially closed in X if for each $x \in X$ and real number α such that $0 < |\alpha| \leq 1$, $\alpha x \in M$ implies $x \in M$. Pumplün and Röhrl considered in [18] the closure operator of **TC** with respect to $\mathcal{N}$ that to each subspace M of a totally convex space X, associates the smallest radially closed subspace $\hat{M}$ containing M. They used this operator to describe the epimorphisms of the subcategory of separated totally convex spaces. In [17], $\hat{M}$ is characterized as being the smallest subspace of X containing M that for every $x \in X$ has the following property: if for every $0 < \epsilon < 1$, there exist $a \in M$, $y \in X$ and $0 < \alpha < 1$ with $\alpha x = \alpha(\epsilon y) + (1 - \alpha)a$, then $x \in \hat{M}$.

Let us consider now the forgetful functor $\mathbf{TC} \xrightarrow{\mathbf{U}} \mathbf{Set}$. If $(\mathcal{E}, \mathcal{M})$ is the usual (surjective, injective)-factorization structure in **Set**, a non-standard U-closure operator of **TC** is defined for every subset M of a totally convex space X by:

$$r_X(M) = \{x \in X : \exists \alpha \text{ with } 0 < |\alpha| \leq 1 \text{ and } \alpha x \in M\}$$

In fact, by (TC1), r is extensive and it is clearly monotone. Also the continuity property is trivially verified (cf. also [19, Propositions 1.2 and 1.4]). The U-closure operator r need not preserve subspaces. So, in particular it is non-standard. However, this operator is useful to describe the closure operator radial hull. In fact, it was shown in [19, 1.7] that the radial hull operator is obtained by alternatively iterating r and the hull operator with respect to $\mathcal{N}$ (i.e., the usual convex-hull operator).

3.6. EXAMPLE. Let us consider the category **DCPO** as in Example 2.5, let $\mathbf{DCPO} \xrightarrow{\mathbf{U}} \mathbf{Set}$ be the usual forgetful functor and let $(\mathcal{E}, \mathcal{M})$ be the (surjective,injective)-factorization structure in **Set**. For every subset M of a dcpo X set

$$\downarrow_X M = \{x \in X : x \leq m, \text{ for some } m \in M\}.$$

It is easy to see that the above down closure is a U-closure operator of **DCPO** and that it is non-standard (in general $\downarrow_X M$ is not a sub-dcpo even if M is). However, the U-closure *scott* in Example 2.5 is obtained by iterating the direct-hull closure (which to every subset associates the smallest directed-complete subset containing it) and the down closure described above.

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