CLOSURE OPERATORS AND CONNECTEDNESS

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ABSTRACT: This paper introduces the notion of connectedness with respect to a closure operator on a construct $\mathcal{X}$. Many classical results about topological connectedness are extended to this setting. Herrlich's connectedness-disconnectedness Galois connection is shown to factor via the collection of all closure operators on $\mathcal{X}$.

KEY WORDS: Closure operator, connectedness, disconnectedness, Galois connection, discrete object, indiscrete object, constant morphism.

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0 INTRODUCTION

The development of a general theory about topological connectedness was started by Preuß (cf. [P_1], [P_2], [P_3]) and by Herrlich ([H]). Further literature on this topic can be found in [AW], [CC], [Cl], [HP], [L], [T] and [SV]. In this paper we present a notion of *connectedness with respect to a closure operator on a construct* $\mathcal{X}$. This notion generalizes the classical concept of connectedness in topology, extending the concept to categories whose objects are structured sets that do not necessarily carry a topological structure. Because of the relationship with closure operators, our notion yields a much closer analog of topological connectedness than the one introduced by Preuß. In fact, many classical results about connectedness in topology can be generalized.

A Galois connection between classes of connectedness and classes of disconnectedness of a given category was presented in [H]. In this paper we show that this Galois connection factors through the collection of all closure operators on the construct $\mathcal{X}$. As a consequence of this factorization, every connectedness class can be seen as the class of indiscrete objects of a closure operator and every class of disconnectedness can be seen as the class of discrete objects of some closure operator.

On first impression, our definition of connectedness with respect to a closure operator might appear to be a special case of Preuß's definition. However, our factorization of Herrlich's Galois connection enables us to see any connectedness class in the sense of Preuß as a connectedness

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class with respect to a closure operator, i.e., in our sense. Therefore, with our approach we gain the advantage of having a closure operator on the category $\mathcal{X}$ with all its nice implications, without losing anything from Preuß's definition.

We use the terminology of [AHS] throughout the paper.

1 PRELIMINARIES

The symbol $\mathcal{X}$ will denote a construct, that is, a concrete category whose objects are structured sets and whose morphisms are structure preserving functions (cf. [AHS]).

We begin by recalling the following

DEFINITION 1.1

$\mathcal{X}$ is called an $(\mathbf{E}, \mathcal{M})$-category for sinks, if there exists a collection $\mathbf{E}$ of $\mathcal{X}$-sinks, and a class $\mathcal{M}$ of $\mathcal{X}$-morphisms such that:

(a) each of $\mathbf{E}$ and $\mathcal{M}$ is closed under compositions with isomorphisms;

(b) $\mathcal{X}$ has $(\mathbf{E}, \mathcal{M})$-factorizations (of sinks); i.e., each sink $\mathbf{s}$ in $\mathcal{X}$ has a factorization $\mathbf{s} = m \circ \mathbf{e}$ with $\mathbf{e} \in \mathbf{E}$ and $m \in \mathcal{M}$, and

(c) $\mathcal{X}$ has the unique $(\mathbf{E}, \mathcal{M})$-diagonalization property; i.e., if $B \xrightarrow{g} D$ and $C \xrightarrow{m} D$ are $\mathcal{X}$-morphisms with $m \in \mathcal{M}$, and $\mathbf{e} = (A_i \xrightarrow{e_i} B)_I$ and $\mathbf{s} = (A_i \xrightarrow{s_i} C)_I$ are sinks in $\mathcal{X}$ with $\mathbf{e} \in \mathbf{E}$, such that $m \circ \mathbf{s} = g \circ \mathbf{e}$, then there exists a unique diagonal $B \xrightarrow{d} C$ such that for every $i \in I$ the following diagrams commute:

$$\begin{array}{ccc} A_i & \xrightarrow{e_i} & B \\ {\scriptstyle s_i}\downarrow & \swarrow{\scriptstyle d} & \\ C & & \end{array} \qquad \text{and} \qquad \begin{array}{ccc} & & B \\ & {\scriptstyle d}\swarrow & \downarrow{\scriptstyle g} \\ C & \xrightarrow[m]{} & D \end{array}$$

These requirements yield the following features of the class $\mathcal{M}$ (cf. [AHS] for the dual case):

PROPOSITION 1.2

(a) Every m in $\mathcal{M}$ is a monomorphism.

(b) $\mathcal{M}$ is closed under $\mathcal{M}$-relative first factors; i.e., if $n \circ m \in \mathcal{M}$, and $n \in \mathcal{M}$, then $m \in \mathcal{M}$.

(c) $\mathcal{M}$ is closed under composition.

(d) $\mathcal{M}$ is closed under intersections

(e) Pullbacks of $\mathcal{X}$-morphisms in $\mathcal{M}$ exist and belong to $\mathcal{M}$.

(f) *The $\mathcal{M}$-subobjects of every $\mathcal{X}$-object form a (possibly large) complete lattice; suprema are formed via* $(\mathbf{E}, \mathcal{M})$*-factorizations*

(g) *If in addition we require $\mathcal{X}$ to have equalizers and $\mathcal{M}$ to contain all regular monomorphisms, then* $\mathbf{E}$ *consists of episinks.* □

Since $(\mathbf{E}, \mathcal{M})$-factorizations are unique up to isomorphism, we will be talking about "the $(\mathbf{E}, \mathcal{M})$-factorization" of a sink.

A pre-order (i.e., a reflexive and transitive relation) on the $\mathcal{M}$-subobjects of every $\mathcal{X}$-object is defined as follows: given two $\mathcal{M}$-subobjects $M \xrightarrow{m} X$ and $N \xrightarrow{n} X$, we say that $m \leq n$ if there exists an $\mathcal{X}$-morphism $M \xrightarrow{t} N$ such that $n \circ t = m$.

Notice that whenever no confusion is likely to arise we use the object-oriented notation $M \leq N$ with the same meaning as $m \leq n$.

We regard $\mathcal{M}$ as a full subcategory of the arrow category of $\mathcal{X}$, with the codomain functor from $\mathcal{M}$ to $\mathcal{X}$ denoted by U. Since U is faithful, $\mathcal{M}$ is concrete over $\mathcal{X}$.

DEFINITION 1.3 ([DG$_2$])

A *closure operator* on $\mathcal{X}$ (with respect to $\mathcal{M}$) is a pair $C = (\gamma, F)$, where F is an endofunctor on $\mathcal{M}$ that satisfies $UF = U$, and γ is a natural transformation from $id_{\mathcal{M}}$ to F that satisfies $(id_U)\gamma = id_U$.

REMARK 1.4

The following characterization of the above definition will be used throughout the paper.

Given a closure operator $C = (\gamma, F)$, every morphism $M \xrightarrow{m} X$ in $\mathcal{M}$ has a canonical factorization

$$\begin{array}{ccc} M & \xrightarrow{]m[_C^X} & [M]_C^X \\ & {\scriptstyle m} \searrow & \downarrow {\scriptstyle [m]_C^X} \\ & & X \end{array}$$

where $[m]_C^X = F(m)$ is called the *C-closure* of m, and $]m[_C^X$ is the domain of the m-component of γ. The functor $[\ ]_C$, that is the endofunctor F, induces an order-preserving expansive function $[\ ]_C^X$ on the $\mathcal{M}$-subobject lattice of every $\mathcal{X}$-object, and these functions are related in the following sense: if p is the pullback of an $\mathcal{M}$-morphism $M \xrightarrow{m} Y$ along some $\mathcal{X}$-morphism $X \xrightarrow{f} Y$, and q is the pullback of $[m]_C^Y$ along f, then $[p]_C^X \leq q$.

Conversely, any family $\{\phi_X\}_{X \in \mathcal{X}}$ of order-preserving expansive functions on the $\mathcal{M}$-subobject lattices with the property: if p is the pullback of an $\mathcal{M}$-morphism $M \xrightarrow{m} Y$ along some $\mathcal{X}$-

morphism $X \xrightarrow{f} Y$, and q is the pullback of $\phi_Y(m)$ along f, then $\phi_X(p) \le q$, uniquely determines a closure operator.

Notice that to denote the C-closure of the $\mathcal{M}$-subobject $M \xrightarrow{m} X$ we normally write $[m]_C^X$ instead of the more complete expression $[M]_C^X \xrightarrow{[m]_C^X} X$. We might simply write $[M]_C^X$ whenever we want to focus on the object part and the morphism $[m]_C^X$ is an obvious one.

DEFINITION 1.5

Given a closure operator C, we say that a morphism $m \in \mathcal{M}$, $M \xrightarrow{m} X$ is C-*closed* if $]m[_C^X$ is an isomorphism. In this case, M will also be called C-*closed*. An $\mathcal{X}$-morphism $X \xrightarrow{f} Y$ is called C-*dense* if for every $(\mathcal{E}, \mathcal{M})$-factorization (e, m) of f we have that $[m]_C^Y$ is an isomorphism. We call C *idempotent* (*weakly hereditary*) provided that that $[m]_C^X$ is C-closed ($]m[_C^X$ is C-dense) for every $\mathcal{M}$-subobject $M \xrightarrow{m} X$. Furthermore, C is said to be *hereditary* if whenever $M \xrightarrow{m} X$, $M \xrightarrow{t} N$ and $N \xrightarrow{n} X$ are morphisms in $\mathcal{M}$ with $n \circ t = m$, we have that $[t]_C^N$ is the pullback of $[m]_C^X$ along n. This is often expressed as: $[M]_C^N \simeq [M]_C^X \cap N$.

A special case of an idempotent closure operator arises in the following way. Given any class $\mathcal{A}$ of $\mathcal{X}$-objects and $M \xrightarrow{m} X$ in $\mathcal{M}$, define $[m]_{\mathcal{A}}^X$ to be the intersection of all equalizers of pairs of $\mathcal{X}$-morphisms r, s from X to some $\mathcal{A}$-object A that satisfy $r \circ m = s \circ m$, and let $]m[_{\mathcal{A}}^X \in \mathcal{M}$ be the unique $\mathcal{X}$-morphism by which m factors through $[m]_{\mathcal{A}}^X$. It is easy to see that the functor $[\ ]_{\mathcal{A}}$ induces an idempotent closure operator $C_{\mathcal{A}}$. This generalizes the Salbany construction of closure operators induced by classes of topological spaces; cf. [S]. To simplify the notation, instead of "$C_{\mathcal{A}}$-dense" we usually write "$\mathcal{A}$-dense".

We denote the collection of all closure operators on $\mathcal{M}$ by $\mathbf{CL}(\mathcal{X}, \mathcal{M})$ pre-ordered as follows: $C \sqsubseteq D$ if $[m]_C^X \le [m]_D^X$ for all $\mathcal{M}$-subobjects $M \xrightarrow{m} X$ and for all $X \in \mathcal{X}$. We say that C and D are equivalent if both $C \sqsubseteq D$ and $D \sqsubseteq C$ hold. Notice that arbitrary suprema and infima exist in $\mathbf{CL}(\mathcal{X}, \mathcal{M})$, they are formed pointwise in the $\mathcal{M}$-subobject fibers.

For more background on closure operators see, e.g., [C_1], [CKS], [DG_2], [DGT] and [K].

DEFINITION 1.6

For pre-ordered classes $\mathcal{X} = (\mathbf{X}, \sqsubseteq)$ and $\mathcal{Y} = (\mathbf{Y}, \sqsubseteq)$, a *Galois connection* $\mathcal{X} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{Y}$ consists of order preserving functions F and G that satisfy $F \dashv G$; i.e., $x \sqsubseteq GF(x)$ for every $x \in \mathbf{X}$ and $FG(y) \sqsubseteq y$ for every $y \in \mathbf{Y}$. (G is adjoint and has F as coadjoint.)

$x \in \mathbf{X}$ (resp. $y \in \mathbf{Y}$) is called a fixed point of the Galois connection $\mathcal{X} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{Y}$ if $GF(x) = x$ (resp. $FG(y) = y$).

2 GENERAL RESULTS ABOUT C-CONNECTEDNESS

The main purpose of this section is to introduce a notion of connectedness with respect to a closure operator C on the construct $\mathcal{X}$ and show that most classical results about topological connectedness can be generalized to this setting.

Throughout the paper we will make the following

ASSUMPTIONS 2.1

(a) The construct $\mathcal{X}$ is an $(\mathbf{E}, \mathcal{M})$-category for sinks;

(b) $\mathcal{X}$ has a superstrong[2] and minimal[3] terminal object T that is a singleton;

(c) $\mathcal{X}$ has equalizers;

(d) $\mathcal{M}$ contains all regular monomorphisms and all morphisms that have T as domain.

Unless otherwise specified, C will always denote a closure operator on $\mathcal{X}$ with respect to the given class $\mathcal{M}$ of $\mathcal{X}$-monomorphisms.

DEFINITION 2.2

(a) An $\mathcal{X}$-object X is called *C-discrete* if $X \neq \emptyset$ and for every $\mathcal{M}$-subobject $M \xrightarrow{m} X$, m is C-closed.

(b) An $\mathcal{X}$-object X is called *C-indiscrete* if $X \neq \emptyset$ and for every $\mathcal{M}$-subobject $M \xrightarrow{m} X$ with $M \neq \emptyset$, m is C-dense.

If $X \xrightarrow{f} Y$ is an $\mathcal{X}$-morphism then $f(X)$ will denote the middle object of the $(\mathbf{E}, \mathcal{M})$-factorization (e_f, m_f) of f and $f^{-1}(f(X))$ will denote the corresponding pullback.

DEFINITION 2.3

An $\mathcal{X}$-morphism $X \xrightarrow{f} Y$ is called constant if $f(X) \simeq T$.

Notice that since $\mathbf{E}$ and $\mathcal{M}$ are closed under composition with isomorphisms, a constant morphism always has an $(\mathbf{E}, \mathcal{M})$-factorization with T as middle object.

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[2] We call a terminal object T *superstrong* if $Hom(T, X) \neq \emptyset$ for every $\mathcal{X}$-object $X \neq \emptyset$ and for every epimorphism $Y \xrightarrow{f} X$ and morphism $T \xrightarrow{t} X$, there exists a morphism $T \xrightarrow{k} Y$ such that $f \circ k = t$.

[3] We call a terminal object T *minimal* if for every $\mathcal{M}$-subobject $M \xrightarrow{m} X$ with $M \neq \emptyset$, $M \leq T$ implies $M \simeq T$.

DEFINITION 2.4

An $\mathcal{X}$-object X is called *C-connected* iff for every C-discrete $\mathcal{X}$-object D, any $\mathcal{X}$-morphism $X \xrightarrow{f} D$ is constant.

PROPOSITION 2.5

Let $X \xrightarrow{f} Y$ be an epimorphism in $\mathcal{X}$. If X is C-connected, then so is Y.

Proof:

Let D be a C-discrete $\mathcal{X}$-object and let $Y \xrightarrow{d} D$ be an $\mathcal{X}$-morphism. Since X is C-connected, $d \circ f$ must be constant. Let (t_X, g) be its $(\mathbf{E}, \mathcal{M})$-factorization with $X \xrightarrow{t_X} T$ being the unique morphism and let (e_d, m_d) be the $(\mathbf{E}, \mathcal{M})$-factorization of d. Since $m_d \circ e_d \circ f = g \circ t_X$, the diagonalization property gives a morphism $T \xrightarrow{k} d(Y)$ such that $k \circ t_X = e_d \circ f$ and $m_d \circ k = g$. Let $d(Y) \xrightarrow{t} T$ be the unique morphism. Clearly $t \circ k = id_T$. Now, $k \circ t \circ e_d \circ f = k \circ t \circ k \circ t_X = k \circ id_T \circ t_X = k \circ t_X = e_d \circ f = id_{d(Y)} \circ e_d \circ f$. By our assumptions, e_d and f are both epimorphisms (cf. Proposition 1.2(g)), which implies that $k \circ t = id_{d(Y)}$. Thus, $d(Y) \simeq T$, i.e., d is constant. $\square$

REMARK 2.6

Suppose that the category $\mathcal{X}$ has products and that the projections are epimorphisms. Then from the above proposition we obtain that if the product of a family of $\mathcal{X}$-objects is C-connected, so is each of its factors. However, the converse is not true. As a counterexample, it is enough to consider in the category **Ab** of Abelian Groups, the subcategory **Tor** consisting of all Torsion Abelian Groups. As Example 4.5 shows, this subcategory is the connectedness class of a certain closure operator, but it is not closed under products.

PROPOSITION 2.7

(a) *Let C be idempotent and let $M \xrightarrow{m} X$ be a C-dense $\mathcal{M}$-subobject of $X \in \mathcal{X}$. If M is C-connected, then so is X.*

(b) *Let C be weakly hereditary and idempotent and let $M \xrightarrow{m} X$ be an $\mathcal{M}$-morphism. If M is C-connected then so is $[M]_C^X$.*

Proof:

(a). Let $X \xrightarrow{d} D$ be a morphism into the C-discrete object D. Since M is C-connected, there is an $(\mathbf{E}, \mathcal{M})$-factorization (t_M, h) of $d \circ m$ with middle object T. Now let $d^{-1}(T) \xrightarrow{\bar{h}} X$ be the pullback of $T \xrightarrow{h} D$ along d. Clearly we have that $M \leq d^{-1}(T)$. The C-denseness of m and the idempotency of C imply that $X \simeq [M]_C^X \leq [d^{-1}(T)]_C^X \simeq d^{-1}(T)$. Notice that this is true, since $T \xrightarrow{h} D$ is C-closed and so is its pullback $d^{-1}(T) \xrightarrow{\bar{h}} X$. Now, $d(X) \simeq (d \circ \bar{h})(d^{-1}(T)) \leq T$. Since $d(X) \neq \emptyset$ and T is minimal, we obtain that $T \simeq d(X)$. Thus, X is C-connected.

(b). Just observe that since C is weakly hereditary, $]m[^X_C$ is C-dense and apply part (a). $\square$

PROPOSITION 2.8

Let $(M_i \xrightarrow{m_i} X)_{i \in I}$ be a family of $\mathcal{M}$-subobjects of $X \in \mathcal{X}$. If each M_i is C-connected and $\cap M_i \neq \emptyset$, then the supremum $\vee M_i$ of the family $\{M_i\}_{i \in I}$ is also C-connected.

Proof:

Let us consider the following commutative diagram

$$\begin{array}{ccc} \cap M_i & \xrightarrow{r_i} & \vee M_i \\ {\scriptstyle n_i}\big\downarrow & \nearrow^{e_i} & \big\downarrow{\scriptstyle m} \\ M_i & \xrightarrow[m_i]{} & X \end{array}$$

where n_i and m are the appropriate subobject morphisms and $r_i = e_i \circ n_i$ for every $i \in I$. Let $\cap M_i \xrightarrow{t} X$ be the morphism that satisfies $m_i \circ n_i = t$ for every $i \in I$ and let $\vee M_i \xrightarrow{d} D$ be a morphism into the C-discrete object D. Since M_i is C-connected for every $i \in I$, we have that $d \circ e_i$ is constant for every $i \in I$, i.e., the following diagram commutes for every $i \in I$

$$\begin{array}{ccc} M_i & \xrightarrow{e_i} & \vee M_i \\ {\scriptstyle t_i}\big\downarrow & & \big\downarrow{\scriptstyle d} \\ T & \xrightarrow[h_i]{} & D \end{array}$$

with (t_i, h_i) being the $(\mathbf{E}, \mathcal{M})$-factorization of $d \circ e_i$.

Notice that $m \circ r_i = m \circ e_i \circ n_i = m_i \circ n_i = t$, for every $i \in I$. This implies that $m \circ r_i = m \circ r_j$ for every $i, j \in I$. Thus $r_i = r_j$, since m is a monomorphism.

Since $x \in \cap M_i \neq \emptyset$, there exists a morphism $T \xrightarrow{f} \cap M_i$. Note that $t_i \circ n_i \circ f = id_T$ and $d \circ e_i \circ n_i = h_i \circ t_i \circ n_i$ for all $i \in I$. Thus $d \circ e_i \circ n_i = h_i \circ t_i \circ n_i = d \circ r_i$. Since $r_i = r_j$ for all $i, j \in I$, we have that $h_i \circ t_i \circ n_i \circ f = h_j \circ t_j \circ n_j \circ f$ for all $i, j \in I$. Consequently, $h_i \circ id_T = h_j \circ id_T$, for all $i, j \in I$, and so $h_i = h_j$ for all $i, j \in I$. Call this morphism h.

Now let (e_d, m_d) be the $(\mathbf{E}, \mathcal{M})$-factorization of d and let $d(\vee M_i) \xrightarrow{q} T$ be the unique morphism. The diagonalization property yields a morphism $T \xrightarrow{k} d(\vee M_i)$ such that $m_d \circ k = h$ and $k \circ t_i = e_d \circ e_i$, for every $i \in I$. Clearly, $q \circ k = id_T$. Now, $k \circ q \circ e_d \circ e_i = k \circ q \circ k \circ t_i = k \circ id_T \circ t_i = k \circ t_i = e_d \circ e_i = id_{d(\vee M_i)} \circ e_d \circ e_i$, for every $i \in I$. Since by our assumptions $(M_i \xrightarrow{e_i} \vee M_i)$ is an episink and e_d is an epimorphism (cf. Proposition 1.2(f), (g)), we obtain that $k \circ q = id_{d(\vee M_i)}$. Therefore $d(\vee M_i) \simeq T$, i.e., d is constant. $\square$

Notice that in view of the above proposition, for every singleton C-connected $\mathcal{M}$-subobject

$\{x\}$ of $X \in \mathcal{X}$, there exists a largest C-connected $\mathcal{M}$-subobject of $\mathcal{X}$ that has $\{x\}$ as subobject. Therefore we can give the following

DEFINITION 2.9

Let $X \in \mathcal{X}$ and let $\{x\}$ be a C-connected $\mathcal{M}$-subobject of X. The largest C-connected $\mathcal{M}$-subobject of X that has $\{x\}$ as subobject will be called the *C-component* of $\{x\}$ in X.

REMARK 2.10

Notice that Proposition 2.8 implies that distinct C-components of the same $\mathcal{X}$-object X must be disjoint.

PROPOSITION 2.11

If C is weakly hereditary and idempotent, then C-components are C-closed.

Proof:

Let $X \in \mathcal{X}$ and let C_X be a C-component in X. Let us consider the commutative diagram

$$\begin{array}{ccc} C_X & \xrightarrow{]m[_C^X} & [C_X]_C^X \\ & m \searrow & \downarrow [m]_C^X \\ & & X \end{array}$$

We know that C_X is C-connected and from Proposition 2.7(b) so is $[C_X]_C^X$. By the maximality of C-components, we have that $C_X \simeq [C_X]_C^X$. Thus C_X is C-closed. $\square$

COROLLARY 2.12

Let C be weakly hereditary and idempotent. Every C-indiscrete $\mathcal{X}$-object X is C-connected.

Proof:

Since $X \neq \emptyset$, there exists an $\mathcal{X}$-morphism $T \xrightarrow{f} X$. Let $T \xrightarrow{d} D$ be a morphism into the C-discrete object D. Since (id_T, d) is an $(\mathbf{E}, \mathcal{M})$-factorization of d, we have that T is a singleton C-connected $\mathcal{M}$-subobject of X. From the above proposition, the C-component of T in X is C-closed and so it must be isomorphic to X. Thus X is C-connected. $\square$

3 A FACTORIZATION OF THE CONNECTEDNESS-DISCONNECTEDNESS GALOIS CONNECTION

In this section we present a factorization of Herrlich's ([H]) connectedness-disconnectedness Galois connection and we show that any connectedness class in Preuß's sense can be seen as a connectedness class with respect to a closure operator, i.e., in our sense.

Let $S(\mathcal{X})$ denote the collection of all full subcategories of $\mathcal{X}$ whose objects are nonempty, ordered by inclusion.

PROPOSITION 3.1

Let $CL(\mathcal{X},\mathcal{M}) \xrightarrow{D} S(\mathcal{X})^{\mathbf{op}}$ and $S(\mathcal{X})^{\mathbf{op}} \xrightarrow{T} CL(\mathcal{X},\mathcal{M})$ be defined as follows:

$D(C)$ is the full subcategory with objects all $X \in \mathcal{X}$ such that X is C-discrete

$T(\mathcal{A}) = Sup\{C \in CL(\mathcal{X},\mathcal{M}) : D(C) \supseteq \mathcal{A}\}$.

Then, $CL(\mathcal{X},\mathcal{M}) \underset{T}{\overset{D}{\rightleftarrows}} S(\mathcal{X})^{\mathbf{op}}$ is a Galois connection.

Proof:

First of all, we recall that suprema exist in $CL(\mathcal{X},\mathcal{M})$, so T is well defined.

Clearly, both D and T preserve the order.

It is immediate to see that $C \sqsubseteq TD(C)$. Now, let $X \in \mathcal{A} \in S(\mathcal{X})^{\mathbf{op}}$. Since the supremum in $CL(\mathcal{X},\mathcal{M})$ is taken pointwise on the $\mathcal{M}$-subobject fibers, for every $\mathcal{M}$-subobject $M \xrightarrow{m} X$, we have that $[M]^X_{T(\mathcal{A})} \simeq M$. Therefore $DT(\mathcal{A}) \leq \mathcal{A}$. □

Similarly we can prove the following

PROPOSITION 3.2

Let $CL(\mathcal{X},\mathcal{M}) \xrightarrow{I} S(\mathcal{X})$ and $S(\mathcal{X}) \xrightarrow{J} CL(\mathcal{X},\mathcal{M})$ be defined as follows:

$I(C)$ is the full subcategory with objects all $X \in \mathcal{X}$ such that X is C-indiscrete

$J(\mathcal{B}) = Inf\{C \in CL(\mathcal{X},\mathcal{M}) : I(C) \supseteq \mathcal{B}\}$.

Then, $S(\mathcal{X}) \underset{I}{\overset{J}{\rightleftarrows}} CL(\mathcal{X},\mathcal{M})$ is a Galois connection. □

The following two results provide a description of how to construct the closure operators $T(\mathcal{A})$ and $J(\mathcal{B})$ defined in Propositions 3.1 and 3.2, respectively. A special case of the construction of $T(\mathcal{A})$ appears in [C$_2$].

PROPOSITION 3.3

Let $\mathcal{A} \in S(\mathcal{X})^{\mathbf{op}}$. For every $X \in \mathcal{X}$, we associate to every $\mathcal{M}$-subobject $M \xrightarrow{m} X$ the $\mathcal{M}$-subobject ${}^X_{\mathcal{A}}[M] \xrightarrow{{}^X_{\mathcal{A}}[m]} X$, where ${}^X_{\mathcal{A}}[M] = \cap\{f^{-1}((f \circ m)(M)) : X \xrightarrow{f} Y, Y \in \mathcal{A}\}$ and ${}^X_{\mathcal{A}}[m]$ is *the corresponding morphism. For every $\mathcal{A} \in S(\mathcal{X})^{\mathbf{op}}$ we have that ${}^{X}_{\mathcal{A}}[\ ]$ defines a closure operator ${}_{\mathcal{A}}C$ on $\mathcal{X}$ and ${}_{\mathcal{A}}C \simeq T(\mathcal{A})$.*

Proof:

Since $(f \circ m)(M)$ is the middle object of the $(\mathbf{E}, \mathcal{M})$-factorization of $f \circ m$, using the diagonalization property, it is easy to show that ${}^{X}_{\mathcal{A}}[\ ]$ is expansive and order-preserving. To show the remaining property, let us consider the following commutative diagram

$$\begin{array}{ccccc} g^{-1}(M) & \xrightarrow{\bar{g}} & M & \xrightarrow{e_1} & (f \circ m)(M) \\ \bar{m}\big\downarrow & & \big\downarrow m & & \big\downarrow m_1 \\ Z & \xrightarrow[g]{} & X & \xrightarrow[f]{} & Y \end{array}$$

where (e_1, m_1) is the $(\mathbf{E}, \mathcal{M})$-factorization of $f \circ m$, $g^{-1}(M) \xrightarrow{\bar{m}} Z$ is the pullback of $M \xrightarrow{m} X$ along g and $Y \in \mathcal{A}$. The diagonalization property yields $(f \circ g \circ \bar{m})(g^{-1}(M)) \leq (f \circ m)(M)$. Therefore $f^{-1}((f \circ g \circ \bar{m})(g^{-1}(M))) \leq f^{-1}((f \circ m)(M))$ and so, $g^{-1}(f^{-1}((f \circ g \circ \bar{m})(g^{-1}(M)))) \leq g^{-1}(f^{-1}((f \circ m)(M)))$. Now, by taking the intersection indexed by all morphisms $X \xrightarrow{f} Y$ with $Y \in \mathcal{A}$, we obtain that $\cap g^{-1}(f^{-1}((f \circ g \circ \bar{m})(g^{-1}(M)))) \leq \cap g^{-1}(f^{-1}((f \circ m)(M))) \simeq g^{-1}(\cap f^{-1}((f \circ m)(M))) \simeq g^{-1}({}^{X}_{\mathcal{A}}[M])$, since pullbacks and intersections commute. However, ${}^{Z}_{\mathcal{A}}[g^{-1}(M)] \leq \cap g^{-1}(f^{-1}((f \circ g \circ \bar{m})(g^{-1}(M))))$. Thus, ${}^{Z}_{\mathcal{A}}[g^{-1}(M)] \leq g^{-1}({}^{X}_{\mathcal{A}}[M])$. Hence, ${}_{\mathcal{A}}C$ is a closure operator.

Now, let $X \in \mathcal{A}$. The existence of $X \xrightarrow{id_X} X$ implies that for every $\mathcal{M}$-subobject $M \xrightarrow{m} X$ we have that ${}^{X}_{\mathcal{A}}[M] \simeq M$, i.e., X is ${}_{\mathcal{A}}C$-discrete. Thus, ${}_{\mathcal{A}}C \sqsubseteq T(\mathcal{A})$.

Finally, let $M \xrightarrow{m} X$ be an $\mathcal{M}$-subobject of $X \in \mathcal{X}$ and let $X \xrightarrow{f} Y$ be an $\mathcal{X}$-morphism with $Y \in \mathcal{A}$. From $M \leq f^{-1}((f \circ m)(M))$, we obtain that $[M]^{X}_{T(\mathcal{A})} \leq [f^{-1}((f \circ m)(M))]^{X}_{T(\mathcal{A})} \simeq f^{-1}((f \circ m)(M))$, since $(f \circ m)(M)$ is $T(\mathcal{A})$-closed and so is its pullback $f^{-1}((f \circ m)(M))$ (cf. Proposition 3.1). Therefore, by considering all morphisms $X \xrightarrow{f} Y$ with $Y \in \mathcal{A}$, we obtain that $[M]^{X}_{T(\mathcal{A})} \leq \cap f^{-1}((f \circ m)(M)) = {}^{X}_{\mathcal{A}}[M]$. Thus, $T(\mathcal{A}) \sqsubseteq {}_{\mathcal{A}}C$. Hence ${}_{\mathcal{A}}C \simeq T(\mathcal{A})$. □

PROPOSITION 3.4

Let $\mathcal{A} \in S(\mathcal{X})$. For every $Y \in \mathcal{X}$, we associate to every $\mathcal{M}$-subobject $M \xrightarrow{m} Y$ the $\mathcal{M}$-subobject ${}^{\mathcal{A}}_{Y}[M] \xrightarrow{{}^{\mathcal{A}}_{Y}[m]} Y = sup(M_i \xrightarrow{m_i} Y)_{i \in I}$, where $(M_i \xrightarrow{m_i} Y)_{i \in I}$ consists of the $\mathcal{M}$-subobject $M \xrightarrow{m} Y$ and all the $\mathcal{M}$-subobjects of the form $f(X) \xrightarrow{m_f} Y$, for every morphism $X \xrightarrow{f} Y$ with $X \in \mathcal{A}$ and $f(X) \cap M \neq \emptyset$. To simplify this expression we will write: ${}^{\mathcal{A}}_{Y}[M] = sup\Big(\{M\} \cup \{f(X) : X \in \mathcal{A}, X \xrightarrow{f} Y, f(X) \cap M \neq \emptyset\}\Big)$.

For every $\mathcal{A} \in S(\mathcal{X})$, ${}^{\mathcal{A}}_{Y}[\ ]$ defines a closure operator ${}^{\mathcal{A}}C$ on $\mathcal{X}$. Moreover, we have that ${}^{\mathcal{A}}C \simeq J(\mathcal{A})$.

Proof:

It is easily seen that ${}^{\mathcal{A}}_{Y}[\ ]$ is expansive and order-preserving. Let us consider the following diagram

$$\begin{array}{ccccccc}
 & & g^{-1}(M) & & \xrightarrow{\quad\quad\quad \bar{g} \quad\quad\quad} & & M \\
 & & & \searrow & & \swarrow & \\
 & & \bar{m}\big\downarrow & & [g^{-1}(M)]^{Y}_{C(\mathcal{A})} \longrightarrow [M]^{Z}_{C(\mathcal{A})} & & \big\downarrow m \\
 & & & \swarrow & & \searrow & \\
X & \xrightarrow[f]{} & Y & & \xrightarrow[\quad\quad\quad g \quad\quad\quad]{} & & Z
\end{array}$$

where the morphism $\bar{m}$ is the pullback of m along g. Since $f(X) \leq g^{-1}((g \circ f)(X))$, we have that $f(X) \cap g^{-1}(M) \neq \emptyset$ implies that $g^{-1}((g \circ f)(X)) \cap g^{-1}(M) \neq \emptyset$. Now, let $X \xrightarrow{h} Z$ be an $\mathcal{X}$-morphism. We have that

$$ {}^{\mathcal{A}}_{Y}[g^{-1}(M)] = sup\left(\{g^{-1}(M)\} \cup \{f(X) : X \in \mathcal{A}, X \xrightarrow{f} Y, f(X) \cap g^{-1}(M) \neq \emptyset\}\right) \leq $$

$$ sup\left(\{g^{-1}(M)\} \cup \{g^{-1}((g \circ f)(X)) : X \in \mathcal{A}, X \xrightarrow{f} Y, g^{-1}((g \circ f)(X)) \cap g^{-1}(M) \neq \emptyset\}\right) \leq $$

$$ sup\left(\{g^{-1}(M)\} \cup \{g^{-1}(h(X)) : X \in \mathcal{A}, X \xrightarrow{h} Z, g^{-1}(h(X)) \cap g^{-1}(M) \neq \emptyset\}\right) \leq $$

$$ g^{-1}\left(sup(\{M\} \cup \{h(X) : X \in \mathcal{A}, X \xrightarrow{h} Z, h(X) \cap M \neq \emptyset\})\right) = g^{-1}({}^{\mathcal{A}}_{Z}[M]) $$

Notice that in the last inequality we have used the fact that $g^{-1}(h(X)) \cap g^{-1}(M) \neq \emptyset$ implies that $h(X) \cap M \neq \emptyset$.

This shows that, for every $\mathcal{A} \in S(\mathcal{X})$, ${}^{\mathcal{A}}C$ is a closure operator.

Let $M \xrightarrow{m} Y$ be an $\mathcal{M}$-subobject of $Y \in \mathcal{A}$ with $M \neq \emptyset$. The existence of the identity morphism yields that ${}^{\mathcal{A}}_{Y}[M] \simeq Y$. Therefore we obtain that $J(\mathcal{A}) \sqsubseteq {}^{\mathcal{A}}C$.

Now, let $M \xrightarrow{m} Y$ be an $\mathcal{M}$-subobject with $M \neq \emptyset$ and let $X \xrightarrow{f} Y$ be an $\mathcal{X}$-morphism with $X \in \mathcal{A}$ and $f(X) \cap M \neq \emptyset$. By our assumptions on $\mathcal{M}$ and T, we have that $f^{-1}(M) \neq \emptyset$. Since X is $J(\mathcal{A})$-indiscrete, we obtain that $f(X) \simeq f([f^{-1}(M)]^{X}_{J(\mathcal{A})}) \leq [f(f^{-1}(M))]^{Y}_{J(\mathcal{A}} \leq [M]^{Y}_{J(\mathcal{A})}$ (cf. [DG$_2$]). Therefore, ${}^{\mathcal{A}}_{Y}[M] \leq [M]^{Y}_{J(\mathcal{A})}$. If $M = \emptyset$, then this last inequality is clearly true. Hence, ${}^{\mathcal{A}}C \sqsubseteq J(\mathcal{A})$ and consequently ${}^{\mathcal{A}}C \simeq J(\mathcal{A})$. □

PROPOSITION 3.5 (cf. [H], [P$_2$])

Let $S(\mathcal{X}) \xrightarrow{\Delta} S(\mathcal{X})^{\mathbf{op}}$ and $S(\mathcal{X})^{\mathbf{op}} \xrightarrow{\nabla} S(\mathcal{X})$ be defined as follows:

$\Delta(\mathcal{B})$ is the full subcategory with objects all $Y \in \mathcal{X}$ such that $X \xrightarrow{f} Y$ is constant for every $X \in \mathcal{B}$,

$\nabla(\mathcal{A})$ is the full subcategory with objects all $X \in \mathcal{X}$ such that $X \xrightarrow{f} Y$ is constant for every $Y \in \mathcal{A}$.

Then, $S(\mathcal{X}) \underset{\nabla}{\overset{\Delta}{\rightleftarrows}} S(\mathcal{X})^{\mathbf{op}}$ is a Galois connection. $\square$

LEMMA 3.6

Let C be a closure operator on $\mathcal{X}$ and let $X, Y \in \mathcal{X}$. If X is C-indiscrete and Y is C-discrete, then any morphism $X \xrightarrow{f} Y$ is constant.

Proof:

Let (e_f, m_f) be the $(\mathbf{E}, \mathcal{M})$-factorization of f. Clearly, $X \neq \emptyset$ implies that $f(X) \neq \emptyset$. Since T is a superstrong terminal object, there exists a morphism $T \xrightarrow{k} f(X)$ with $k \in \mathcal{M}$. Let $t = m_f \circ k$. Since Y is C-discrete, T is C-closed and so is its pullback $f^{-1}(T)$. By our assumptions on $\mathcal{M}$ and T, we have that $f^{-1}(T) \neq \emptyset$. Since X is C-indiscrete, we have that $f^{-1}(T) \simeq [f^{-1}(T)]_C^X \simeq X$. This implies that $f(X) \simeq f(f^{-1}(T)) \leq T$. Since T is minimal, we have that $T \simeq f(X)$. Thus, f is constant. $\square$

THEOREM 3.7

The Galois connection $S(\mathcal{X}) \underset{\nabla}{\overset{\Delta}{\rightleftarrows}} S(\mathcal{X})^{\mathbf{op}}$ factors through $CL(\mathcal{X}, \mathcal{M})$ via the Galois connections $S(\mathcal{X}) \underset{I}{\overset{J}{\rightleftarrows}} CL(\mathcal{X}, \mathcal{M})$ and $CL(\mathcal{X}, \mathcal{M}) \underset{T}{\overset{D}{\rightleftarrows}} S(\mathcal{X})^{\mathbf{op}}$.

Proof:

First of all, it is easy to see that the two compositions $D \circ J$ and $I \circ T$ give rise to a Galois connection between $S(\mathcal{X})$ and $S(\mathcal{X})^{\mathbf{op}}$.

Next we must show that $I \circ T = \nabla$. Let $\mathcal{A} \in S(\mathcal{X})^{\mathbf{op}}$ and let $X \in (I \circ T)(\mathcal{A})$. Since $X \neq \emptyset$ and any object $Y \in \mathcal{A}$ is $T(\mathcal{A})$-discrete (cf. Proposition 3.1), Lemma 3.6 gives us that any morphism $X \xrightarrow{f} Y$ is constant. Thus $X \in \nabla(\mathcal{A})$.

Now, let $X \in \nabla(\mathcal{A})$ and let $X \xrightarrow{f} Y$ be a morphism with $Y \in \mathcal{A}$. Consider an $\mathcal{M}$-subobject $M \xrightarrow{m} X$ with $M \neq \emptyset$. Let (e_f, m_f) and $(e_{f \circ m}, m_{f \circ m})$ be the $(\mathbf{E}, \mathcal{M})$-factorizations of f and $f \circ m$, respectively. Clearly, $m_f \circ e_f \circ m = m_{f \circ m} \circ e_{f \circ m}$. Since f is constant, the diagonalization property yields a morphism $(f \circ m)(M) \xrightarrow{t} T$ such that $m_f \circ t = m_{f \circ m}$. Since T is minimal we obtain that $(f \circ m)(M) \simeq T$. Therefore we have that $f^{-1}((f \circ m)(M)) \simeq f^{-1}(T) \simeq f^{-1}(f(X)) \simeq X$. Proposition 3.3 implies that $[M]_{T(\mathcal{A})}^X \simeq X$, i.e., $X \in (I \circ T)(\mathcal{A})$. Thus $I \circ T = \nabla$.

Now we show that $\Delta = D \circ J$. Let $Y \in \Delta(\mathcal{B})$ and let $M \xrightarrow{m} Y$ be an $\mathcal{M}$-subobject. For every $X \in \mathcal{B}$ consider all those $\mathcal{X}$-morphisms $X \xrightarrow{f} Y$ such that $f(X) \cap M \neq \emptyset$. Since f is constant, $f(X) \simeq T$. Thus $T \cap M \simeq f(X) \cap M \neq \emptyset$ is an $\mathcal{M}$-subobject of T. The minimality of T implies that $T \cap M \simeq T$. From Proposition 3.4 we obtain that $[M]_{J(\mathcal{B})}^Y \simeq M$. Thus, $Y \in (D \circ J)(\mathcal{B})$.

Finally, let $Y \in (D \circ J)(\mathcal{B})$ and let $X \xrightarrow{f} Y$ be an $\mathcal{X}$-morphism with $X \in \mathcal{B}$. Consider the

$(\mathbf{E}, \mathcal{M})$-factorization (e_f, m_f) of f. Since $f(X) \neq \emptyset$, there exists a morphism $T \xrightarrow{k} f(X)$. Let $t = m_f \circ k$. Clearly from $T \leq f(X)$ we obtain that $f(X) \cap T \simeq T \neq \emptyset$. Since Y is $J(\mathcal{B})$-discrete, we must have that $[T]^{f(X)}_{J(\mathcal{B})} \simeq T$. From Proposition 3.4, this implies that $f(X) \leq T$. Therefore, $f(X) \simeq T$, i.e., f is constant. Thus $Y \in \Delta(\mathcal{B})$ and hence $\Delta = D \circ J$. □

Next we show that if $\mathcal{A}$ is a full reflective subcategory of $\mathcal{X}$, then the closure operator $T(\mathcal{A})$ can be described in a rather simple form. This turns out to be very useful in constructing examples.

PROPOSITION 3.8

Let $\mathcal{A}$ be a full reflective subcategory of $\mathcal{X}$, let $X \in \mathcal{X}$ and let $X \xrightarrow{r_X} rX$ be the reflection morphism. The assignment ${}^X[\ ]^{\mathcal{A}}$ that to each $\mathcal{M}$-subobject $M \xrightarrow{m} X$ associates the $\mathcal{M}$-subobject of X, ${}^X[M]^{\mathcal{A}} \xrightarrow{{}^X[m]^{\mathcal{A}}} X$, where ${}^X[M]^{\mathcal{A}} = r_X^{-1}((r_X \circ m)(M))$ and ${}^X[m]^{\mathcal{A}}$ is the induced morphism, defines a closure operator $C^{\mathcal{A}}$ on $\mathcal{X}$.

Proof:

It is rather easy to show that ${}^X[\ ]^{\mathcal{A}}$ is expansive and order-preserving. Let us consider the following commutative diagram

$$\begin{array}{ccccccc} (r_X \circ \bar{m})(f^{-1}(M)) & \xleftarrow{e_2} & f^{-1}(M) & \xrightarrow{\bar{f}} & M & \xrightarrow{e_1} & (r_Y \circ m)(M) \\ {\scriptstyle m_2}\big\downarrow & & \big\downarrow{\scriptstyle \bar{m}} & & \big\downarrow{\scriptstyle m} & & \big\downarrow{\scriptstyle m_1} \\ rX & \xleftarrow[r_X]{} & X & \xrightarrow[f]{} & Y & \xrightarrow[r_Y]{} & rY \end{array}$$

where, (e_1, m_1) is the $(\mathbf{E}, \mathcal{M})$-factorization of $r_Y \circ m$, $\bar{m}$ is the pullback of m along f and (e_2, m_2) is the $(\mathbf{E}, \mathcal{M})$-factorization of $r_X \circ \bar{m}$. Since $\mathcal{A}$ is reflective in $\mathcal{X}$, there exists a unique morphism $rX \xrightarrow{f'} rY$ such that $f' \circ r_X = r_Y \circ f$. Therefore, we have that $f' \circ m_2 \circ e_2 = f' \circ r_X \circ \bar{m} = r_Y \circ f \circ \bar{m} = r_Y \circ m \circ \bar{f} = m_1 \circ e_1 \circ \bar{f}$. From the $(\mathbf{E}, \mathcal{M})$-diagonalization property, there exists a morphism $(r_X \circ \bar{m})(f^{-1}(M)) \xrightarrow{d} (r_Y \circ m)(M)$ such that $d \circ e_2 = e_1 \circ \bar{f}$ and $m_1 \circ d = f' \circ m_2$. Let us consider the following two pullback squares

$$\begin{array}{ccc} (r_X \circ \bar{m})(f^{-1}(M)) & \xleftarrow{\bar{r}_X} & r_X^{-1}((r_X \circ \bar{m})(f^{-1}(M))) \\ {\scriptstyle m_2}\big\downarrow & & \big\downarrow{\scriptstyle \bar{m}_2} \\ rX & \xleftarrow[r_X]{} & X \end{array} \qquad \begin{array}{ccc} r_Y^{-1}((r_Y \circ m)(M)) & \xrightarrow{\bar{r}_Y} & (r_Y \circ m)(M) \\ {\scriptstyle \bar{m}_1}\big\downarrow & & \big\downarrow{\scriptstyle m_1} \\ Y & \xrightarrow[r_Y]{} & rY \end{array}$$

Now, $m_1 \circ d \circ \bar{r}_X = f' \circ m_2 \circ \bar{r}_X = f' \circ r_X \circ \bar{m}_2 = r_Y \circ f \circ \bar{m}_2$, i.e., the following diagram commutes

$$\begin{array}{ccccc}
r_X^{-1}((r_X \circ \bar{m})(f^{-1}(M))) & \xrightarrow{\bar{r}_X} & (r_X \circ \bar{m})(f^{-1}(M)) & \xrightarrow{d} & (r_Y \circ m)(M) \\
\bar{m}_2 \big\downarrow & & & & \big\downarrow m_1 \\
X & \xrightarrow[f]{} & Y & \xrightarrow[r_Y]{} & rY
\end{array}$$

As a consequence of the universal property of pullbacks we obtain the existence of a unique morphism $r_X^{-1}((r_X \circ \bar{m})(f^{-1}(M))) \xrightarrow{d'} r_Y^{-1}((r_Y \circ m)(M))$ such that $\bar{r}_Y \circ d' = d \circ \bar{r}_X$ and $\bar{m}_1 \circ d' = f \circ \bar{m}_2$. Therefore, the following diagram commutes

$$\begin{array}{ccc}
r_X^{-1}((r_X \circ \bar{m})(f^{-1}(M))) & \xrightarrow{d'} & r_Y^{-1}((r_Y \circ m)(M)) \\
\bar{m}_2 \big\downarrow & & \big\downarrow \bar{m}_1 \\
X & \xrightarrow[f]{} & Y
\end{array}$$

Again, as a consequence of the universal property of pullbacks we obtain the desired morphism $r_X^{-1}((r_X \circ \bar{m})(f^{-1}(M))) \xrightarrow{d''} f^{-1}(r_Y^{-1}((r_Y \circ m)(M)))$. Therefore, we have that ${}^X[f^{-1}(M)]^{\mathcal{A}} \leq f^{-1}({}^X[M]^{\mathcal{A}})$. This completes the proof. $\square$

PROPOSITION 3.9

Let $\mathcal{A}$ be a full reflective subcategory of $\mathcal{X}$. If $M \xrightarrow{m} X$ is an $\mathcal{M}$-subobject of an $\mathcal{A}$-object X, then ${}^X[M]^{\mathcal{A}} \simeq M$, i.e., each nonempty object $X \in \mathcal{A}$ is $C^{\mathcal{A}}$-discrete.

Proof:

Let us consider the following commutative diagram

$$\begin{array}{ccccc}
M & & \xrightarrow{\quad m \quad} & & X \\
& & & \nearrow \bar{m}_1 & \\
e_1 \big\downarrow & & r_X^{-1}((r_X \circ m)(M)) & & \big\downarrow r_X \\
& \swarrow \bar{r}_X & & & \\
(r_X \circ m)(M) & & \xrightarrow[\quad m_1 \quad]{} & & rX
\end{array}$$

Since $X \in \mathcal{A}$, we have that r_X is an isomorphism and consequently so is its pullback $\bar{r}_X$ along m_1. Since $\mathcal{M}$ is closed under composition with isomorphisms, we have that $r_X \circ m \in \mathcal{M}$. This implies that e_1 is an isomorphism. Therefore we have that $M \simeq r_X^{-1}((r_X \circ m)(M)) = {}^X[M]^{\mathcal{A}}$. $\square$

COROLLARY 3.10

Let $\mathcal{A}$ be a full reflective subcategory of $\mathcal{X}$. Then $C^{\mathcal{A}} \simeq {}_{\mathcal{A}}C$.

Proof:

From Proposition 3.3, it is straightforward to see that $T(\mathcal{A}) \simeq {}_{\mathcal{A}}C \sqsubseteq C^{\mathcal{A}}$. However, from the definition of $T(\mathcal{A})$ (cf. Proposition 3.1) and from Proposition 3.9, we have that $C^{\mathcal{A}} \sqsubseteq T(\mathcal{A})$. Thus, $C^{\mathcal{A}} \simeq {}_{\mathcal{A}}C$. □

4 EXAMPLES

We now present some examples to illustrate the general theory.

EXAMPLE 4.1

Let $\mathcal{X}$ be the category **Top** of topological spaces and let $\mathcal{M}$ be the class of all embeddings. If C is the closure operator induced by the topology, then the class of C-discrete objects agrees with the class **DISCR** of nonempty discrete topological spaces and the C-connected objects are exactly the classical nonempty connected topological spaces.

Let $M \xrightarrow{m} X$ be an $\mathcal{M}$-subobject of $X \in \mathbf{TOP}$. Clearly, $[M]^X_{T(\mathbf{DISCR})}$ equals the intersection of all clopen subsets of X containing M. If M is a singleton subobject, then $[M]^X_{T(\mathbf{DISCR})}$ is exactly the quasicomponent of M. From Theorem 3.7, connected nonempty topological spaces form the indiscrete class of such a closure operator.

Now, let $\mathcal{A}$ be the class of all connected nonempty topological spaces. From Proposition 3.4, $[M]^X_{J(\mathcal{A})}$ is the union of M with all connected subsets of X which intersect M. It is easy to check that the subcategory of all Totally Disconnected nonempty topological spaces form the discrete class of $J(\mathcal{A})$. Thus from Theorem 3.7, Connected nonempty topological spaces and Totally Disconnected nonempty topological spaces are fixed points of the Galois connection (Δ, ∇) of Proposition 3.5 (cf. [H]).

EXAMPLE 4.2

Let $\mathcal{X}$ be the category **Top** of topological spaces and let $\mathcal{M}$ be the class of all embeddings. Let $\mathcal{A} = \mathbf{TOP_0} \in S(\mathcal{X})^{\mathbf{op}}$ and let $\mathcal{B} = \mathbf{IND} \in S(\mathcal{X})$. **IND** and $\mathbf{TOP_0}$ are corresponding fixed points of the Galois connection (Δ, ∇) of Proposition 3.5 (cf. [AW]).

Let $M \xrightarrow{m} X$ be an $\mathcal{M}$-subobject of $X \in \mathbf{TOP}$ and let

$$c(M) = \{y \in X : \exists x \in M \ with \ \bar{\{x\}} = \bar{\{y\}}\}$$

.

where, $\bar{\{x\}}$ denotes the usual topological closure of $\{x\}$. If $X \xrightarrow{r_0} r_0X$ is the $\mathbf{TOP_0}$-reflection, then $c(M) = r_0^{-1}((r_0 \circ m)(M))$. Thus, from Corollary 3.10, $[M]^X_{T(\mathbf{TOP_0})} = c(M)$. It is easy to see that $[M]^X_{T(\mathbf{TOP_0})} \subseteq b(M)$, where $b(M)$ is the **b**-closure of M. We recall that $b(M)$ consists of all those points $x \in X$ such that for every neighborhood U of x, $M \cap Cl(x) \cap U \neq \emptyset$, where $Cl(x)$ denotes the topological closure of the subset $\{x\}$ (cf. [B], [NW]).

If $Y \in \mathbf{IND}$ and $Y \xrightarrow{f} X$ is continuous, then $f(Y) \in \mathbf{IND}$. Thus $[M]^X_{J(\mathbf{IND})}$ is the union of M with all indiscrete subobjects of X which intersect M.

If Z is an object of $(D \circ J)(\mathbf{IND})$, then the only indiscrete subspaces of Z are the singletons. This means that $Z \in \mathbf{TOP_0}$. Clearly, if $Z \in \mathbf{TOP_0}$, then it cannot have indiscrete subspaces with more than one point. Therefore, J is discrete on $\mathbf{TOP_0}$, i.e., $(D \circ J)(\mathbf{IND}) = \mathbf{TOP_0}$, as we expected.

EXAMPLE 4.3

Let $\mathcal{X}$ be the category **Top** of topological spaces and let $\mathcal{M}$ be the class of all embeddings. Suppose that $\mathcal{A} = \mathbf{TOP_1} \in S(\mathcal{X})^{\mathbf{op}}$ and let $\mathcal{B}$ be the full subcategory whose objects are all absolutely connected nonempty topological spaces, i.e., $\mathcal{B} = \{X \in \mathbf{TOP}$ such that X cannot be decomposed into any disjoint family $\mathcal{L}$ of nonempty closed subsets with $|\mathcal{L}| > 1\}$ (cf. [P_1]). $\mathcal{A}$ and $\mathcal{B}$ are corresponding fixed points of the Galois connection (Δ, ∇) of Proposition 3.5. Let $X \in \mathbf{TOP}$ and let $M \xrightarrow{m} X$ be an $\mathcal{M}$-subobject. Since every topological space in $\mathbf{TOP_1}$ is $C_{\mathbf{TOP_1}}$-discrete (cf. [G]), by definition of $T(\mathcal{A})$, we have that $[M]^X_{\mathbf{TOP_1}} \leq [M]^X_{T(\mathbf{TOP_1})}$. Now, let $X \xrightarrow{r_1} r_1X$ be the $\mathbf{TOP_1}$-reflection morphism. Then, from Corollary 3.10, $[M]^X_{T(\mathbf{TOP_1})} \simeq r_1^{-1}((r_1 \circ m)(M))$. However, $[M]^X_{\mathbf{TOP_1}} \simeq r_1^{-1}((r_1 \circ m)(M))$ (cf. [DGT, Proposition 3.11]). Therefore, $[M]^X_{\mathbf{TOP_1}} \simeq [M]^X_{T(\mathbf{TOP_1})}$, i.e., the $T(\mathbf{TOP_1})$-closure agrees with the Salbany closure induced by $\mathbf{TOP_1}$. So, from Theorem 3.7, $\mathcal{B}$ is the class of indiscrete objects of $C_{\mathbf{TOP_1}}$.

EXAMPLE 4.4

Let $\mathcal{X}$ be the category **Grp** and let $\mathcal{M}$ be the class of all monomorphisms. Consider the full subcategory $\mathcal{A} = \mathbf{Ab}$. Since **Ab** is closed under quotients, every $X \in \mathbf{Ab}$ is $C_{\mathbf{Ab}}$-discrete. Therefore, by the definition of the functor T, we have that $C_{\mathbf{Ab}} \sqsubseteq T(\mathbf{Ab})$. Let $M \xrightarrow{m} X$ be a monomorphism in **Grp** and let $X \underset{g}{\overset{f}{\rightrightarrows}} Y$ be two homomorphisms such that $f \circ m = g \circ m$, with $Y \in \mathcal{A}$. Since $equ(f,g) = ker(f-g) = (f-g)^{-1}(f-g)(M)$, we have that $[M]^X_{T(\mathbf{Ab})} \leq [M]^X_{\mathbf{Ab}}$. This, together with the above inequality gives that $C_{\mathbf{Ab}} \simeq T(\mathbf{Ab})$. Consequently, the subcategory $\nabla(\mathbf{Ab})$ of all $C_{\mathbf{Ab}}$-connected objects agrees with the subcategory of all $C_{\mathbf{Ab}}$-indiscrete objects which is equal to the subcategory of all groups G such that G has no proper normal subgroup N with G/N abelian. Notice that $\nabla(\mathbf{Ab})$ is the subcategory of perfect groups, i.e., $X \in \nabla(\mathbf{Ab})$ iff $X = X'$, where X' denotes the subgroup generated by the commutators.

If $Y \in \nabla(\mathbf{Ab})$ and $Y \xrightarrow{f} X$ is a homomorphism, then $f(Y) \in \nabla(\mathbf{Ab})$. Thus $[M]^X_{J(\nabla(\mathbf{Ab}))}$ is the subgroup generated by M and all perfect subgroups of X. Finally, it is easy to see that

$(D \circ J)(\nabla(\mathbf{Ab}))$ is the class of all groups which do not have any non-trivial perfect subgroup.

EXAMPLE 4.5

Let $\mathcal{X}$ be the category **Ab** and let $\mathcal{M}$ be the class of all monomorphisms. Let $\mathcal{T}$ and $\mathcal{F}$ be corresponding fixed points of the Galois connection (Δ, ∇) of Proposition 3.5. (The pair $(\mathcal{T}, \mathcal{F})$ is normally called a torsion theory.) Let $X \in \mathbf{Ab}$ and let $X \xrightarrow{r_X} rX$ be its $\mathcal{F}$-reflection. For every subobject $M \xrightarrow{m} X$ we have that $M + Ker(r_X) \simeq r_X^{-1}(r_X(M))$. This, together with Corollary 3.10, gives us that $[M]^X_{T(\mathcal{F})} \simeq r_X^{-1}(r_X(M)) \simeq M + Ker(r_X)$. Clearly, $\mathcal{T}$ is the class of $T(\mathcal{F})$-indiscrete objects (cf. Theorem 3.7). Also notice that $T(\mathcal{F}) \sqsubseteq C_{\mathcal{F}}$ (cf. Example 4.4). In particular, if $(\mathcal{T}, \mathcal{F})$ =(Torsion,Torsion-free), then $[M]^X_{T(\mathcal{F})} \simeq M + Tor(X)$, where $Tor(X)$ denotes the torsion subgroup of X. If $(\mathcal{T}, \mathcal{F}) =$ (Divisible,Reduced), then $[M]^X_{T(\mathcal{F})} \simeq M + Div(X)$, where $Div(X)$ denotes the largest divisible subgroup of X. It is interesting to notice that in both cases, $[M]^X_{J(\mathcal{T})} = [M]^X_{T(\mathcal{F})}$. Therefore, the subcategory **Tor** (**Div**) of Torsion Abelian Groups (Divisible Abelian Groups) is the connectedness class with respect to the closure operator $T(\mathcal{T})$, where $\mathcal{T}$ denotes the subcategory of Torsion-free Abelian Groups (Reduced Abelian Groups).

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