CONNECTEDNESS, DISCONNECTEDNESS AND CLOSURE OPERATORS, A MORE GENERAL APPROACH

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Abstract: Let $\mathcal{X}$ be an arbitrary category with an $(\mathbf{E}, \mathcal{M})$-factorization structure for sinks. A notion of constant morphism that depends on a chosen class of monomorphisms is introduced. This notion yields a Galois connection that can be seen as a generalization of the classical connectedness-disconnectedness correspondence (also called torsion-torsion free in algebraic contexts). It is shown that this Galois connection factors through the collection of all closure operators on $\mathcal{X}$ with respect to $\mathcal{M}$.

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Introduction

This paper presents in the setting of an arbitrary category some ideas that were introduced by D. Hajek and the author in a category whose objects were structured sets (cf. [3]).

Let $\mathcal{X}$ be an arbitrary category with an $(\mathbf{E}, \mathcal{M})$-factorization structure for sinks and let $\mathcal{N} \subseteq \mathcal{M}$. An $\mathcal{X}$-morphism $X \xrightarrow{f} Y$ is called $\mathcal{N}$-constant if its direct image is isomorphic to the direct image under f of every $\mathcal{N}$-subobject of X. So, if $S(\mathcal{X})$ denotes the class of all subclasses of objects of $\mathcal{X}$, ordered by inclusion, for every $\mathcal{N} \subseteq \mathcal{M}$, the relation: $X \mathcal{R}_{\mathcal{N}} Y$ if and only if every $\mathcal{X}$-morphism $X \xrightarrow{f} Y$ is $\mathcal{N}$-constant yields a Galois connection $S(\mathcal{X}) \underset{\nabla_{\mathcal{N}}}{\overset{\Delta_{\mathcal{N}}}{\rightleftarrows}} S(\mathcal{X})^{op}$. If $\mathcal{N}$ is closed under direct images, we have that this Galois connection factors through $CL(\mathcal{X}, \mathcal{M})$, i.e., the collection of all closure operators on $\mathcal{X}$ with respect to $\mathcal{M}$, via two Galois connections $CL(\mathcal{X}, \mathcal{M}) \underset{T_{\mathcal{N}}}{\overset{D_{\mathcal{N}}}{\rightleftarrows}} S(\mathcal{X})^{op}$ and $S(\mathcal{X}) \underset{I_{\mathcal{N}}}{\overset{J_{\mathcal{N}}}{\rightleftarrows}} CL(\mathcal{X}, \mathcal{M})$.

The development of a general theory of topological connectedness was started by Preuß (cf. [15-17]) and by Herrlich [10]. We recently became aware that an introduction to the study of connectedness and disconnectedness in an arbitrary category, using an approach similar to ours was made by Petz (cf. [14]).

We would like to point out that the definition of $\mathcal{N}$-constant morphism that

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appears in this paper was not chosen with the intention of developing a general theory of connectedness and disconnectedness, but rather to support certain constructions with closure operators in an arbitrary setting. A general theory of connectedness and disconnectedness was recently presented by Clementino in [6], where she extends results in [12] to an arbitrary category.

Examples and further study of the Galois connections mentioned in this paper will appear in a subsequent paper (cf. [2]).

We use the terminology of [1] throughout the paper. We also acknowledge that Paul Taylor's commutative diagrams macro package was used to typeset most of the diagrams in this paper.

1. Preliminaries

Throughout we consider a category $\mathcal{X}$ together with a fixed class $\mathcal{M}$ of $\mathcal{X}$-monomorphisms and a class $\mathbf{E}$ of $\mathcal{X}$-sinks such that $\mathcal{X}$ is an $(E,\mathcal{M})$-*category for sinks*, (cf. [1] for the dual case), that is:

(a) each of $\mathbf{E}$ and $\mathcal{M}$ is closed under compositions with isomorphisms;

(b) $\mathcal{X}$ has $(E,\mathcal{M})$-factorizations (of sinks); i.e., each sink $\mathbf{s}$ in $\mathcal{X}$ has a factorization $s = m \circ e$ with $e \in E$ and $m \in \mathcal{M}$, and

(c) $\mathcal{X}$ has the unique $(E,\mathcal{M})$-diagonalization property; i.e., if $B \xrightarrow{g} D$ and $C \xrightarrow{m} D$ are $\mathcal{X}$-morphisms with $m \in \mathcal{M}$, and $e = (A_i \xrightarrow{e_i} B)_I$ and $s = (A_i \xrightarrow{s_i} C)_I$ are sinks in $\mathcal{X}$ with $e \in E$, such that $m \circ s = g \circ e$, then there exists a unique diagonal $B \xrightarrow{d} C$ such that $m \circ d = g$ and for every $i \in I$, $d \circ e_i = s_i$.

DEFINITION 1.1. A *closure operator* C on $\mathcal{X}$ (with respect to $\mathcal{M}$) is a family $\{(\ )^{C}_{X}\}_{X\in\mathcal{X}}$ of functions on the $\mathcal{M}$-subobject lattices of $\mathcal{X}$ with the following properties that hold for each $X \in \mathcal{X}$:

(a) [*expansiveness*] $m \le (m)^{C}_{X}$, for every $\mathcal{M}$-subobject $M \xrightarrow{m} X$;

(b) [*order-preservation*] $m \le n \Rightarrow (m)^{C}_{X} \le (n)^{C}_{X}$ for every pair of $\mathcal{M}$-subobjects of X;

(c) [*morphism-consistency*] If p is the pullback of the $\mathcal{M}$-subobject $M \xrightarrow{m} Y$ along some $\mathcal{X}$-morphism $X \xrightarrow{f} Y$ and q is the pullback of $(m)^{C}_{Y}$ along f, then $(p)^{C}_{X} \le q$, i.e., the closure of the inverse image of m is less than or equal to the inverse image of the closure of m.

Condition (a) implies that for every closure operator C on $\mathcal{X}$, every $\mathcal{M}$-subobject $M \xrightarrow{m} X$ has a canonical factorization $m = (m)^{C}_{X} \circ t$, where $(M)^{C}_{X} \xrightarrow{(m)^{C}_{X}} X$ is called the C-*closure* of the subobject (M,m).

When no confusion is likely we will write m^{C} rather than $(m)^{C}_{X}$ and for notational symmetry we will denote the morphism t by m_{C}.

If $X \xrightarrow{f} Y$ is an $\mathcal{X}$-morphism and $M \xrightarrow{m} X$ is an $\mathcal{M}$-subobject, then the $(E, \mathcal{M})$-factorization of $f \circ m$ will be denoted by $X \xrightarrow{e_{f \circ m}} M_f \xrightarrow{m_f} Y$. $M_f \xrightarrow{m_f} Y$ will be called the direct image of m along f. If $N \xrightarrow{n} Y$ is an $\mathcal{M}$-subobject, the pullback $f^{-1}(N) \xrightarrow{f^{-1}(n)} X$ of n along f will be called the inverse image of n along f. Whenever no confusion is likely to arise, to simplify the notation we will denote the morphism $e_{f \circ m}$ simply e_f.

REMARK 1.2. Notice that in the above definition, under condition (b), the morphism-consistency condition (c) is equivalent to the following statement concerning direct images: if $M \xrightarrow{m} X$ is an $\mathcal{M}$-subobject and $X \xrightarrow{f} Y$ is a morphism, then $((m)^{C}_{Y})_f \leq (m_f)^{C}_{Y}$, i.e., the direct image of the closure of m is less than or equal to the closure of the direct image of m; (cf. [7]).

DEFINITION 1.3. Given a closure operator C, we say that $m \in \mathcal{M}$ is *C-closed* (*C-dense*) if m_{C} (m^{C}) is an isomorphism. We call C *idempotent* provided that m^{C} is C-closed for every $m \in \mathcal{M}$. C is called *weakly hereditary* if m_{C} is C-dense for every $m \in \mathcal{M}$.

Notice that Definition 1.1(c) implies that pullbacks of C-closed $\mathcal{M}$-subobjects are C-closed.

We denote the collection of all closure operators on $\mathcal{M}$ by $CL(\mathcal{X}, \mathcal{M})$ preordered as follows: $C \sqsubseteq D$ if $m^{C} \leq m^{D}$ for all $m \in \mathcal{M}$ (where $\leq$ is the usual order on subobjects). Notice that arbitrary suprema and infima exist in $CL(\mathcal{X}, \mathcal{M})$, they are formed pointwise in the $\mathcal{M}$-subobject fibers.

For more background on closure operators see, e.g., [4], [5], [7], [8] and [13]. For a detailed survey on the same topic one could check [11].

DEFINITION 1.4. For pre-ordered classes $\mathcal{X} = (X, \sqsubseteq)$ and $\mathcal{Y} = (Y, \sqsubseteq)$, a *Galois connection* $\mathcal{X} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{Y}$ consists of order preserving functions F and G that satisfy $F \dashv G$, i.e., $x \sqsubseteq GF(x)$ for every $x \in X$ and $FG(y) \sqsubseteq y$ for every $y \in Y$. (G is adjoint and has F as coadjoint).

If $x \in \mathcal{X}$ and $y \in \mathcal{Y}$ are such that $F(x) = y$ and $G(y) = x$, then x and y are said to be corresponding fixed points of the Galois connection $(\mathcal{X}, F, G, \mathcal{Y})$.

Properties and many examples of Galois connections can be found in [9].

2. General results

Throughout the paper we assume that $\mathcal{X}$ is an $(E, \mathcal{M})$-category for sinks.

Let $S(\mathcal{X})$ be the collection of all subcategories of $\mathcal{X}$, ordered by inclusion and let $\mathcal{N}$ be a fixed subclass of $\mathcal{M}$. For every $X \in \mathcal{X}$, we denote by $\mathcal{N}_X$ all the $\mathcal{N}$-subobjects that have X as codomain.

DEFINITION 2.1. Let $\mathcal{N} \subseteq \mathcal{M}$. An $\mathcal{X}$-morphism $X \xrightarrow{f} Y$ is called $\mathcal{N}$-constant if for every $\mathcal{N}$-subobject $N \xrightarrow{n} X$, we have that $n_f \simeq (id_X)_f$.

As a consequence we have the following

PROPOSITION 2.2. (cf. [10]) *Let $\mathcal{N} \subseteq \mathcal{M}$. Define $S(\mathcal{X}) \xrightarrow{\Delta_{\mathcal{N}}} S(\mathcal{X})^{op}$ and $S(\mathcal{X})^{op} \xrightarrow{\nabla_{\mathcal{N}}} S(\mathcal{X})$ as follows:*

$$\nabla_{\mathcal{N}}(\mathcal{A}) = \{X \in \mathcal{X} : \forall Y \in \mathcal{A}, X \xrightarrow{f} Y \textit{ is } \mathcal{N}\textit{-constant}\}$$

$$\Delta_{\mathcal{N}}(\mathcal{B}) = \{Y \in \mathcal{X} : \forall X \in \mathcal{B}, X \xrightarrow{f} Y \textit{ is } \mathcal{N}\textit{-constant}\}$$

Then, $S(\mathcal{X}) \underset{\nabla_{\mathcal{N}}}{\overset{\Delta_{\mathcal{N}}}{\rightleftarrows}} S(\mathcal{X})^{op}$ is a Galois connection. □

As in [3], with some minor modifications we have the following two Galois connections.

PROPOSITION 2.3. *Let $CL(\mathcal{X},\mathcal{M}) \xrightarrow{D_{\mathcal{N}}} S(\mathcal{X})^{op}$ and $S(\mathcal{X})^{op} \xrightarrow{T_{\mathcal{N}}} CL(\mathcal{X},\mathcal{M})$ be defined by:*

$$D_{\mathcal{N}}(C) = \{X \in \mathcal{X} : \textit{ every } n \in \mathcal{N}_X \textit{ is } C\textit{-closed}\}$$

$$T_{\mathcal{N}}(\mathcal{A}) = Sup\{C \in CL(\mathcal{X},\mathcal{M}) : D_{\mathcal{N}}(C) \supseteq \mathcal{A}\}.$$

Then, $CL(\mathcal{X},\mathcal{M}) \underset{T_{\mathcal{N}}}{\overset{D_{\mathcal{N}}}{\rightleftarrows}} S(\mathcal{X})^{op}$ is a Galois connection. □

PROPOSITION 2.4. *Let $CL(\mathcal{X},\mathcal{M}) \xrightarrow{I_{\mathcal{N}}} S(\mathcal{X})$ and $S(\mathcal{X}) \xrightarrow{J_{\mathcal{N}}} CL(\mathcal{X},\mathcal{M})$ be defined by:*

$$I_{\mathcal{N}}(C) = \{X \in \mathcal{X} : \textit{ every } n \in \mathcal{N}_X \textit{ is } C\textit{-dense}\}$$

$$J_{\mathcal{N}}(\mathcal{B}) = Inf\{C \in CL(\mathcal{X},\mathcal{M}) : I_{\mathcal{N}}(C) \supseteq \mathcal{B}\}.$$

Then, $S(\mathcal{X}) \underset{I_{\mathcal{N}}}{\overset{J_{\mathcal{N}}}{\rightleftarrows}} CL(\mathcal{X},\mathcal{M})$ is a Galois connection. □

Clearly, we have the following

COROLLARY 2.5. *The composition functions $D_{\mathcal{N}} \circ J_{\mathcal{N}}$ and $I_{\mathcal{N}} \circ T_{\mathcal{N}}$ give rise to a Galois connection between $S(\mathcal{X})$ and $S(\mathcal{X})^{op}$.* □

The following result provides a description of how to construct the closure operator $T_{\mathcal{N}}(\mathcal{A})$ defined in Proposition 2.3.

PROPOSITION 2.6. *Let $\mathcal{A} \in S(\mathcal{X})^{op}$ and let $\mathcal{N}$ be a subclass of $\mathcal{M}$. For every*

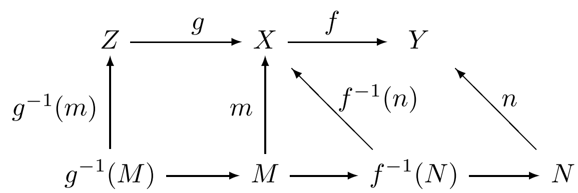
$X \in \mathcal{X}$ and for every $\mathcal{M}$-subobject $M \xrightarrow{m} X$, we define

$$_{\mathcal{A}}m = \cap\{f^{-1}(n) : Y \in \mathcal{A}, X \xrightarrow{f} Y, N \xrightarrow{n} Y \in \mathcal{N}_Y \text{and } m \leq f^{-1}(n)\}.$$

For every $\mathcal{A} \in S(\mathcal{X})^{op}$ we have that the function $_{\mathcal{A}}(\ )$ that to every $\mathcal{M}$-subobject $M \xrightarrow{m} X$ associates $_{\mathcal{A}}m$ is an idempotent closure operator on $\mathcal{X}$ and $_{\mathcal{A}}m \simeq m^{T_{\mathcal{N}}(\mathcal{A})}$.

Proof: We first observe that as a consequence of $\mathcal{X}$ being an $(\mathbf{E}, \mathcal{M})$-category for sinks, $\mathcal{M}$ is closed under the formation of pullbacks and intersections and so $_{\mathcal{A}}m \in \mathcal{M}$.

We begin by showing that $_{\mathcal{A}}(\ )$ is a closure operator on $\mathcal{X}$. It is straightforward to show that $_{\mathcal{A}}(\ )$ is expansive and order-preserving. To show the remaining property, let us consider the following commutative diagram



where $Y \in \mathcal{A}$, $n \in \mathcal{N}_Y$ and $m \leq f^{-1}(n)$.

Now, $g^{-1}(m) \leq g^{-1}(_{\mathcal{A}}m) = g^{-1}(\cap f^{-1}(n)) \simeq \cap g^{-1}(f^{-1}(n)) \simeq \cap(f \circ g)^{-1}(n)$, since pullbacks and intersections commute. Now since not all the morphisms from Z to Y are of the form $f \circ g$ we obtain that $_{\mathcal{A}}(g^{-1}(m)) \leq \cap(f \circ g)^{-1}(n) \simeq g^{-1}(f^{-1}(n)) \simeq g^{-1}(_{\mathcal{A}}m)$. Hence, $_{\mathcal{A}}(\ )$ is a closure operator.

To show idempotency it is enough to observe that if $n \in \mathcal{N}_Y$ and $_{\mathcal{A}}m \leq f^{-1}(n)$ then clearly $m \leq f^{-1}(n)$. On the other hand, if $m \leq f^{-1}(n)$, then by definition of $_{\mathcal{A}}m$ we also have that $_{\mathcal{A}}m \leq f^{-1}(n)$. Thus we obtain that $_{\mathcal{A}}m \simeq {_{\mathcal{A}}}(_{\mathcal{A}}m)$

Now, let $X \in \mathcal{A}$. The existence of $X \xrightarrow{id_X} X$ implies that for every $\mathcal{N}$-subobject $N \xrightarrow{n} X$ we have that $_{\mathcal{A}}n \simeq n$. Thus, $_{\mathcal{A}}(\ ) \sqsubseteq T_{\mathcal{N}}(\mathcal{A})$.

Finally, let $X \in \mathcal{X}$, let $X \xrightarrow{f} Y$ be an $\mathcal{X}$-morphism with $Y \in \mathcal{A}$ and let $M \xrightarrow{m} X$ be an $\mathcal{M}$-subobject. Consider the $\mathcal{N}$-subobject $N \xrightarrow{n} Y$ with $m \leq f^{-1}(n)$. Then, we obtain that $m^{T_{\mathcal{N}}(\mathcal{A})} \leq (f^{-1}(n))^{T_{\mathcal{N}}(\mathcal{A})} \simeq f^{-1}(n)$, since n is $T_{\mathcal{N}}(\mathcal{A})$-closed and so is its pullback $f^{-1}(n)$. Therefore, by considering all morphisms $X \xrightarrow{f} Y$ with $Y \in \mathcal{A}$ and $n \in \mathcal{N}_Y$ with $m \leq f^{-1}(n)$, we obtain that $m^{T_{\mathcal{N}}(\mathcal{A})} \leq \cap f^{-1}(n) = {_{\mathcal{A}}m}$. Thus, $T_{\mathcal{N}}(\mathcal{A}) \sqsubseteq {_{\mathcal{A}}}(\ )$. Hence $_{\mathcal{A}}(\ ) \simeq T_{\mathcal{N}}(\mathcal{A})$. □

Next we provide a description of how to construct the closure operator $J_{\mathcal{N}}(\mathcal{B})$ defined in Proposition 2.4.

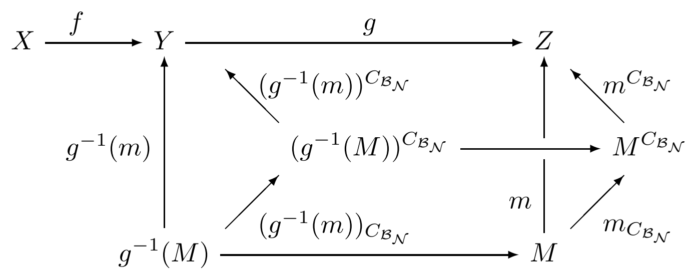
PROPOSITION 2.7. *Let $\mathcal{B} \in S(\mathcal{X})$ and let $\mathcal{N} \subseteq \mathcal{M}$. For every $Y \in \mathcal{X}$ and for every $\mathcal{M}$-subobject $M \xrightarrow{m} Y$, we define*

$$C_{\mathcal{B}_{\mathcal{N}}}(m) = sup\left(\{m\} \cup \{(id_X)_f : X \in \mathcal{B}, X \xrightarrow{f} Y \text{ and } \exists n \in \mathcal{N}_X \text{ with } n_f \leq m\}\right).$$

For every $\mathcal{B} \in S(\mathcal{X})$, the function $C_{\mathcal{B}_\mathcal{N}}$ is a weakly hereditary closure operator on $\mathcal{X}$. Moreover, we have that $C_{\mathcal{B}_\mathcal{N}}(m) \simeq m^{J_\mathcal{N}(\mathcal{B})}$.

Proof: We first notice that due to the fact that $\mathcal{X}$ is an $(\mathbf{E}, \mathcal{M})$-category for sinks, suprema are formed via $(\mathbf{E}, \mathcal{M})$-factorizations. Therefore, $C_{\mathcal{B}_\mathcal{N}}(m) \in \mathcal{M}$.

It is easily seen that $C_{\mathcal{B}_\mathcal{N}}$ is expansive and order-preserving. Let us consider the following commutative diagram



Let $X \xrightarrow{h} Z$ be an $\mathcal{X}$-morphism. We have that $(g^{-1}(m))^{C_{\mathcal{B}_\mathcal{N}}} =$

$$sup\Big(\{g^{-1}(m)\} \cup \{(id_X)_f : X \in \mathcal{B}, X \xrightarrow{f} Y \text{ and } \exists n \in \mathcal{N}_X \text{ with } n_f \leq g^{-1}(m)\}\Big) \simeq$$

$$sup\Big(\{g^{-1}(m)\} \cup \{(id_X)_f : X \in \mathcal{B}, X \xrightarrow{f} Y \text{ and } \exists n \in \mathcal{N}_X \text{ with } n_{g \circ f} \leq m\}\Big) \leq$$

$$sup\Big(\{g^{-1}(m)\} \cup \{g^{-1}((id_X)_{g \circ f}) : X \in \mathcal{B}, X \xrightarrow{f} Y \text{ and } \exists n \in \mathcal{N}_X \text{ with } n_{g \circ f} \leq m\}\Big) \simeq$$

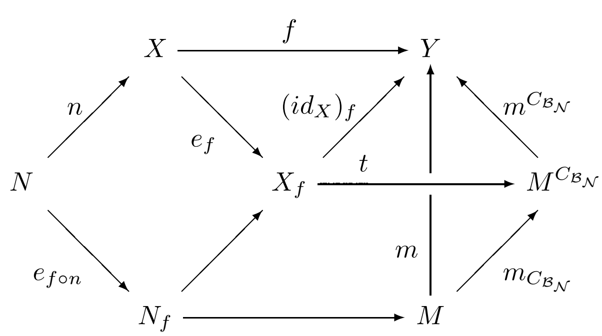
$$g^{-1}\Big(sup(\{m\} \cup \{(id_X)_{g \circ f} : X \in \mathcal{B}, X \xrightarrow{f} Y \text{ and } \exists n \in \mathcal{N}_X \text{ with } n_{g \circ f} \leq m\}\Big) \leq$$

$$g^{-1}\Big(sup(\{m\} \cup \{(id_X)_h : X \in \mathcal{B}, X \xrightarrow{h} Z \text{ and } \exists n \in \mathcal{N}_X \text{ with } n_h \leq m\}\Big) = g^{-1}(m^{C_{\mathcal{B}_\mathcal{N}}}).$$

Notice that above we have used the fact that pullbacks and suprema commute.

This shows that, for every $\mathcal{B} \in S(\mathcal{X})$, $C_{\mathcal{B}_\mathcal{N}}$ is a closure operator.

To see that $C_{\mathcal{B}_\mathcal{N}}$ is weakly hereditary, let $M \xrightarrow{m} Y$ be an $\mathcal{M}$-subobject and let $X \xrightarrow{f} Y$, with $X \in \mathcal{B}$, be such that there exists $N \xrightarrow{n} X \in \mathcal{N}_X$ with $n_f \leq m$. Let us consider the following commutative diagram



Notice that the morphism t exists by construction of $m^{C_{\mathcal{B}_{\mathcal{N}}}}$.

Consider the $(\mathbf{E},\mathcal{M})$-factorization of $t\circ e_f$, $(e_{t\circ e_f},(id_X)_{t\circ e_f})$. Due to the $(\mathbf{E},\mathcal{M})$-diagonaliza-tion property, from the following commutative diagram

$$\begin{array}{ccc}
N & \xrightarrow{e_{t\circ e_f}} & N_{t\circ e_f} \\
 & & \downarrow{\scriptstyle n_{t\circ e_f}} \\
e_{f\circ n}\ \Big\downarrow & & M^{C_{\mathcal{B}_{\mathcal{N}}}} \\
 & & \downarrow{\scriptstyle m^{C_{\mathcal{B}_{\mathcal{N}}}}} \\
N_f & \xrightarrow[n_f]{} & Y
\end{array}$$

we obtain that $n_{t\circ e_f} \le n_f \le m$ and so $(id_X)_{t\circ e_f}$ occurs in the construction of $(m_{C_{\mathcal{B}_{\mathcal{N}}}})^{C_{\mathcal{B}_{\mathcal{N}}}}$. However, since $(id_X)_f$ and $C_{\mathcal{B}_{\mathcal{N}}}(m)$ both belong to $\mathcal{M}$, a property of $(\mathbf{E},\mathcal{M})$-categories implies that also $t\in\mathcal{M}$. Therefore, we have that $t\simeq(id_X)_{t\circ e_f}$. We conclude that $(id_X)_f\le(m_{C_{\mathcal{B}_{\mathcal{N}}}})^{C_{\mathcal{B}_{\mathcal{N}}}}$, which implies $m^{C_{\mathcal{B}_{\mathcal{N}}}}\le(m_{C_{\mathcal{B}_{\mathcal{N}}}})^{C_{\mathcal{B}_{\mathcal{N}}}}$. Consequently, $m^{C_{\mathcal{B}_{\mathcal{N}}}}\simeq(m_{C_{\mathcal{B}_{\mathcal{N}}}})^{C_{\mathcal{B}_{\mathcal{N}}}}$, i.e.; $C_{\mathcal{B}_{\mathcal{N}}}$ is weakly hereditary.

If $Y\in\mathcal{B}$ and $N\xrightarrow{n}Y\in\mathcal{N}_Y$, the identity morphism yields that $n^{C_{\mathcal{B}_{\mathcal{N}}}}\simeq id_Y$. Therefore we obtain that $J_{\mathcal{N}}(\mathcal{B})\sqsubseteq C_{\mathcal{B}_{\mathcal{N}}}$.

Now, let $M\xrightarrow{m}Y$ be an $\mathcal{M}$-subobject of Y and let $X\xrightarrow{f}Y$ be an $\mathcal{X}$-morphism with $X\in\mathcal{B}$ and such that there exists $N\xrightarrow{n}X\in\mathcal{N}_X$ with $n_f\le m$. By definition of $J_{\mathcal{N}}(\mathcal{B})$, we have that $n^{J_{\mathcal{N}}(\mathcal{B})}\simeq id_X$. This implies that $(id_X)_f\simeq(n^{J_{\mathcal{N}}(\mathcal{B})})_f\le(n_f)^{J_{\mathcal{N}}(\mathcal{B})}\le m^{J_{\mathcal{N}}(\mathcal{B})}$ (cf. Remark 1.2). Therefore, $C_{\mathcal{B}_{\mathcal{N}}}(m)\le m^{J_{\mathcal{N}}(\mathcal{B})}$. Thus, $J_{\mathcal{N}}(\mathcal{B})\simeq C_{\mathcal{B}_{\mathcal{N}}}$. □

THEOREM 2.8. *Let $\mathcal{N}$ be a subclass of $\mathcal{M}$ closed under the formation of direct images. Then the Galois connection* $S(\mathcal{X})\underset{\nabla_{\mathcal{N}}}{\overset{\Delta_{\mathcal{N}}}{\rightleftarrows}}S(\mathcal{X})^{op}$ *factors through* $CL(\mathcal{X},\mathcal{M})$ *via the two Galois connections* $S(\mathcal{X})\underset{I_{\mathcal{N}}}{\overset{J_{\mathcal{N}}}{\rightleftarrows}}CL(\mathcal{X},\mathcal{M})$ *and* $CL(\mathcal{X},\mathcal{M})\underset{T_{\mathcal{N}}}{\overset{D_{\mathcal{N}}}{\rightleftarrows}}S(\mathcal{X})^{op}$.

Proof: Let $\mathcal{A}\in S(\mathcal{X})^{op}$ and let $X\in(I_{\mathcal{N}}\circ T_{\mathcal{N}})(\mathcal{A})$. Consider $X\xrightarrow{f}Y$ with $Y\in\mathcal{A}$ and let $N\xrightarrow{n}X$ belong to $\mathcal{N}_X$. From the properties of closure operators, we have that $(n^{T_{\mathcal{N}}(\mathcal{A})})_f\le(n_f)^{T_{\mathcal{N}}(\mathcal{A})}$. Since n is $T_{\mathcal{N}}(\mathcal{A})$-dense, we have that $(n^{T_{\mathcal{N}}(\mathcal{A})})_f\simeq(id_X)_f$. From Proposition 2.3, we have that every $\mathcal{N}$-subobject of Y is $T_{\mathcal{N}}(\mathcal{A})$-closed, and since $\mathcal{N}$ is closed under direct images, we obtain that $(n_f)^{T_{\mathcal{N}}(\mathcal{A})}\simeq n_f$. Therefore we have that $(id_X)_f\le n_f$. However, the reverse inequality always holds, thus we have that $n_f\simeq(id_X)_f$, i.e., $X\in\nabla_{\mathcal{N}}(\mathcal{A})$. Thus $(I_{\mathcal{N}}\circ T_{\mathcal{N}})(\mathcal{A})\subseteq\nabla_{\mathcal{N}}(\mathcal{A})$.

Now, let $X\in\nabla_{\mathcal{N}}(\mathcal{A})$. Let us consider $X\xrightarrow{f}Y$ with $Y\in\mathcal{A}$ and an $\mathcal{N}$-subobject $N\xrightarrow{n}X$. By hypothesis we have that $n_f\simeq(id_X)_f$. Since $\mathcal{N}$ is closed under direct images, we have that n_f belongs to $\mathcal{N}_Y$. Consequently, $id_X\simeq f^{-1}((id_X)_f)\simeq$

$f^{-1}(n_f)$. Notice that, due to the existence of a Galois connection between direct and inverse images, we have that if $n \leq f^{-1}(n')$ then $n_f \leq n'$ and so $id_X \simeq f^{-1}(n_f) \leq f^{-1}(n')$, for every $n' \in \mathcal{N}_Y$ satisfying $n \leq f^{-1}(n')$. Therefore we obtain that $n^{T_{\mathcal{N}}(\mathcal{A})} \simeq id_X$, for every $N \xrightarrow{n} X$ in $\mathcal{N}_X$. So, $X \in (I_{\mathcal{N}} \circ T_{\mathcal{N}})(\mathcal{A})$. Thus $I_{\mathcal{N}} \circ T_{\mathcal{N}} = \nabla_{\mathcal{N}}$.

Let $\mathcal{B} \in S(\mathcal{X})$ and let $Y \in (D_{\mathcal{N}} \circ J_{\mathcal{N}})(\mathcal{B})$. Consider $X \xrightarrow{f} Y$ with $X \in \mathcal{B}$. If $N \xrightarrow{n} X$ belongs to $\mathcal{N}_X$, then from Proposition 2.4 we have that n is $J_{\mathcal{N}}(\mathcal{B})$-dense, i.e., $n^{J_{\mathcal{N}}(\mathcal{B})} \simeq id_X$. This implies that $(n^{J_{\mathcal{N}}(\mathcal{B})})_f \simeq (id_X)_f$. ¿From the properties of closure operators, we have that $(n^{J_{\mathcal{N}}(\mathcal{B})})_f \leq (n_f)^{J_{\mathcal{N}}(\mathcal{B})}$. Since every $\mathcal{N}$-subobject of Y is $J_{\mathcal{N}}(\mathcal{B})$-closed and $\mathcal{N}$ is closed under direct images, we have that $(n_f)^{J_{\mathcal{N}}(\mathcal{B})} \simeq n_f$. Therefore, we obtain that $(id_X)_f \leq n_f$. Since we always have that $n_f \leq (id_X)_f$, we conclude that $n_f \simeq (id_X)_f$. Thus, $X \in \Delta_{\mathcal{N}}(\mathcal{B})$ and so $(D_{\mathcal{N}} \circ J_{\mathcal{N}})(\mathcal{B}) \subseteq \Delta_{\mathcal{N}}(\mathcal{B})$.

Now, let $Y \in \Delta_{\mathcal{N}}(\mathcal{B})$ and let $M \xrightarrow{m} Y \in \mathcal{N}_Y$. Consider $X \xrightarrow{f} Y$ with $X \in \mathcal{B}$ and $n \in \mathcal{N}_X$ such that $n_f \leq m$. Since f is $\mathcal{N}$-constant, we have that $(id_X)_f \simeq n_f \leq m$. Consequently, from Proposition 2.7 we have that $m^{J_{\mathcal{N}}(\mathcal{B})} \simeq m$ for every $M \xrightarrow{m} X$ in $\mathcal{N}_Y$. So, $X \in (D_{\mathcal{N}} \circ J_{\mathcal{N}})(\mathcal{B})$. Thus $D_{\mathcal{N}} \circ J_{\mathcal{N}} = \Delta_{\mathcal{N}}$. □

REMARK 2.9. (1) Notice that, since $(D_{\mathcal{N}} \circ J_{\mathcal{N}}, I_{\mathcal{N}} \circ T_{\mathcal{N}})$ is an adjoint situation, we could have proved either of the two equalities $I_{\mathcal{N}} \circ T_{\mathcal{N}} = \nabla_{\mathcal{N}}$ or $D_{\mathcal{N}} \circ J_{\mathcal{N}} = \Delta_{\mathcal{N}}$ and obtain the other one (up to isomorphism) by the uniqueness of adjoint situations (cf. [1, Proposition 19.9] and [9, Proposition 1.04]).

(2) Also notice that in the case that $\mathcal{X}$ is a construct satisfying all the assumptions of [3] and $\mathcal{N} = \mathcal{M}$, then our present notion of $\mathcal{N}$-constant morphism falls short of agreeing with the one of constant morphism in [3]. The only difference being the fact that in this case, morphisms with constant domain are $\mathcal{M}$-constant but they are not constant morphisms according to [3]. However, in the category **Top** of topological spaces (resp. **Ab** of abelian groups), by taking $\mathcal{M}$={ all extremal monomorphisms} and $\mathcal{N}$={ all extremal monomorphisms with nonempty domain} (obviously in **Ab** $\mathcal{N}$=$\mathcal{M}$), Proposition 2.2 produces connectedness and disconnectedness (resp. torsion theories). More detailed examples that illustrate Theorem 2.8 can be found in [2].

We conclude with some results that show how the various constructions presented above interact with each other.

PROPOSITION 2.10. *Let $\mathcal{N} \subseteq \mathcal{M}$ and let $\mathcal{B} \in S(\mathcal{X})$ and $\mathcal{A} \in S(\mathcal{X})^{op}$ be two corresponding fixed points of the Galois connection $(S(\mathcal{X}), \Delta_{\mathcal{N}}, \nabla_{\mathcal{N}}, S(\mathcal{X})^{op})$. Then, the following hold:*

(a) If $Y \in \mathcal{A}$ and $m \in \mathcal{M}_Y$, then m is $J_{\mathcal{N}}(\mathcal{B})$-closed;

(b) Let $\mathcal{N}$ be closed under the formation of pullbacks. If $X \in \mathcal{B}$ and $m \in \mathcal{M}_X$, then m is $T_{\mathcal{N}}(\mathcal{A})$-dense;

(c) $J_{\mathcal{N}}(\mathcal{B}) \sqsubseteq T_{\mathcal{N}}(\mathcal{A})$;

(d) If $\mathcal{N}$ is closed under direct images, then for every $C \in CL(\mathcal{X}, \mathcal{M})$, we have that $D_{\mathcal{N}}(C) \subseteq (\Delta_{\mathcal{N}} \circ I_{\mathcal{N}})(C)$;

(e) *If $\mathcal{N}$ is closed under direct images, then for every $C \in CL(\mathcal{X}, \mathcal{M})$, we have that* $I_{\mathcal{N}}(C) \subseteq (\nabla_{\mathcal{N}} \circ D_{\mathcal{N}})(C)$.

Proof: (a). Let $Y \in \mathcal{A}$ and let $X \xrightarrow{f} Y$ be an $\mathcal{X}$-morphism. Let $M \xrightarrow{m} Y$ be an $\mathcal{M}$-subobject and let $N \xrightarrow{n} X$ be an $\mathcal{N}$-subobject such that $X \in \mathcal{B}$ and $n_f \leq m$. Then, f is $\mathcal{N}$-constant and therefore we have that $(id_X)_f \simeq n_f \leq m$. Thus, $m^{J_{\mathcal{N}}(\mathcal{B})} \simeq m$, i.e.; m is $J_{\mathcal{N}}(\mathcal{B})$-closed.

(b). Let $X \in \mathcal{B}$ and let $M \xrightarrow{m} X$ be an $\mathcal{M}$-subobject. Suppose that $X \xrightarrow{f} Y$ is an $\mathcal{X}$-morphism with $Y \in \mathcal{A}$ and let $N \xrightarrow{n} Y$ be an $\mathcal{N}$-subobject with $m \leq f^{-1}(n)$. Since $\mathcal{N}$ is closed under the formation of pullbacks, then $f^{-1}(n) \in \mathcal{N}_X$. Now, f is $\mathcal{N}$-constant and so we have that $(id_X)_f \simeq (f^{-1}(n))_f \leq n$. This implies that $id_X \leq f^{-1}((id_X)_f) \leq f^{-1}(n)$. Thus, $m^{T_{\mathcal{N}}(\mathcal{A})} \simeq id_X$. Hence, m is $T_{\mathcal{N}}(\mathcal{A})$-dense.

(c). It is a direct consequence of (a).

(d). Let $X \in I_{\mathcal{N}}(C)$, let $Y \in D_{\mathcal{N}}(C)$ and let $X \xrightarrow{f} Y$ be a morphism. From the properties of closure operators and the fact that $\mathcal{N}$ is closed under direct images, we have that for every $N \xrightarrow{n} X \in \mathcal{N}$, $(id_X)_f \simeq (n^C)_f \leq (n_f)^C \simeq n_f$. Thus we obtain that $Y \in (\Delta_{\mathcal{N}} \circ I_{\mathcal{N}})(C)$.

(e). Similarly to (d). □

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