CONNECTEDNESS, DISCONNECTEDNESS AND CLOSURE OPERATORS: FURTHER RESULTS

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ABSTRACT: Let \( \mathcal{X} \) be an arbitrary category with an \((E, M)\)-factorization structure for sinks. A notion of constant morphism that depends on a chosen class of monomorphisms was previously used to provide a generalization of the connectedness-disconnectedness Galois connection (also called torsion-torsion free in algebraic contexts). This Galois connection was shown to factor through the class of all closure operators on \( \mathcal{X} \) with respect to \( M \). Here, properties and implications of this factorization are investigated. In particular, it is shown that this factorization can be further factored. Examples are provided.

KEY WORDS: Closure operator, Galois connection, connectedness, disconnectedness.

AMS CLASSIFICATION: 18A20, 18A32, 06A15.

0 INTRODUCTION

This paper is a continuation of the work started in [C2].

Let \( \mathcal{X} \) be an arbitrary category with an \((E, M)\)-factorization structure for sinks and let \( N \subseteq M \). An \( \mathcal{X} \)-morphism \( X \rightarrow Y \) is called \( N \)-constant if the direct image of \( X \) under \( f \) is isomorphic to the direct image under \( f \) of every \( N \)-subobject of \( X \). If \( S(\mathcal{X}) \) denotes the collection of all subclasses of objects of \( \mathcal{X} \), ordered by inclusion, for every \( N \subseteq M \), the relation: \( X R_N Y \) if and only if every \( \mathcal{X} \)-morphism \( X \rightarrow Y \) is \( N \)-constant yields a Galois connection \( S(\mathcal{X}) \xrightarrow{\Delta_{N'}} \xleftarrow{\nabla_{N'}} S(\mathcal{X})^{op} \).

It was proved in [C2] that if \( N \) is closed under direct images, we have that this Galois connection factors through \( CL(\mathcal{X}, M) \), i.e., the collection of all closure operators on \( \mathcal{X} \) with respect to \( M \), via two Galois connections \( S(\mathcal{X}) \xrightarrow{J_N} CL(\mathcal{X}, M) \) and \( CL(\mathcal{X}, M) \xrightarrow{D_N} S(\mathcal{X})^{op} \).

The development of a general theory of topological connectedness was started by Preuß (cf. [P1-P3]) and by Herrlich [H]. However, our definition of \( N \)-constant morphism was not chosen with the intention of developing a general theory of connectedness and disconnectedness, but rather to support certain constructions with closure operators. A general theory of connectedness and disconnectedness was recently presented by Clementino in [Cl], where she extends results in [HP] to an arbitrary category.

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In this paper we present some closedness properties of the Galois closed classes of the Galois connection $S(\mathcal{X}) \xrightarrow{\Delta_N} \xrightarrow{\nabla_N} S(\mathcal{X})^{\text{op}}$. We show that if $\mathcal{A}$ is a reflective subcategory of $\mathcal{X}$ and $\mathcal{B}$ is coreflective in $\mathcal{X}$, then a simpler characterization of the closure operators $T_N(\mathcal{A})$ and $J_N(\mathcal{B})$ can be given. Moreover, following some ideas introduced in [CKS], we bring more insight into the theory by showing that the Galois connections $S(\mathcal{X}) \xrightarrow{\text{CL}(\mathcal{X}, \mathcal{M})} \xrightarrow{\text{CL}(\mathcal{X}, \mathcal{M})} S(\mathcal{X})^{\text{op}}$ can also be factored.

Section 3 includes a number of examples that illustrate the theory.

We use the terminology of [AHS] throughout the paper.

1 PRELIMINARIES

Throughout we consider a category $\mathcal{X}$ and a fixed class $\mathcal{M}$ of $\mathcal{X}$-monomorphisms, which contains all $\mathcal{X}$-isomorphisms. It is assumed that $\mathcal{X}$ is $\mathcal{M}$-complete; i.e.,

1. $\mathcal{M}$ is closed under composition

2. Pullbacks of $\mathcal{M}$-morphisms exist and belong to $\mathcal{M}$, and multiple pullbacks of (possibly large) families of $\mathcal{M}$-morphisms with common codomain exist and belong to $\mathcal{M}$.

One of the consequences of the above assumptions is that there is a uniquely determined class $\mathcal{E}$ of sinks in $\mathcal{X}$ such that $\mathcal{X}$ is an $(\mathcal{E}, \mathcal{M})$-category for sinks, that is:

1. each of $\mathcal{E}$ and $\mathcal{M}$ is closed under compositions with isomorphisms;
2. $\mathcal{X}$ has $(\mathcal{E}, \mathcal{M})$-factorizations (of sinks); i.e., each sink $s$ in $\mathcal{X}$ has a factorization $s = m \circ e$ with $e \in \mathcal{E}$ and $m \in \mathcal{M}$, and
3. $\mathcal{X}$ has the unique $(\mathcal{E}, \mathcal{M})$-diagonalization property; i.e., if $B \xrightarrow{g} D$ and $C \xrightarrow{m} D$ are $\mathcal{X}$-morphisms with $m \in \mathcal{M}$, and $e = (A_i \xrightarrow{e_i} B)_I$ and $s = (A_i \xrightarrow{s_i} C)_I$ are sinks in $\mathcal{X}$ with $e \in \mathcal{E}$, such that $m \circ s = g \circ e$, then there exists a unique diagonal $B \xrightarrow{d} C$ such that for every $i \in I$ the following diagrams commute:

\[
\begin{array}{ccc}
A_i & \xrightarrow{e_i} & B \\
\downarrow s_i & \quad \quad \quad & \quad \quad \downarrow g \\
C & \xrightarrow{d} & C \xrightarrow{m} D
\end{array}
\]

That $\mathcal{X}$ is an $(\mathcal{E}, \mathcal{M})$-category implies the following features of $\mathcal{M}$ and $\mathcal{E}$ (cf. [AHS] for the dual case):

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2 Paul Taylor’s commutative diagrams macro package was used to typeset most of the diagrams in this paper.
PROPOSITION 1.1

(0) Every isomorphism is in both \( \mathcal{M} \) and \( \mathcal{E} \) (as a singleton sink).

(1) Every \( m \) in \( \mathcal{M} \) is a monomorphism.

(2) \( \mathcal{M} \) is closed under \( \mathcal{M} \)-relative first factors, i.e., if \( n \circ m \in \mathcal{M} \), and \( n \in \mathcal{M} \), then \( m \in \mathcal{M} \).

(3) \( \mathcal{M} \) is closed under composition.

(4) Pullbacks of \( \mathcal{X} \)-morphisms in \( \mathcal{M} \) exist and belong to \( \mathcal{M} \).

(5) The \( \mathcal{M} \)-subobjects of every \( \mathcal{X} \)-object form a (possibly large) complete lattice; suprema are formed via \( (\mathcal{E}, \mathcal{M}) \)-factorizations and infima are formed via intersections.

Notice that in the above proposition, the word “lattice” is to be understood in a generalized sense for not necessarily antisymmetric pre-orders. Moreover, throughout the paper we will use the expression \( \mathcal{M} \)-subobject for both \( m \in \mathcal{M} \) and the corresponding equivalence class of elements of \( \mathcal{M} \).

DEFINITION 1.2

A closure operator \( C \) on \( \mathcal{X} \) (with respect to \( \mathcal{M} \)) is a family \( \{ (\ )^C_X \}_{X \in \mathcal{X}} \) of functions on the \( \mathcal{M} \)-subobject lattices of \( \mathcal{X} \) with the following properties that hold for each \( X \in \mathcal{X} \):

(a) [expansiveness] \( m \leq (m)^C_X \), for every \( \mathcal{M} \)-subobject \( M \xrightarrow{m} X \);

(b) [order-preservation] \( m \leq n \Rightarrow (m)^C_X \leq (n)^C_X \) for every pair of \( \mathcal{M} \)-subobjects of \( X \);

(c) [morphism-consistency] If \( p \) is the pullback of the \( \mathcal{M} \)-subobject \( M \xrightarrow{m} Y \) along some \( \mathcal{X} \)-morphism \( X \xrightarrow{f} Y \) and \( q \) is the pullback of \( (m)^C_Y \) along \( f \), then \( (p)^C_X \leq q \), i.e., the closure of the inverse image of \( m \) is less than or equal to the inverse image of the closure of \( m \).

Condition (a) implies that for every closure operator \( C \) on \( \mathcal{X} \), every \( \mathcal{M} \)-subobject \( M \xrightarrow{m} X \) has a canonical factorization

\[
M \xrightarrow{t} (M)^C_X \xrightarrow{m} X
\]

where \( ((M)^C_X, (m)^C_X) \) is called the \( C \)-closure of the subobject \( (M, m) \).
When no confusion is likely we will write $m^C$ rather than $(m)_X^C$ and for notational symmetry we will denote the morphism $t$ by $m_c$.

**REMARK 1.3**

1. Notice that in the above definition, under condition (b), the morphism-consistency condition (c) is equivalent to the following statement concerning direct images: if $M \xrightarrow{m} X$ is an $\mathcal{M}$-subobject and $X \xrightarrow{\ell} Y$ is a morphism, then $((m^C)_Y)\ell \leq (m_f)^C_Y$, i.e., the direct image of the closure of $m$ is less than or equal to the closure of the direct image of $m$; (cf. [DG]).

2. Under condition (a), both order-preservation and morphism-consistency, i.e., conditions (b) and (c) together are equivalent to the following: given $(M, m)$ and $(N, n)$ $\mathcal{M}$-subobjects of $X$ and $Y$, respectively, if $f$ and $g$ are morphisms such that $n \circ g = f \circ m$, then there exists a unique morphism $d$ such that the following diagram

$$
\begin{array}{ccc}
M & \xrightarrow{g} & N \\
m \downarrow & & \downarrow n \\
M^C & \xrightarrow{d} & N^C \\
m_c \downarrow & & \downarrow n_c \\
X & \xrightarrow{f} & Y
\end{array}
$$

commutes.

3. If we regard $\mathcal{M}$ as a full subcategory of the arrow category of $\mathcal{X}$, with the codomain functor from $\mathcal{M}$ to $\mathcal{X}$ denoted by $U$, then the above definition can also be stated in the following way: A closure operator on $\mathcal{X}$ (with respect to $\mathcal{M}$) is a pair $C = (\gamma, F)$, where $F$ is an endofunctor on $\mathcal{M}$ that satisfies $U F = U$, and $\gamma$ is a natural transformation from $id_{\mathcal{M}}$ to $F$ that satisfies $(id_U)\gamma = id_U$ (cf. [DG]).

**DEFINITION 1.4**

Given a closure operator $C$, we say that $m \in \mathcal{M}$ is $C$-closed if $m_c$ is an isomorphism. An $\mathcal{X}$-morphism $f$ is called $C$-dense if for some (and hence every) $(\mathcal{E}, \mathcal{M})$-factorization $(e, m)$ of $f$ we have that $m_c^C$ is an isomorphism. We call $C$ idempotent provided that $m_c^C$ is $C$-closed for every $m \in \mathcal{M}$. $C$ is called weakly hereditary if $m_c$ is $C$-dense for every $m \in \mathcal{M}$. Furthermore, if $\mathcal{M}' \subseteq \mathcal{M}$, then $C$ is said to be hereditary with respect to $\mathcal{M}'$ if whenever $M \xrightarrow{m} X$, $M \xrightarrow{t} N$ and $N \xrightarrow{n} X$ are morphisms in $\mathcal{M}$ with $n \circ t = m$ and $n \in \mathcal{M}'$, we have that $t^C$ is the pullback of $m_c^C$ along $n$ (cf. [CG]). If $\mathcal{M}' = \mathcal{M}$, then $C$ is simply called hereditary.

Notice that Definition 1.2(c) implies that pullbacks of $C$-closed $\mathcal{M}$-subobjects are $C$-closed.
A special case of an idempotent closure operator arises in the following way. Given any class \( A \) of \( X \)-objects and \( M \xrightarrow{m} X \) in \( M \), define \( m^A \) to be the intersection of all equalizers of pairs of \( X \)-morphisms \( r, s \) from \( X \) to some \( A \)-object \( A \) that satisfy \( r \circ m = s \circ m \), and let \( m^A \in M \) be the unique \( X \)-morphism by which \( m \) factors through \( m^A \). It is easy to see that this gives rise to an idempotent closure operator that we will denote by \( S_A \). This generalizes the Salbany construction of closure operators induced by classes of topological spaces, cf. [S].

We denote the collection of all closure operators on \( M \) by \( CL(X, M) \) pre-ordered as follows: \( C \subseteq D \) if \( m^C \leq m^D \) for all \( m \in M \) (where \( \leq \) is the usual order on subobjects). Notice that arbitrary suprema and infima exist in \( CL(X, M) \), they are formed pointwise in the \( M \)-subobject fibers.

For more background on closure operators see, e.g., [C1], [CKS1], [CKS2], [DG], [DGT] and [K]. For a detailed survey on the same topic, one could check [Ho].

**DEFINITION 1.5**

For pre-ordered classes \( \mathcal{X} = (X, \sqsubseteq) \) and \( \mathcal{Y} = (Y, \sqsubseteq) \), a Galois connection \( \mathcal{X} \xrightarrow{F} \mathcal{Y} \) consists of order preserving functions \( F \) and \( G \) that satisfy \( F \dashv G \), i.e., \( x \sqsubseteq GF(x) \) for every \( x \in X \) and \( FG(y) \sqsubseteq y \) for every \( y \in Y \). (\( G \) is adjoint and has \( F \) as coadjoint).

If \( x \in \mathcal{X} \) and \( y \in \mathcal{Y} \) are such that \( F(x) = y \) and \( G(y) = x \), then \( x \) and \( y \) are said to be corresponding fixed points of the Galois connection \( (\mathcal{X}, F, G, \mathcal{Y}) \) (we may use at times the shorter notation \( (F, G) \)). To be more precise, we may sometimes make use of the expressions “left fixed point” and “right fixed point” for \( x \) and \( y \), respectively.

Properties and many examples of Galois connections can be found in [EKMS].

**2 GENERAL RESULTS**

Throughout the paper we assume that \( \mathcal{X} \) is an \( (E, \mathcal{M}) \)-category for sinks.

Let \( S(\mathcal{X}) \) denote the collection of all subcategories of \( \mathcal{X} \), ordered by inclusion and let \( \mathcal{N} \) be a fixed subclass of \( \mathcal{M} \). Throughout, for every \( X \in \mathcal{X} \), \( \mathcal{M}_X (\mathcal{N}_X) \) will denote the “lattice” of all \( \mathcal{M} \)-subobjects (\( \mathcal{N} \)-subobjects) of \( X \). We begin by recalling the following definition and results from [C2].

**DEFINITION 2.1**

Let \( \mathcal{N} \subseteq \mathcal{M} \). An \( \mathcal{X} \)-morphism \( X \xrightarrow{f} Y \) is called \( \mathcal{N} \)-constant if for every \( \mathcal{N} \)-subobject \( N \xrightarrow{n} X \), we have that \( nf \simeq (id_X)_f \).
PROPOSITION 2.2 (cf. [H])
Let \( N \subseteq M \). Define \( S(X) \xrightarrow{\Delta_N} S(X)^{\text{op}} \) and \( S(X)^{\text{op}} \xrightarrow{\nabla_N} S(X) \) as follows:
\[
\nabla_N(A) = \{ X \in X : \forall Y \in A, \text{ every } X \xrightarrow{f} Y \text{ is } N\text{-constant} \}
\]
\[
\Delta_N(B) = \{ Y \in X : \forall X \in B, \text{ every } X \xrightarrow{f} Y \text{ is } N\text{-constant} \}
\]
Then, \( S(X) \xrightarrow{\Delta_N} S(X)^{\text{op}} \xrightarrow{\nabla_N} S(X) \) is a Galois connection.

PROPOSITION 2.3
Let \( \text{CL}(X, M) \xrightarrow{D_N} S(X)^{\text{op}} \) and \( S(X)^{\text{op}} \xrightarrow{T_N} \text{CL}(X, M) \) be defined by:
\[
D_N(C) = \{ X \in X : \text{ every } n \in N_X \text{ is } C\text{-closed} \}
\]
\[
T_N(A) = \text{Sup}\{ C \in \text{CL}(X, M) : D_N(C) \supseteq A \}
\]
Then, \( \text{CL}(X, M) \xrightarrow{D_N} S(X)^{\text{op}} \xrightarrow{T_N} \text{CL}(X, M) \) is a Galois connection.

PROPOSITION 2.4
Let \( \text{CL}(X, M) \xrightarrow{I_N} S(X)^{\text{op}} \) and \( S(X)^{\text{op}} \xrightarrow{J_N} \text{CL}(X, M) \) be defined by:
\[
I_N(C) = \{ X \in X : \text{ every } n \in N_X \text{ is } C\text{-dense} \}
\]
\[
J_N(B) = \text{Inf}\{ C \in \text{CL}(X, M) : I_N(C) \supseteq B \}
\]
Then, \( S(X)^{\text{op}} \xrightarrow{J_N} \text{CL}(X, M) \xrightarrow{I_N} S(X)^{\text{op}} \) is a Galois connection.

PROPOSITION 2.5
Let \( A \in S(X)^{\text{op}} \). For every \( X \in X \) and for every \( M\)-subobject \( M \xrightarrow{m} X \), we define
\[
\mathcal{A}m = \cap \{ f^{-1}(n) : Y \in \mathcal{A}, X \xrightarrow{f} Y, N \xrightarrow{n} Y \in N_Y \text{ and } m \leq f^{-1}(n) \}.
\]
For every \( A \in S(X)^{\text{op}} \) we have that the function \( \mathcal{A}( ) \) that to every \( M\)-subobject \( M \xrightarrow{m} X \) associates \( \mathcal{A}m \) is an idempotent closure operator on \( X \) and \( \mathcal{A}m \simeq m^{T_N}(\mathcal{A}) \).

PROPOSITION 2.6
Let \( B \in S(X) \). For every \( Y \in X \) and for every \( M\)-subobject \( M \xrightarrow{m} Y \), we define
\[
C_{B_N}(m) = \text{sup} \left( \{ m \} \cup \{ (id_X)_f : X \in B, X \xrightarrow{f} Y \text{ and } \exists n \in N_X \text{ with } n_f \leq m \} \right).
\]
For every \( B \in S(X) \), the function \( C_{B_N} \) is a weakly hereditary closure operator on \( X \). Moreover, we have that \( C_{B_N}(m) \simeq m^{J_N}(B) \).
THEOREM 2.7

Let \( N \) be a subclass of \( M \) closed under the formation of direct images. Then the Galois connection \( S(\mathcal{X}) \xrightarrow{\Delta_N} S(\mathcal{X})^{op} \) factors through \( CL(\mathcal{X}, M) \) via the two Galois connections

\[
S(\mathcal{X}) \xrightarrow{\Delta_N} CL(\mathcal{X}, M) \quad \text{and} \quad CL(\mathcal{X}, M) \xrightarrow{D_N} S(\mathcal{X})^{op}.
\]

\[\square\]

PROPOSITION 2.8

Let \( N \subseteq M \) and let \( X \xrightarrow{f} Y \) be an \( X \)-morphism. The following are equivalent:

(a) For every \( N \)-subobject \( N \xrightarrow{n} X \) we have that \( n \circ f \simeq (id_X) \circ f \);
(b) \( f \) factors through \( n \circ f \), for every \( N \)-subobject \( N \xrightarrow{n} X \).

Proof:

(a) \( \Rightarrow \) (b). Consider the \( N \)-subobject \( N \xrightarrow{n} X \) and let \( (e, (id_X) \circ f) \) and \( (e_f, n \circ f) \) be \((E, M)\)-factorizations of \( f \) and \( f \circ n \), respectively. From the hypothesis, there is an isomorphism \( X \xrightarrow{i} N \) such that \( n \circ f \circ i = (id_X) \circ f \). Consequently we have that \( f = (id_X) \circ f \circ i \circ e \).

Thus (b) holds.

(b) \( \Rightarrow \) (a). We will use here the same notation as in the first part of the proof. From the hypothesis we obtain a morphism \( X \xrightarrow{d} N \) such that \( n \circ d = f \). Now we have the following two commutative diagrams

\[
\begin{array}{ccc}
X & \xrightarrow{e} & X_f \\
\downarrow & & \downarrow \text{(id}_X) \circ f \\
N_f & \xrightarrow{n \circ f} & Y \\
& & \downarrow n_f \\
\end{array}
\quad \quad \begin{array}{ccc}
N & \xrightarrow{e_f} & N_f \\
\downarrow \text{con} & & \downarrow n_f \\
N_f & \xrightarrow{n} & Y \\
& & \downarrow \text{id}_X \\
\end{array}
\]

The \((E, M)\)-diagonalization property yields a morphism \( r \) such that \( n \circ r = (id_X) \circ f \) and a morphism \( s \) such that \( (id_X) \circ f \circ s = n \). Therefore we obtain that \( (id_X) \circ f \circ s \circ r = (id_X) \circ f = (id_X) \circ id_X \). Since \((id_X) \circ f \) is a monomorphism, we conclude that \( s \circ r = id_X \). This shows that \( s \) is a retraction and a monomorphism and consequently is actually an isomorphism.

\[\square\]

REMARK 2.9

Part (b) of Proposition 2.8 provides an alternative formulation of Definition 2.1. Since this formulation of the definition of \( N \)-constant morphism simplifies the writing of the proofs, we will be using it throughout the paper.

Next we examine the behavior of the Galois closed classes of \( S(\mathcal{X}) \xrightarrow{\Delta_N} S(\mathcal{X})^{op} \) with respect to some classical constructions.
PROPOSITION 2.10

For every subcategory $B \in S(X)$, $\Delta_N(B)$ is closed under $M$-subobjects.

Proof:

Let $M \xrightarrow{m} Y$ be an $M$-subobject with $Y \in \Delta_N(B)$, let $X \in B$, let $N \xrightarrow{n} X$ be an $N$-subobject of $X$ and let $X \xrightarrow{f} M$ be an $X$-morphism. If $(e_f, n_f)$ and $(e_{mof}, n_{mof})$, denote the $(E, M)$-factorizations of $f \circ n$ and $m \circ f \circ n$, respectively, then we obtain the following commutative diagram

![Commutative Diagram]

We first observe that since $Y \in \Delta_N(B)$, from Remark 2.9, there is a morphism $X \xrightarrow{d} N_{mof}$ such that $n_{mof} \circ d = m \circ f$.

Now we have that $m \circ n_f \circ e_f = m \circ f \circ n = n_{mof} \circ e_{mof}$. The $(E, M)$-diagonalization property implies the existence of a unique morphism $N_{mof} \xrightarrow{t} N_f$ that satisfies $t \circ e_{mof} = e_f$ and $m \circ n_f \circ t = n_{mof}$. So, we have that $m \circ n_f \circ t \circ d = n_{mof} \circ d = m \circ f$. Since $m$ is a monomorphism, we obtain that $n_f \circ t \circ d = f$. Thus $f$ factors through $n_f$ and again from Remark 2.9, $M \in \Delta_N(B)$. 

PROPOSITION 2.11

Let $X$ have a terminal object $T$ and assume that any morphism with $T$ as domain belongs to $M$. If $E$ is contained in the class of episinks and $N$ consists of all morphisms having $T$ as domain, then for every subclass $B \in S(X)$, we have that $\Delta_N(B)$ is closed under monosources.

Proof:

Let us consider the following commutative diagram

![Commutative Diagram]
where \((e_i, n_i)\) and \((e_f, n_f)\) are the \((E, M)\)-factorizations of \(p_i \circ f \circ t_X\) and \(f \circ t_X\), respectively and \((Y \overset{p_i}{\to} Y_i)_{i \in I}\) is a monosource with \(Y_i \in \Delta_N(B)\) for every \(i \in I\). Let \(X \overset{d_X}{\to} T\) be the unique morphism with codomain \(T\). By hypothesis, for every \(i \in I\), there exists a morphism \(X \overset{t_i}{\to} T_i\) such that \(p_i \circ f = m_i \circ t_i\). Notice that since epimorphisms with terminal domain are already isomorphisms, we have that \(e_i\) is an isomorphism for every \(i \in I\). Since \(T\) is a terminal object, we have that \(d_X = e_i^{-1} \circ t_i\) for every \(i \in I\) and so \(t_i = e_i \circ d_X\), for every \(i \in I\). Now, \(p_i \circ f = m_i \circ t_i = m_i \circ e_i \circ d_X = p_i \circ f \circ t_X \circ d_X = p_i \circ n_f \circ e_f \circ d_X\) for every \(i \in I\). Since \((p_i)_{i \in I}\) is a monosource, we have that \(f = n_f \circ e_f \circ d_X\). Thus, \(f\) factors through \(n_f\) and so \(Y \in \Delta_N(B)\).

We recall that an \(x\)-morphism \(X \overset{q}{\to} Q\) is called an \(E\)-quotient if the singleton sink \(q\) belongs to \(E\).

**Proposition 2.12**

If \(N\) is closed under pullbacks along \(E\)-morphisms, then for every \(A \in S(A)^{op}, \nabla_N(A)\) is closed under \(E\)-quotients.

**Proof:**

Let us consider the following commutative diagram

\[
\begin{array}{ccc}
X & \overset{f}{\longrightarrow} & Y \\
q^{-1}(n) \downarrow & & \downarrow n \\
q^{-1}(N) & \overset{t}{\longrightarrow} & N \\
\end{array}
\]

where \(q \in E, Y \in A, X \in \nabla_N(A)\) and \(q^{-1}(n)\) is the pullback of \(n \in N\) along \(q\). Notice that by hypothesis \(q^{-1}(n) \in N\). Since \(f \circ q\) is \(N\)-constant, it factors through \((q^{-1}(n))_{foq}\) via a morphism \(X \overset{d}{\longrightarrow} (q^{-1}(N))_{foq}\). Now let us consider the commutative diagram
From the \((E, M)\)-diagonalization property we obtain a morphism \(d'\) such that \(d' \circ q = d\) and \((q^{-1}(n))_{f \circ q} \circ d' = f\) and a morphism \(t'\) such that \(t' \circ e_{f \circ q} = e_{f} \circ t\) and \(n_{f} \circ t' = (q^{-1}(n))_{f \circ q}\). Therefore we obtain that \(f = n_{f} \circ t' \circ d'\), and so \(Q \in \nabla_{\mathcal{N}}(A)\).

\[
egin{array}{c}
X \\ d \\
\downarrow \quad \downarrow q \\
Y \\
\downarrow f \\
\qquad f \circ q \\
q^{-1}(N) \\
\uparrow e_{f \circ q} \\
\downarrow e_{f} \circ t \\
N_{f}
\end{array}
\]

PROPOSITION 2.13

Let \(\mathcal{N}\) be closed under pullbacks and let \(X_{i} \in \nabla_{\mathcal{N}}(A)\) for every \(i \in I\). If the coproduct \(\amalg X_{i}\) exists, then it also belongs to \(\nabla_{\mathcal{N}}(A)\).

Proof:

Let us consider the following commutative diagram

\[
egin{array}{c}
X_{i} \\ k_{i} \\
\downarrow k_{i}^{-1}(n) \\
N_{i} \\
\downarrow h_{i} \\
k_{i}^{-1}(N) \\
\uparrow e_{f} \\
N_{f}
\end{array}
\]

where \((e_{f}, n_{f})\) is an \((E, M)\)-factorization of \(f \circ n\), \(k_{i}^{-1}(n)\) is the pullback of \(n \in \mathcal{N}\) along \(k_{i}\) and \(Y \in A\).

Now, let \((e_{i}, n_{i})\) be an \((E, M)\)-factorization of \(f \circ n \circ k_{i} \circ k_{i}^{-1}(n)\). Since, for every \(i \in I\), \(X_{i} \in \nabla_{\mathcal{N}}(A)\) and \(k_{i}^{-1}(n) \in \mathcal{N}\), we have that for every \(i \in I\) there exists a morphism \(t_{i}\) having \(X_{i}\) as domain such that \(n_{i} \circ t_{i} = f \circ k_{i}\). Since for every \(i \in I\) we have that \(n_{i} \circ e_{i} = n_{f} \circ e_{f} \circ h_{i}\), then the \((E, M)\)-diagonalization property implies that for every \(i \in I\) there exists a morphism \(d_{i}\) with codomain \(N_{f}\), such that \(n_{f} \circ d_{i} = n_{i}\) and \(d_{i} \circ e_{i} = e_{f} \circ h_{i}\). Therefore, there exists a unique \(\amalg X_{i} \xrightarrow{t} N_{f}\) such that \(t \circ k_{i} = d_{i} \circ t_{i}\). Now we have that \(f \circ k_{i} = n_{f} \circ t_{i} = n_{f} \circ d_{i} \circ t_{i} = n_{f} \circ t \circ k_{i}\), for every \(i \in I\). The uniqueness condition in the definition of coproduct implies that \(n_{f} \circ t = f\).

Thus, we have that \(\amalg X_{i} \in \nabla_{\mathcal{N}}(A)\).
Remark 2.14

Notice that if in the category TOP of topological spaces (Grp of groups), we choose for instance the (episink, extremal monomorphism)-factorization structure, and \( \mathcal{N} \) consists of all inclusions of singleton subobjects, then Propositions 2.10 and 2.11 imply that for any subcategory \( \mathcal{B} \) of TOP (Grp), \( \Delta_\mathcal{N}(\mathcal{B}) \) is an epireflective subcategory of TOP (Grp). Moreover, if we choose as \( \mathcal{N} \) any class of extremal monomorphisms that is closed under pullbacks, then Propositions 2.12 and 2.13 imply that for any subcategory \( \mathcal{A} \) of TOP (Grp), \( \nabla_\mathcal{N}(\mathcal{A}) \) is a coreflective subcategory of TOP (Grp).

Proposition 2.15

Let \( \mathcal{N} \) be a subclass of \( \mathcal{M} \) closed under the formation of direct images. Then the following hold:

(a) Suppose that \( A \in S(X)^{\text{op}} \) is a right fixed point of the Galois connection \( (\Delta_\mathcal{N}, \nabla_\mathcal{N}) \). Then, \( A \) is also a right fixed point of the Galois connection \( (D_\mathcal{N}, T_\mathcal{N}) \).

(b) Suppose that \( B \in S(X) \) is a left fixed point of the Galois connection \( (\Delta_\mathcal{N}, \nabla_\mathcal{N}) \). Then, \( B \) is also a left fixed point of the Galois connection \( (J_\mathcal{N}, I_\mathcal{N}) \).

(c) If \( C \in CL(X, \mathcal{M}) \) satisfies \( C = T_\mathcal{N}(A) = J_\mathcal{N}(B) \), for some \( A \in S(X)^{\text{op}} \) that is a right fixed point of the Galois connection \( (D_\mathcal{N}, T_\mathcal{N}) \) and for some \( B \in S(X) \) that is a left fixed point of the Galois connection \( (J_\mathcal{N}, I_\mathcal{N}) \), then \( A \) and \( B \) are corresponding fixed points of the Galois connection \( (\Delta_\mathcal{N}, \nabla_\mathcal{N}) \).

Proof:

(a). Suppose that \( A \in S(X)^{\text{op}} \) is a right fixed point of the Galois connection \( (\Delta_\mathcal{N}, \nabla_\mathcal{N}) \). In [C2], Proposition 2.10, we proved that under the assumption of \( \mathcal{N} \) being closed under direct images, we have that for every \( C \in CL(X, \mathcal{M}) \), \( D_\mathcal{N}(C) \subseteq (\Delta_\mathcal{N} \circ I_\mathcal{N})(C) \). So, we have that \( A \subseteq D_\mathcal{N}(T_\mathcal{N}(A)) \subseteq (\Delta_\mathcal{N} \circ I_\mathcal{N})(T_\mathcal{N}(A)) = (\Delta_\mathcal{N} \circ \nabla_\mathcal{N})(A) = A \). Thus, \( A = (D_\mathcal{N} \circ T_\mathcal{N})(A) \).

(b). Similarly to (a).

c). Under our hypotheses we have that \( C = T_\mathcal{N}(A) = J_\mathcal{N}(B) \). Then we have that \( \Delta_\mathcal{N}(B) = D_\mathcal{N}(J_\mathcal{N}(B)) = D_\mathcal{N}(T_\mathcal{N}(A)) = A \). Analogously we obtain that \( \nabla_\mathcal{N}(A) = B \). \( \square \)

Proposition 2.16

Let \( \mathcal{N} \) be a subclass of \( \mathcal{M} \) closed under composition and let \( \mathcal{M}' \subseteq \mathcal{M} \). If \( C \) is a closure operator that is hereditary with respect to \( \mathcal{M}' \) and \( \mathcal{M}' \) satisfies the condition that for every \( n \in \mathcal{N} \) and \( m' \in \mathcal{M}' \), the morphism \( m' \circ n \) belongs to \( \mathcal{N} \), then \( D_\mathcal{N}(C) \) and \( I_\mathcal{N}(C) \) are both closed under the formation of \( \mathcal{M}' \)-subobjects.
Proof:
Let $X \in D_N(C)$ and let $N \rightarrow X$ be an $\mathcal{M}'$-subobject of $X$. If $M \rightarrow N$ is an $\mathcal{N}$-subobject of $N$ then, by our assumption on $\mathcal{M}'$, we have that $m = n \circ t \in \mathcal{N}$. Let us consider the following commutative diagram

$$
\begin{array}{c}
M^C \\
m_N^C \\
M \\
t_C \\
t_N^C \\
N \\
n \\
m \\
m_C \\
X \\
m_C^C \\
X
\end{array}
$$

The fact that $C$ is hereditary with respect to $\mathcal{M}'$ implies that $t^C$ is the pullback of $m_C^C$ along $n$. However, since $X \in D_N(C)$, we have that $m_C$ is an isomorphism, i.e., $m$ is $C$-closed. Now, since $n$ is a monomorphism, $t$ is the pullback of $m$ along $n$ and so $t$ is also $C$-closed. Thus $N \in D_N(C)$.

Let $X \in I_N(C)$, and let $N \rightarrow X$ be an $\mathcal{M}'$-subobject of $X$. If $M \rightarrow N$ is an $\mathcal{N}$-subobject of $N$ then, by our assumption on $\mathcal{M}'$, we have that $m = n \circ t \in \mathcal{N}$. Let us consider again the commutative diagram in the proof of part (a). Since $X \in I_N(C)$, we have that $m_C$ is an isomorphism. Again, the hereditary property of $C$ with respect to $\mathcal{M}'$ implies that $t^C$ is the pullback of $m^C$ along $n$. Since $t^C$, as a pullback of an isomorphism, is an isomorphism, we can conclude that $N \in I_N(C)$.

Next we show that under some additional assumptions on the subclass $\mathcal{A}$, the descriptions of the closure operators $T_N(A)$ and $J_N(A)$ given in Propositions 2.5 and 2.6 can be further simplified. First we need the following

**Lemma 2.17**

Let $N \subseteq \mathcal{M}$ and let $\mathcal{A} \subseteq X$. Denote by $N^{pb}_A$ the union of $N$ with all inverse images (pullbacks) of elements of $N$ along all $X$-morphisms having codomain in $\mathcal{A}$. Then, we have that $T_N(A) \simeq T_{N^{pb}_A}(A)$.

Proof:
Clearly, since $N \subseteq N^{pb}_A$, from Proposition 2.5 we have that $T_{N^{pb}_A}(A) \subseteq T_N(A)$. On the other hand, since pullbacks of $C$-closed subobjects are $C$-closed, any closure operator $C$ that satisfies $D_N(C) \supseteq \mathcal{A}$ also satisfies $D_{N^{pb}_A}(C) \supseteq \mathcal{A}$. Thus, $T_N(A) \subseteq T_{N^{pb}_A}(A)$. This, together with the previous inequality yields that $T_N(A) \simeq T_{N^{pb}_A}(A)$. 

12
PROPOSITION 2.18

Let $\mathcal{A}$ be a full, reflective subcategory of $\mathcal{X}$ and for $X \in \mathcal{X}$, let $X \xrightarrow{r_X} rX$ denote the reflection morphism. If $\mathcal{N}$ is closed under the formation of pullbacks along $\mathcal{X}$-morphisms with codomain in $\mathcal{A}$ then, for every $\mathcal{M}$-subobject $M \xrightarrow{m} X$, we have that

$$m_{\mathcal{N}(A)} \simeq \bigcap \{ r_X^{-1}(n) : N \xrightarrow{n} rX \in \mathcal{N} \text{ and } m \leq r_X^{-1}(n) \}.$$ 

Proof:

First of all, it is straightforward to notice that $m_{\mathcal{N}(A)} \leq \bigcap \{ r_X^{-1}(n) : N \xrightarrow{n} rX \in \mathcal{N} \text{ and } m \leq r_X^{-1}(n) \}$ (cf. Proposition 2.5). Now, let $X \xrightarrow{f} Y$ be an $\mathcal{X}$-morphism with $Y \in \mathcal{A}$ and let $N \xrightarrow{n} Y$ be an $\mathcal{N}$-subobject of $Y$. Since $\mathcal{A}$ is reflective in $\mathcal{X}$, there exists a unique $\mathcal{X}$-morphism $rX \xrightarrow{g} Y$ such that $g \circ r_X = f$. Clearly, if the $\mathcal{M}$-subobject $M \xrightarrow{m} X$ satisfies $m \leq f^{-1}(n)$, then we have that $m \leq f^{-1}(n) \simeq (g \circ r_X)^{-1}(n) \simeq r_X^{-1}(g^{-1}(n))$. From the hypothesis, $g^{-1}(n) \in \mathcal{N}$. Thus we have that $m_{\mathcal{N}(A)} \geq \bigcap \{ r_X^{-1}(n) : N \xrightarrow{n} rX \in \mathcal{N} \text{ and } m \leq r_X^{-1}(n) \}$. This, together with the other inequality proves the result.

COROLLARY 2.19

Let $\mathcal{A}$ be a full, reflective subcategory of $\mathcal{X}$ and for $X \in \mathcal{X}$, let $X \xrightarrow{r_X} rX$ denote the reflection morphism. Then, for every $\mathcal{N} \subseteq \mathcal{M}$ and for every $\mathcal{M}$-subobject $M \xrightarrow{m} X$, we have that

$$m_{\mathcal{N}(A)} \simeq \bigcap \{ r_X^{-1}(n) : N \xrightarrow{n} rX \in \mathcal{N} \text{ and } m \leq r_X^{-1}(n) \}.$$ 

Proof:

From Lemma 2.17 we can replace $\mathcal{N}$ by $\mathcal{N}^b_{\mathcal{A}}$ and apply the previous proposition.

PROPOSITION 2.20

Let $\mathcal{A}$ be a full, coreflective subcategory of $\mathcal{X}$ and for $Y \in \mathcal{X}$, let $cY \xrightarrow{cY} Y$ denote the coreflection morphism. If $\mathcal{N}$ is closed under the formation of direct images then, for every $\mathcal{M}$-subobject $M \xrightarrow{m} Y$, we have that

$$m_{\mathcal{N}(A)} \simeq \text{sup} \left( \{ m \} \cup \{ (id_{cY})_{cY} : \exists n \in \mathcal{N}_{cY} \text{ with } n_{cY} \leq m \} \right).$$

Proof:

Clearly we have that $m_{\mathcal{N}(A)} \geq \text{sup} \left( \{ m \} \cup \{ (id_{cY})_{cY} : \exists n \in \mathcal{N}_{cY} \text{ with } n_{cY} \leq m \} \right)$ (cf. Proposition 2.6).

Now, let $M \xrightarrow{m} Y$ be an $\mathcal{M}$-subobject and let $X \xrightarrow{f} Y$ be such that $X \in \mathcal{A}$ and there exists an $\mathcal{N}$-morphism $N \xrightarrow{n} X$ with $n_f \leq m$. Since $\mathcal{A}$ is a coreflective subcategory of $\mathcal{X}$, there exists a unique morphism $X \xrightarrow{g} cY$ such that $cY \circ g = f$. Let $(e_1, m_1)$ and $(e_2, m_2)$
be the \((E, M)\)-factorizations of \(f\) and \(c_Y\), respectively. Clearly, \(m_2 \circ e_2 \circ g = m_1 \circ e_1\), from the \((E, M)\)-diagonalization property, we obtain a monomorphism \((id_X)_f \xrightarrow{d} (id_{c_Y})_c\), such that \(m_2 \circ d = m_1\). Again, using the \((E, M)\)-diagonalization property, from the following commutative diagram

we obtain that \((n_g)_{c_Y} \leq m\). Notice that since \(N\) is closed under direct images, we have that \(n_g \in N\). Therefore every \((id_X)_f\) that occurs in the formation of \(m^{J_N(A)}\) is dominated by some \((id_{c_Y})_c\) that satisfies the condition that there exists a morphism \(n \in N_{c_Y}\) such that \(n_{c_Y} \leq m\). Thus, we have that \(m^{J_N(A)} \leq \sup \{\{m\} \cup \{(id_{c_Y})_c : \exists n \in N_{c_Y} \text{ with } n_{c_Y} \leq m\}\}\) This, together with the previous inequality, gives the wanted result.

Now, following the ideas presented in [CKS2], we would like to show how the two Galois connections \(S(\mathcal{X}) \xrightarrow{H_N} CL(\mathcal{X}, M)\) and \(CL(\mathcal{X}, M) \xrightarrow{K_N} S(\mathcal{X})^{op}\) can each be factored via three other Galois connections.

Let \(S(M)\) denote the collection of all subclasses of \(M\), ordered by inclusion. We begin with the following result.

**PROPOSITION 2.21**

Let \(N \subseteq M\) and let \(S(\mathcal{X}) \xrightarrow{H_N} S(M)\) and \(S(M) \xrightarrow{K_N} S(\mathcal{X})^{op}\) be defined by:

\[
H_N(A) = \{ n \in N_X : X \in A \}
\]

\[
K_N(M') = \{ X \in \mathcal{X} : n \in N_X \Rightarrow n \in M' \}.
\]

Then \(S(\mathcal{X}) \xrightarrow{H_N} S(M)\) is a Galois connection.

**Proof:**

Clearly both \(H_N\) and \(K_N\) are order-preserving.

Now, if \(X \in A\), then every \(n \in N_X\) also belongs to \(H_N(A)\). Consequently we have that \(X \in (K_N \circ H_N)(A)\).
On the other hand, if \( n \in (H_N \circ K_N)(\mathcal{M}') \), then \( n \in \mathcal{N}_X \) with \( X \in K_N(\mathcal{M}') \). This implies that \( n \in \mathcal{M}' \).

As a consequence we obtain that \( S(\mathcal{M})^\text{op} \xrightarrow{K_N^\text{op}} H_N^\text{op} S(\mathcal{X})^\text{op} \) is also a Galois connection.

We recall the following definition and the next three results from [CKS2].

**DEFINITION 2.22**

(1) A subclass \( \mathcal{N} \) of \( \mathcal{M} \) is called \( \mathbf{E} \)-sink stable, if for every commutative square

\[
\begin{array}{ccc}
M & \xrightarrow{f} & N \\
\downarrow m & & \downarrow n \\
X & \xrightarrow{g} & Y
\end{array}
\]

with \( n \in \mathcal{M} \) and the 2-sink \((g, n) \in \mathbf{E}\) we have that \( m \in \mathcal{N} \) implies \( n \in \mathcal{N} \).

(2) \( P_{es}(\mathcal{M}) \) denotes the collection of all \( \mathbf{E} \)-sink stable subclasses of \( \mathcal{M} \), ordered by inclusion.

(3) \( P_{pb}(\mathcal{M}) \) denotes the collection of all pullback-stable subclasses of \( \mathcal{M} \), ordered by inclusion.

**THEOREM 2.23** [CKS2, Theorem 2.3]

(1) Let \( \mathcal{N} \in P_{pb}(\mathcal{M}) \). If for every \( \mathcal{M} \)-subobject \( M \xrightarrow{m} X \), we define:

\[
m^{S_N} = \inf \{ m' \in \mathcal{N} : M' \xrightarrow{m'} X \text{ and } m \leq m' \}
\]

then \( S_{\mathcal{N}} \) is an idempotent closure operator with respect to \( \mathcal{M} \).

(2) Let \( \mathcal{N} \in P_{es}(\mathcal{M}) \). If for every \( \mathcal{M} \)-subobject \( M \xrightarrow{m} X \), we define:

\[
m^{C_N} = \sup \{ (N \xrightarrow{n} X) : \exists (M \xrightarrow{t} N) \in \mathcal{N} \text{ with } n \circ t = m \}
\]

then \( C_{\mathcal{N}} \) is a weakly hereditary closure operator with respect to \( \mathcal{M} \).

**THEOREM 2.24** [cf. CKS2, Theorem 2.4]

(1) Let \( CL(\mathcal{X}, \mathcal{M}) \xrightarrow{R_*} P_{pb}(\mathcal{M})^\text{op} \) and \( P_{pb}(\mathcal{M})^\text{op} \xrightarrow{R_*^*} CL(\mathcal{X}, \mathcal{M}) \) be defined by:

\[
R_*(C) = \{ m \in \mathcal{M} : m \text{ is } C\text{-closed} \}
\]

\[
R^*(\mathcal{N}) = P_{N}. 
\]

Then, \( CL(\mathcal{X}, \mathcal{M}) \xrightarrow{R_*} P_{pb}(\mathcal{M})^\text{op} \) is a Galois connection;

(2) Let \( CL(\mathcal{X}, \mathcal{M}) \xrightarrow{K_*} P_{es}(\mathcal{M}) \) and \( P_{es}(\mathcal{M}) \xrightarrow{K_*} CL(\mathcal{X}, \mathcal{M}) \) be defined by:

\[
K^*(C) = \{ m \in \mathcal{M} : m \text{ is } C\text{-dense} \}
\]
\[ K_*(N) = C_N. \]
Then, \( P_{cs}(M) \xrightarrow{K_*} CL(X, M) \) is a Galois connection. \[ \square \]

**PROPOSITION 2.25** [CKS2, Proposition 2.7]

(1) Let \( P_{ph}(M)^{op} \xrightarrow{Q_*} S(M)^{op} \) and \( S(M)^{op} \xrightarrow{Q^*} P_{ph}(M)^{op} \) be defined by:

\[ Q_*(M') = M' \]

\[ Q^*(M'') = \{ m \in M : m \text{ is a pullback of some } n \in M'' \}. \]

Then, \( P_{ph}(M)^{op} \xrightarrow{Q_*} S(M)^{op} \) is a Galois connection.

(2) Let \( S(M) \xrightarrow{L_*} P_{cs}(M) \) and \( P_{cs}(M) \xrightarrow{L^*} S(M) \) be defined by:

\[ L_* (M') = \{ n \in M : n \circ f = g \circ m' \text{ for some } m' \in M' \text{ and some } \mathcal{X} \text{-morphisms } f \text{ and } g \text{ with } (g, n) \in E \} \]

\[ L^*(M'') = M''. \]

Then, \( S(M) \xrightarrow{L_*} P_{cs}(M) \) is a Galois connection. \[ \square \]

Now we can prove the following

**THEOREM 2.26**

For every \( N \subseteq M \), we have the following two commutative diagrams of Galois connections

\[
\begin{array}{c c c c c c}
S(\mathcal{X}) & L_\mathcal{X} & CL(\mathcal{X}, M) & CL(\mathcal{X}, M) & D_\mathcal{X} & S(\mathcal{X})^{op} \\
\Downarrow K_N & & \Downarrow K_* & & \Downarrow T_N & \\
H_N & K_* & K^* & R_* & K^{op}_N \& H^{op}_N \\
\hline
S(M) & L_* & P_{cs}(M) & P_{ph}(M)^{op} & Q_* & S(M)^{op} \\
\end{array}
\]

**Proof:**

Let us start with the left diagram. Given \( C \in CL(\mathcal{X}, M) \), \( K^*(C) = \{ m \in M : m \text{ is } C\text{-dense} \} \) and so \( L^*(K^*(C)) = K^*(C) \). Thus, \( K_N(L^*(K^*(C))) = K_N(K^*(C)) = \{ X \in \mathcal{X} : n \in N_X \Rightarrow n \text{ is } C\text{-dense} \} = I_N(C) \).

Now let \( \mathcal{B} \in S(\mathcal{X}) \). Then, \( H_N(\mathcal{B}) = \{ n \in N_X : X \in \mathcal{B} \} \) and therefore we have that \( L_*(H_N(\mathcal{B})) = \{ t \in M : t \circ f = g \circ m' \text{ for some } m' \in N_X, X \in \mathcal{B} \text{ and } (g, t) \in E \} \). Now, notice that \( I_N(K_*(L_*(H_N(\mathcal{B})))) = (K_N \circ L^* \circ K^*)(K_*(L_* \circ H_N)(\mathcal{B})) \supseteq \mathcal{B} \). Thus, from Proposition 2.4 we have that \( J_N(\mathcal{B}) \subseteq K_*(L_*(H_N(\mathcal{B}))). \)

To show that \( K_*(L_*(H_N(\mathcal{B}))) \subseteq J_N(\mathcal{B}) \), consider the \( \mathcal{M} \)-subobject \( M \overset{m}{\longrightarrow} Y \). Let \( m' \in M_Y \) be such that there exists \( t \in L_*(H_N(\mathcal{B})) \) with \( m = m' \circ t \). Therefore we obtain the following
where \( n \in \mathcal{N} \), \((g, t) \in \mathcal{E}, X \in \mathcal{B}\) and \( n_f\) is the direct image of \( n \) along the morphism \( f = m' \circ g \).

Now, since \( m \circ h = n_f \circ e_f \), the \((\mathcal{E}, \mathcal{M})\)-diagonalization property yields the existence of a morphism \( N_f \xrightarrow{d'} M \) such that \( m \circ d = n_f \) and \( d \circ e_f = h \). This implies that the morphism \( X_f \xrightarrow{(id_X)_f} Y \) occurs in the construction of \( m^{J_N(\mathcal{B})} \) (cf. Proposition 2.6). Let \( X_f \vee M \xrightarrow{\gamma} Y \) be the supremum of \((id_X)_f \) and \( m \), and let \( i_M \) and \( i_X \) be the morphisms with codomain \( X_f \vee M \) induced by the supremum construction. Notice that \( \gamma \leq m^{J_N(\mathcal{B})} \). From the following commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{m' \circ g} & Y \\
\downarrow g & & \downarrow \gamma \\
M' & \xrightarrow{id_M} & M' \\
\end{array}
\]

it is easily seen that \( m' \circ t = \gamma \circ i_M \) and \( m' \circ g = (id_X)_f \circ e = \gamma \circ i_X \circ e \). Again the \((\mathcal{E}, \mathcal{M})\)-diagonalization property yields a morphism \( d' \) that makes the following diagram commute

\[
\begin{array}{ccc}
X_f \vee M & \xrightarrow{(i_X \circ e, i_M)} & M' \\
\downarrow (g, t) & & \downarrow m' \\
\end{array}
\]

Notice that the bullet in the above diagram represents the objects \( X \) and \( M \). Thus, \( m' \leq \gamma \leq m^{J_N(\mathcal{B})} \) and so \( K_*(l_*(H_N(\mathcal{B}))) \subseteq J_N(\mathcal{B}) \).
Now let us prove the commutativity of the diagram on the right. If \( C \in CL(\mathcal{X}, \mathcal{M}) \), then \( R_*(C) = \{ m \in \mathcal{M} : m \text{ is } C\text{-closed} \} \). From the properties of closure operators, \( R_*(C) \) is pullback-stable. Consequently, \( Q_*(R_*(C)) = R_*(C) \) and \( K_\mathcal{N}(Q_*(R_*(C))) = \{ X \in \mathcal{X} : n \in \mathcal{N}_X \Rightarrow n \in R_*(C) \} = \{ X \in \mathcal{X} : n \in \mathcal{N}_X \Rightarrow n \text{ is } C\text{-closed} \} = D_\mathcal{N}(C) \).

Now let \( \mathcal{A} \in S(\mathcal{X})^{op} \). \( H_\mathcal{N}^{op}(\mathcal{A}) = \{ n \in \mathcal{N}_Y : Y \in \mathcal{A} \} \). \( Q_*(H_\mathcal{N}^{op}(\mathcal{A})) = \{ m \in \mathcal{M}_X : m \text{ is a pullback of some } n \in H_\mathcal{N}^{op}(\mathcal{A}) \text{ along some } \mathcal{X}\text{-morphism } X \xrightarrow{f} Y \} = \{ m \in \mathcal{M}_X : m \text{ is a pullback of some } n \in \mathcal{N}_Y \text{ along some } \mathcal{X}\text{-morphism } X \xrightarrow{f} Y, Y \in \mathcal{A} \} \). Consequently, given the \( \mathcal{M}\)-subobject \( M \xrightarrow{m} X \), we have that \( m^{\mathcal{R}(Q_*(H_\mathcal{N}^{op}(\mathcal{A})))} = \inf \{ m' \in Q_*(H_\mathcal{N}^{op}(\mathcal{A})) : m \leq m' \} = \inf \{ m' \in \mathcal{M}_X : m \leq m' \text{ and } m' = f^{-1}(n), n \in \mathcal{N}_Y, X \xrightarrow{f} Y, Y \in \mathcal{A} \} = m^{T_\mathcal{N}(\mathcal{A})} \). This concludes the proof.

\[ \square \]

**PROPOSITION 2.27**

(a) Let \( \mathcal{X} \) have squares and equalizers and let \( \mathcal{M} \) contain all regular subobjects. Assume that \( \mathcal{N} \) is the class of diagonal morphisms, i.e., morphisms of the form \( Y \xrightarrow{\delta_Y} Y \times Y \), with \( Y \in \mathcal{X} \), where \( \delta_Y \) is the equalizer of the projections of the square \( Y \times Y \) into \( Y \). Then for every subcategory \( \mathcal{A} \) of \( \mathcal{X} \) that is closed under squares and \( \mathcal{M}\)-subobjects, \( T_\mathcal{N}(\mathcal{A}) \) agrees with \( S_\mathcal{A} \), that is the Sambany closure induced by \( \mathcal{A} \).

(b) Let \( \mathcal{X} \) have squares and equalizers and let \( \mathcal{M} \) contain all regular subobjects. If \( \mathcal{A} \) is closed under squares and \( \mathcal{N} \) consists of all \( \mathcal{A}\)-regular subobjects, then \( T_\mathcal{N}(\mathcal{A}) \) agrees with \( S_\mathcal{A} \).

**Proof:**

(a). Let \( M \xrightarrow{m} X \) be an \( \mathcal{M}\)-subobject and let \( E \xrightarrow{e} X \) be the equalizer of a pair of morphisms \( f, g \) with codomain \( Y \in \mathcal{A} \) such that \( f \circ m = g \circ m \). It is easy to see that \( e \) is the pullback of the diagonal morphism \( Y \xrightarrow{\delta_Y} Y \times Y \) along the morphism \( X \xleftarrow{<f,g>} Y \times Y \). Since \( m \leq e \), \( \mathcal{N} \) contains all diagonal morphisms and \( Y \times Y \in \mathcal{A} \), we have that \( e \) occurs in the construction of \( T_\mathcal{N}(\mathcal{A}) \). Therefore, \( m^{T_\mathcal{N}(\mathcal{A})} \leq m^{S_\mathcal{A}} \).

Now, let \( X \xrightarrow{f} Y \times Y \) be an \( \mathcal{X}\)-morphism with \( Y \times Y \in \mathcal{A} \). Notice that, since \( \mathcal{M} \) contains all regular subobjects and \( \mathcal{A} \) is closed under \( \mathcal{M}\)-subobjects, we have that \( Y \in \mathcal{A} \). If \( m \leq f^{-1}(\delta_Y) \), then since \( f^{-1}(\delta_Y) = eqv(\pi_1 \circ f, \pi_2 \circ f) \) and \( Y \in \mathcal{A} \), we have that \( f^{-1}(\delta_Y) \) occurs in the construction of \( S_\mathcal{A} \). Thus, we have that \( m^{S_\mathcal{A}} \leq m^{T_\mathcal{N}(\mathcal{A})} \) and therefore we can conclude that \( T_\mathcal{N}(\mathcal{A}) \simeq S_\mathcal{A} \).

(b). Let \( M \xrightarrow{m} X \) be an \( \mathcal{M}\)-subobject, let \( X \xrightarrow{f} Y \) be an \( \mathcal{X}\)-morphism with \( Y \in \mathcal{A} \) and let \( n \in \mathcal{N}_Y \). Since pullbacks of equalizers are equalizers, we have that if \( m \leq f^{-1}(n) \) then \( f^{-1}(n) \) is the equalizer of a pair of morphisms with codomain in \( \mathcal{A} \) that agree on \( m \). Thus, \( f^{-1}(n) \) occurs in the construction of \( S_\mathcal{A} \). This implies that \( m^{S_\mathcal{A}} \leq m^{T_\mathcal{N}(\mathcal{A})} \).

On the other hand, since \( \delta_Y \in \mathcal{N} \), exactly as in part a), we can show that \( m^{T_\mathcal{N}(\mathcal{A})} \leq m^{S_\mathcal{A}} \). Consequently \( S_\mathcal{A} \simeq T_\mathcal{N}(\mathcal{A}) \).
\section{Examples}

**Example 3.1**

Let $\mathcal{N}$ be the class of all $\mathcal{X}$-isomorphisms.

For every $\mathcal{A} \in S(\mathcal{X})^{\text{op}}$, it follows from Propositions 2.3 and 2.4 that $T_\mathcal{N}(\mathcal{A})$ is the indiscrete closure operator and $(I_\mathcal{N} \circ T_\mathcal{N})(\mathcal{A}) = \mathcal{X}$. Moreover, for every $\mathcal{B} \in S(\mathcal{X})$, $J_\mathcal{N}(\mathcal{B})$ is the discrete closure operator and clearly $(D_\mathcal{N} \circ J_\mathcal{N})(\mathcal{B}) = \mathcal{X}$.

Notice that, although $\mathcal{N}$ does not satisfy the hypotheses of Theorem 2.7, we still have that $\nabla_\mathcal{N} = I_\mathcal{N} \circ T_\mathcal{N}$ and $\Delta_\mathcal{N} = D_\mathcal{N} \circ J_\mathcal{N}$. This is due to the fact that in this case every $\mathcal{X}$-morphism is $\mathcal{N}$-constant.

In what follows, for the category $\text{Top}$ of topological spaces we will choose as $\mathcal{M}$ the class of all extremal monomorphisms (embeddings). We recall that if $E$ is the class of episinks in $\text{Top}$, then $\text{Top}$ is an $(E, \mathcal{M})$-category. For the category $\text{Grp}$ of groups and $\text{Ab}$ of abelian groups we will use the (episink,monomorphism)-factorization structure.

**Example 3.2** (cf. [CH])

Let $\mathcal{X}$ be the category $\text{Top}$ and let $\mathcal{N}$ be the class of all extremal monomorphisms with nonempty domain. Notice that since $\mathcal{N}$ contains all singleton monomorphisms (i.e., morphisms with singleton domain), to say that a morphism $X \xrightarrow{f} Y$ is $\mathcal{N}$-constant simply means that $f(X)$ is a singleton.

(a). If $C$ is the closure operator induced by the topology, then the class $D_\mathcal{N}(C)$ agrees with the class $\text{Discr}$ of discrete topological spaces and $\nabla_\mathcal{N}(\text{Discr})$ consists of the classical connected topological spaces.

If $M \xrightarrow{m} X$ be an $\mathcal{M}$-subobject of $X \in \text{Top}$, then $M^{T_\mathcal{N}(\text{Discr})}$ equals the intersection of all clopen subsets of $X$ containing $M$. Since $\mathcal{M}$ satisfies the conditions of Theorem 2.7, we have that the class $(I_\mathcal{N} \circ T_\mathcal{N})(\text{Discr})$ consists of all connected topological spaces.

Now, let $\mathcal{B}$ be the class of all connected topological spaces. From Proposition 2.6, $M^{J_\mathcal{N}(\text{Discr})}$ is the union of $M$ with all connected subsets of $X$ which intersect $M$ and the subcategory of all totally disconnected topological spaces agrees with $(D_\mathcal{N} \circ J_\mathcal{N})(\mathcal{B})$. Thus, again from Theorem 2.7, connected topological spaces and totally disconnected topological spaces are fixed points of the Galois connection $(\Delta_\mathcal{N}, \nabla_\mathcal{N})$ of Proposition 2.2.

(b). Let $\mathcal{A} = \text{Top}_0 \in S(\mathcal{X})^{\text{op}}$. $\text{Ind}$ and $\text{Top}_0$ are corresponding fixed points of the Galois connection $(\Delta_\mathcal{N}, \nabla_\mathcal{N})$ of Proposition 2.2 (cf. [AW]).

Let $M \xrightarrow{m} X$ be an $\mathcal{M}$-subobject of $X \in \text{Top}$ and let $c(M) = \{y \in X : \exists x \in M \text{ with } \bar{\{x\}} = \{y\}\}$. where, $\bar{\{x\}}$ denotes the usual topological closure of $\{x\}$. If $X \xrightarrow{r_0} r_0X$ is the $\text{Top}_0$-
reflection, then \( M^{T_X(\text{Top}_0)} = c(M) = r_0^{-1} r_0(M) \). Moreover, \( M^{T_X(\text{Top}_0)} \subseteq b(M) \), where \( b(M) \) is the \( b \)-closure of \( M \) (cf. \([B], [NW]\)). \( M^{J_N(\text{Ind})} \) is the union of \( M \) with all indiscrete subobjects of \( X \) which intersect \( M \) and \( (D_N \circ J_N)(\text{Ind}) = \text{Top}_0. \)

(c). Let \( A = \text{Top}_1 \in S(\mathcal{X})^{\text{op}} \) and let \( B \) be the class of all absolutely connected topological spaces, i.e., \( B = \{X \in \text{Top} \mid X \text{ cannot be decomposed into any disjoint family } \mathcal{L} \text{ of nonempty closed subsets with } |\mathcal{L}| > 1\} \) (cf. \([P_1]\)). \( A \) and \( B \) are corresponding fixed points of the Galois connection \((\Delta_N, \nabla_N)\) of Proposition 2.2. Let \( M \xrightarrow{m} X \) be an \( M \)-subobject of \( X \in \text{Top}. \)

We have that \( M^{T_{\text{Top}_1}} = M^{T_X(\text{Top}_0)} \) ([CH, Example 4.3]), i.e., the \( T_N(\text{Top}_1) \)-closure agrees with the Saldany closure induced by \( \text{Top}_1 \). So, from Theorem 2.7, we have that \( B = I_N(S_{\text{Top}_1}). \)

From Proposition 2.6 one can see that \( M^{J_N(B)} \) is the union of \( M \) with all absolutely connected subsets of \( X \) that intersect \( M \). Theorem 2.7 implies that \( \text{Top}_1 = (D_N \circ J_N)(B). \) This can be also easily verified directly.

We observe that, from Proposition 2.15, \( \text{Top}_0 \), \( \text{Top}_1 \) and all totally disconnected topological spaces are fixed points of the Galois connection \((D_N, T_N)\). Moreover, \( \text{Ind} \), connected topological spaces and absolutely connected topological spaces are fixed points of the Galois connection \((J_N, I_N)\).

**EXAMPLE 3.3**

Let \( \mathcal{X} \) be the category \( \text{Top} \) and let \( \mathcal{N} \) be the the class of all singleton monomorphisms.

Let \( A = \text{Top}_1 \in S(\mathcal{X})^{\text{op}} \) and let \( B \) be the class of all absolutely connected topological spaces, i.e., \( B = \{X \in \text{Top} \mid X \text{ cannot be decomposed into any disjoint family } \mathcal{L} \text{ of nonempty closed subsets with } |\mathcal{L}| > 1\} \). Since \( \mathcal{N} \)-constant in this case simply means constant, \( A \) and \( B \) are corresponding fixed points of the Galois connection \((\Delta_N, \nabla_N)\) of Proposition 2.2 (cf. \([P_1]\)). Let \( X \in \text{Top}_1 \) and let \( M \xrightarrow{m} X \) be an \( M \)-subobject. First notice that \( M^{T_X(A)} \) is closed in the usual topology of \( X \). Now let \( C \) be a closed subspace of \( X \) and let \( X \) denote the topological spaces with underlying set \(|X|\) endowed with the cofinite topology. Choose \( x \in C \) and define \( X \xrightarrow{f} C \) by \( f(C) = \{x\} \subseteq C \) and \( f|_{X-C} = id_X|_{X-C}. \) Clearly \( f \) is continuous and \( C = f^{-1}(C). \) Therefore, \( M^{T_X(A)} \) is the intersection of all closed sets containing \( M \), i.e., it is the closure of \( M \) in the topology of \( X \). However, notice that if \( X \not\in \text{Top}_1 \), then the \( T_N(A) \)-closure might be larger than the topological closure of \( X \). For example if \( X = \{0, 1\} \) with \( \{0\} \) open and \( \{1\} \) closed (Sierpinski space) and \( M = \{1\} \) then, clearly \( M^{T_X(A)} = X. \)

Since \( \mathcal{N} \) satisfies the conditions of Theorem 2.2, we obtain that \( B = (I_N \circ T_N)(\text{Top}_1). \) As in Example 3.2(c), \( M^{J_N(B)} \) is the union of \( M \) with all absolutely connected subsets of \( X \) that intersect \( M \).
EXAMPLE 3.4

Let $\mathcal{X}$ be the category $\text{Top}$ of topological spaces and let $\mathcal{N}$ be the class of all diagonal morphisms. If $\mathcal{A}$ is any epireflective subcategory of $\text{Top}$, then it satisfies the hypotheses of Proposition 2.27(a) and so $T_{\mathcal{N}}(\mathcal{A}) = S_{\mathcal{A}}$. Therefore if $\mathcal{A} = \text{Top}_0$, then $T_{\mathcal{N}}(\mathcal{A})$ is the b-closure ([B], [NW]). If $\mathcal{A} = \text{Top}_1$ or any bireflective subcategory of $\text{Top}$, then $T_{\mathcal{N}}(\mathcal{A})$ is discrete inside $\mathcal{A}$ ([G]). If $\mathcal{A} = \text{Top}_2$, then $T_{\mathcal{N}}(\mathcal{A})$ agrees with the usual topological closure inside $\text{Top}_2$.

EXAMPLE 3.5

Let $\mathcal{X}$ be the category $\text{Grp}$ and let $\mathcal{N} = \mathcal{M}$ be the class of all monomorphisms in $\text{Grp}$. Clearly, to say that a $\text{Grp}$-morphism $X \xrightarrow{f} Y$ is $\mathcal{N}$-constant simply means that the image of $X$ under $f$ is a singleton.

(a). If $\mathcal{A} = \text{Grp} \in S(\text{Grp})^{\text{op}}$, the category $\text{Sng}$ of singleton groups and $\text{Grp}$ are corresponding fixed points of the Galois connection $(\Delta_{\mathcal{N}}, \nabla_{\mathcal{N}})$. Now, from Proposition 2.27 b) $T_{\mathcal{N}}(\text{Grp}) = S_{\text{Grp}}$, which is the discrete operator. Actually, in this case, $J_{\mathcal{N}}(\text{Sng}) = T_{\mathcal{N}}(\text{Grp})$. On the other side, if we take $\mathcal{A} = \text{Grp} \in S(\text{Grp})$, then again $\text{Grp}$ and $\text{Sng}$ are corresponding fixed points of the same Galois connection. Moreover, $J_{\mathcal{N}}(\text{Grp})$ and $T_{\mathcal{N}}(\text{Sng})$ both agree with the indiscrete operator.

(b). Let $\mathcal{A}$ be the subcategory $\text{Ab}$ of abelian groups. We have that $S_{\text{Ab}} \simeq T_{\mathcal{N}}(\text{Ab})$ (cf. [CH, Example 4.4]). $\mathcal{N}$ satisfies the hypotheses of Theorem 2.7 and consequently, $\nabla_{\mathcal{N}}(\text{Ab})$ agrees with $I_{\mathcal{N}}(S_{\text{Ab}})$ which is equal to the class of perfect groups, i.e., $X \in \nabla_{\mathcal{N}}(\text{Ab})$ iff $X = X'$, where $X'$ denotes the subgroup generated by the commutators. Thus $M^{J_{\mathcal{N}}(\nabla(\text{Ab}))}$ is the subgroup generated by $M$ and all perfect subgroups of $X$. and $(D_{\mathcal{N}} \circ J_{\mathcal{N}})(\nabla_{\mathcal{N}}(\text{Ab}))$ is the class of all groups which do not have any non-trivial perfect subgroup.

EXAMPLE 3.6

Let $\mathcal{X} = \text{Grp}$, let $\mathcal{N}$ be the class of all singleton monomorphisms. Clearly in this case $\mathcal{N}$-constant simply means constant.

(a). $(\text{Sng}, \text{Grp})$ is a pair of corresponding fixed points of $(\Delta_{\mathcal{N}}, \nabla_{\mathcal{N}})$. It is easy to see that $T_{\mathcal{N}}(\text{Grp})$ is the normal closure and $J_{\mathcal{N}}(\text{Sng})$ is the discrete operator. On the other side, if we consider the pair of corresponding fixed points $(\text{Grp}, \text{Sng})$, then both $J_{\mathcal{N}}(\text{Grp})$ and $T_{\mathcal{N}}(\text{Sng})$ agree with the indiscrete operator.

(b). As in Example 3.5(b), the class $\mathcal{B}$ of perfect groups and the class $\mathcal{A}$ that consists of all groups that do not have any non-trivial perfect subgroup form a pair $(\mathcal{B}, \mathcal{A})$ of corresponding fixed points of $(\Delta_{\mathcal{N}}, \nabla_{\mathcal{N}})$. For every $M \leq X$, $m^{T_{\mathcal{N}}(\mathcal{A})}$ is the intersection of all normal subgroups of $X$ containing $M$ such that $X/M \in \mathcal{A}$. Moreover, $M^{J_{\mathcal{N}}(\mathcal{B})}$ is the subgroup generated by $M$ and all perfect subgroups of $X$. 

21
(c). Part b) can be generalized as follows. Let \( A \in S(\text{Grp}) \). If \( A \) is closed under subgroups, then \( T_X(A) \) agrees with the \( A \)-normal closure (cf. [FJ], [FW]). \( B = (I_N \circ T_X)(A) \) consists of all those groups \( X \) that do not have any proper normal subgroup \( N \) such that \( X/N \in A \). For every subgroup \( M \) of \( Y \), \( M^{J_N(B)} \) is the subgroup generated by \( M \) and by those subgroups \( S \) of \( Y \) which do not have any proper normal subgroup \( N \) such that \( S/N \in A \).

**EXAMPLE 3.7**

(a). Let \( \mathcal{X} \) be the category \( \textbf{Ab} \) and let \( \mathcal{N} = \mathcal{M} \) be the class of all monomorphisms in \( \textbf{Ab} \). Let \( (T, \mathcal{F}) \) be a torsion theory. Clearly, \( T \) and \( \mathcal{F} \) are corresponding fixed points of the Galois connection \( (\Delta_N, \nabla_N) \) of Proposition 2.2. Let \( X \in \textbf{Ab} \) and let \( X \xrightarrow{r} X \) be its \( \mathcal{F} \)-reflection. For every subobject \( M \xrightarrow{m} X \) we have that \( M^{T_N(\mathcal{F})} \simeq r_X^{-1}(r_X(M)) \simeq M + Ker(r_X) \). Since \( T \) is closed under quotients, \( M^{J_N(T)} \) is the subgroup generated by \( M \) and all subgroups \( S \leq X \) such that \( S \in T \). In particular, if \( (T, \mathcal{F}) = (\text{Torsion,}\text{Torsion-free}) \), then \( M^{T_N(\mathcal{F})} \simeq M + Tor(X) \), where \( Tor(X) \) denotes the torsion subgroup of \( X \). If \( (T, \mathcal{F}) = (\text{Divisible},\text{Reduced}) \), then \( M^{T_N(\mathcal{F})} \simeq M + Div(X) \), where \( Div(X) \) denotes the largest divisible subgroup of \( X \). It is interesting to notice that in both cases, \( M^{J_N(T)} = M^{T_N(\mathcal{F})} \) (cf. [CH]).

(b). Now let \( \mathcal{N} \) be the class of all inclusions of divisible subgroups. Again \( \mathcal{N} \)-constant means constant. As above, if \( (T, \mathcal{F}) \) is a torsion theory, then \( T \) and \( \mathcal{F} \) are corresponding fixed points of the Galois connection \( (\Delta_N, \nabla_N) \) of Proposition 2.2. If \( \text{Red} \) is the subcategory of reduced abelian groups, then for every subgroup \( M \xrightarrow{m} X \), \( M^{T_N(\text{Red})} \) is the intersection of all subgroups of \( \mathcal{X} \) containing \( M \) such that \( X/M \) is reduced. As it is easily seen, this agrees with the Salbany closure \( S_{\text{Red}} \). Moreover, if \( \text{Div} \) is the subcategory of divisible abelian groups, then for every subgroup \( M \xrightarrow{m} X \), \( M^{J_N(\text{Div})} \simeq M + Div(X) \).

(c). If \( \mathcal{N} \) is the class of all inclusions of torsion subgroups, then also in this case \( \mathcal{N} \)-constant means constant. If we consider the torsion theory \( (T, TF) \) where \( T \) is the subcategory of all torsion abelian groups and \( TF \) is the subcategory of all torsion free abelian groups, then for every subgroup \( M \xrightarrow{m} X \), \( M^{T_N(TF)} \) is the intersection of all subgroups of \( \mathcal{X} \) containing \( M \) such that \( X/M \) is torsion free. As it is easily seen, this agrees with theSalbany closure \( S_{TF} \). Moreover, for every subgroup \( M \xrightarrow{m} X \), \( M^{J_N(T)} \simeq M + Tor(X) \).

(d). Let \( \mathcal{N} \) consist of all singleton monomorphisms and let \( (T, \mathcal{F}) \) be a torsion theory. Then, for every subgroup \( M \leq X \), \( M^{T_N(\mathcal{F})} = S_{\mathcal{F}} \) and \( M^{J_N(T)} \) is the subgroup generated by \( M \) and all subgroups \( S \leq X \) such that \( S \in T \).

(e). Notice that if \( \mathcal{N} \) consists of all diagonal morphisms and \( (T, \mathcal{F}) \) is a torsion theory then, from Proposition 2.27(a), for every subgroup \( M \leq X \), \( M^{T_N(\mathcal{F})} = S_{\mathcal{F}} \).
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