CONNECTEDNESS, DISCONNECTEDNESS AND CLOSURE OPERATORS: FURTHER RESULTS

G. Castellini [1]

ABSTRACT: Let $\mathcal{X}$ be an arbitrary category with an $(\mathbf{E},\mathcal{M})$-factorization structure for sinks. A notion of constant morphism that depends on a chosen class of monomorphisms was previously used to provide a generalization of the connectedness-disconnectedness Galois connection (also called torsion-torsion free in algebraic contexts). This Galois connection was shown to factor throught the class of all closure operators on $\mathcal{X}$ with respect to $\mathcal{M}$. Here, properties and implications of this factorization are investigated. In particular, it is shown that this factorization can be further factored. Examples are provided.

KEY WORDS: Closure operator, Galois connection, connectedness, disconnectedness.

AMS CLASSIFICATION: 18A20, 18A32, 06A15.

0 INTRODUCTION

This paper is a continuation of the work started in [C_2].

Let $\mathcal{X}$ be an arbitrary category with an $(\mathbf{E},\mathcal{M})$-factorization structure for sinks and let $\mathcal{N}\subseteq\mathcal{M}$. An $\mathcal{X}$-morphism $X \xrightarrow{f} Y$ is called $\mathcal{N}$-constant if the direct image of X under f is isomorphic to the direct image under f of every $\mathcal{N}$-subobject of X. If $S(\mathcal{X})$ denotes the collection of all subclasses of objects of $\mathcal{X}$, ordered by inclusion, for every $\mathcal{N}\subseteq\mathcal{M}$, the relation: $X\mathcal{R}_{\mathcal{N}}Y$ if and only if every $\mathcal{X}$-morphism $X \xrightarrow{f} Y$ is $\mathcal{N}$-constant yields a Galois connection $S(\mathcal{X}) \underset{\nabla_{\mathcal{N}}}{\overset{\Delta_{\mathcal{N}}}{\rightleftarrows}} S(\mathcal{X})^{\mathbf{op}}$. It was proved in [$\mathrm{C}_2$] that if $\mathcal{N}$ is closed under direct images, we have that this Galois connection factors through $CL(\mathcal{X},\mathcal{M})$, i.e., the collection of all closure operators on $\mathcal{X}$ with respect to $\mathcal{M}$, via two Galois connections $S(\mathcal{X}) \underset{I_{\mathcal{N}}}{\overset{J_{\mathcal{N}}}{\rightleftarrows}} CL(\mathcal{X},\mathcal{M})$ and $CL(\mathcal{X},\mathcal{M}) \underset{T_{\mathcal{N}}}{\overset{D_{\mathcal{N}}}{\rightleftarrows}} S(\mathcal{X})^{\mathbf{op}}$.

The development of a general theory of topological connectedness was started by Preuß (cf. [P_1-P_3]) and by Herrlich [H]. However, our definition of $\mathcal{N}$-constant morphism was not chosen with the intention of developing a general theory of connectedness and disconnectedness, but rather to support certain constructions with closure operators. A general theory of connectedness and disconnectedness was recently presented by Clementino in [Cl], where she extends results in [HP] to an arbitrary category.

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In this paper we present some closedness properties of the Galois closed classes of the Galois connection $S(\mathcal{X}) \underset{\nabla_{\mathcal{N}}}{\overset{\Delta_{\mathcal{N}}}{\rightleftarrows}} S(\mathcal{X})^{\mathbf{op}}$. We show that if $\mathcal{A}$ is a reflective subcategory of $\mathcal{X}$ and $\mathcal{B}$ is coreflective in $\mathcal{X}$, then a simpler characterization of the closure operators $T_{\mathcal{N}}(\mathcal{A})$ and $J_{\mathcal{N}}(\mathcal{B})$ can be given. Moreover, following some ideas introduced in [CKS$_2$], we bring more insight into the theory by showing that the Galois connections $S(\mathcal{X}) \underset{I_{\mathcal{N}}}{\overset{J_{\mathcal{N}}}{\rightleftarrows}} CL(\mathcal{X},\mathcal{M})$ and $CL(\mathcal{X},\mathcal{M}) \underset{T_{\mathcal{N}}}{\overset{D_{\mathcal{N}}}{\rightleftarrows}} S(\mathcal{X})^{\mathbf{op}}$ can also be factored.

Section 3 includes a number of examples that illustrate the theory.

We use the terminology of [AHS] throughout the paper[2].

1 PRELIMINARIES

Throughout we consider a category $\mathcal{X}$ and a fixed class $\mathcal{M}$ of $\mathcal{X}$-monomorphisms, which contains all $\mathcal{X}$-isomorphisms. It is assumed that $\mathcal{X}$ is $\mathcal{M}$-complete; i.e.,

(1) $\mathcal{M}$ is closed under composition

(2) Pullbacks of $\mathcal{M}$-morphisms exist and belong to $\mathcal{M}$, and multiple pullbacks of (possibly large) families of $\mathcal{M}$-morphisms with common codomain exist and belong to $\mathcal{M}$.

One of the consequences of the above assumptions is that there is a uniquely determined class $\mathbf{E}$ of sinks in $\mathcal{X}$ such that $\mathcal{X}$ is an $(\mathbf{E},\mathcal{M})$-*category for sinks*, that is:

(a) each of $\mathbf{E}$ and $\mathcal{M}$ is closed under compositions with isomorphisms;

(b) $\mathcal{X}$ has $(\mathbf{E},\mathcal{M})$-factorizations (of sinks); i.e., each sink $\mathbf{s}$ in $\mathcal{X}$ has a factorization $\mathbf{s} = m \circ \mathbf{e}$ with $\mathbf{e} \in \mathbf{E}$ and $m \in \mathcal{M}$, and

(c) $\mathcal{X}$ has the unique $(\mathbf{E},\mathcal{M})$-diagonalization property; i.e., if $B \xrightarrow{g} D$ and $C \xrightarrow{m} D$ are $\mathcal{X}$-morphisms with $m \in \mathcal{M}$, and $\mathbf{e} = (A_i \xrightarrow{e_i} B)_I$ and $\mathbf{s} = (A_i \xrightarrow{s_i} C)_I$ are sinks in $\mathcal{X}$ with $\mathbf{e} \in \mathbf{E}$, such that $m \circ \mathbf{s} = g \circ \mathbf{e}$, then there exists a unique diagonal $B \xrightarrow{d} C$ such that for every $i \in I$ the following diagrams commute:

$$\begin{tikzcd} A_i \arrow[r, "e_i"] \arrow[d, "s_i"'] & B \arrow[dl, "d"] \\ C & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & B \arrow[dl, "d"'] \arrow[d, "g"] \\ C \arrow[r, "m"'] & D \end{tikzcd}$$

That $\mathcal{X}$ is an $(\mathbf{E},\mathcal{M})$-category implies the following features of $\mathcal{M}$ and $\mathbf{E}$ (cf. [AHS] for the dual case):

[2] Paul Taylor's commutative diagrams macro package was used to typeset most of the diagrams in this paper.

PROPOSITION 1.1

(0) Every isomorphism is in both $\mathcal{M}$ and $\mathbf{E}$ (as a singleton sink).

(1) Every m in $\mathcal{M}$ is a monomorphism.

(2) $\mathcal{M}$ is closed under $\mathcal{M}$-relative first factors, i.e., if $n \circ m \in \mathcal{M}$, and $n \in \mathcal{M}$, then $m \in \mathcal{M}$.

(3) $\mathcal{M}$ is closed under composition.

(4) Pullbacks of $\mathcal{X}$-morphisms in $\mathcal{M}$ exist and belong to $\mathcal{M}$.

(5) The $\mathcal{M}$-subobjects of every $\mathcal{X}$-object form a (possibly large) complete lattice; suprema are formed via $(\mathbf{E}, \mathcal{M})$-factorizations and infima are formed via intersections. $\square$

Notice that in the above proposition, the word "lattice" is to be understood in a generalized sense for not necessarily antisymmetric pre-orders. Moreover, throughout the paper we will use the expression $\mathcal{M}$-subobject for both $m \in \mathcal{M}$ and the corresponding equivalence class of elements of $\mathcal{M}$.

If $X \xrightarrow{f} Y$ is an $\mathcal{X}$-morphism and $M \xrightarrow{m} X$ is an $\mathcal{M}$-subobject, then $M \xrightarrow{e_{f\circ m}} M_f \xrightarrow{m_f} Y$ will denote the $(\mathbf{E}, \mathcal{M})$-factorization of $f \circ m$. $M_f \xrightarrow{m_f} Y$ will be called the direct image of m along f. If $N \xrightarrow{n} Y$ is an $\mathcal{M}$-subobject, then the pullback $f^{-1}(N) \xrightarrow{f^{-1}(n)} X$ of n along f will be called the inverse image of n along f. Whenever no confusion is likely to arise, to simplify the notation we will denote the morphism $e_{f\circ m}$ simply e_f.

DEFINITION 1.2

A *closure operator* C on $\mathcal{X}$ (with respect to $\mathcal{M}$) is a family $\{(\ )_X^C\}_{X\in\mathcal{X}}$ of functions on the $\mathcal{M}$-subobject lattices of $\mathcal{X}$ with the following properties that hold for each $X \in \mathcal{X}$:

(a) [*expansiveness*] $m \leq (m)_X^C$, for every $\mathcal{M}$-subobject $M \xrightarrow{m} X$;

(b) [*order-preservation*] $m \leq n \Rightarrow (m)_X^C \leq (n)_X^C$ for every pair of $\mathcal{M}$-subobjects of X;

(c) [*morphism-consistency*] If p is the pullback of the $\mathcal{M}$-subobject $M \xrightarrow{m} Y$ along some $\mathcal{X}$-morphism $X \xrightarrow{f} Y$ and q is the pullback of $(m)_Y^C$ along f, then $(p)_X^C \leq q$, i.e., the closure of the inverse image of m is less than or equal to the inverse image of the closure of m.

Condition (a) implies that for every closure operator C on $\mathcal{X}$, every $\mathcal{M}$-subobject $M \xrightarrow{m} X$ has a canonical factorization

$$\begin{tikzcd} M \arrow[r, "t"] \arrow[dr, "m"'] & (M)_X^C \arrow[d, "(m)_X^C"] \\ & X \end{tikzcd}$$

where $((M)_X^C, (m)_X^C)$ is called the *C-closure* of the subobject (M, m).

When no confusion is likely we will write m^{C} rather than $(m)^{C}_{X}$ and for notational symmetry we will denote the morphism t by m_{C}.

REMARK 1.3

(1) Notice that in the above definition, under condition (b), the morphism-consistency condition (c) is equivalent to the following statement concerning direct images: if $M \xrightarrow{m} X$ is an $\mathcal{M}$-subobject and $X \xrightarrow{f} Y$ is a morphism, then $((m)^{C}_{Y})_f \leq (m_f)^{C}_{Y}$, i.e., the direct image of the closure of m is less than or equal to the closure of the direct image of m; (cf. [DG]).

(2) Under condition (a), both order-preservation and morphism-consistency, i.e., conditions (b) and (c) together are equivalent to the following: given (M, m) and (N, n) $\mathcal{M}$-subobjects of X and Y, respectively, if f and g are morphisms such that $n \circ g = f \circ m$, then there exists a unique morphism d such that the following diagram

$$\begin{tikzcd}
M \arrow[rr, "g"] \arrow[dd, "m"'] \arrow[dr, "m_C"] & & N \arrow[dd, "n"'] \arrow[dr, "n_C"] & \\
& M^{C} \arrow[rr, "d"] \arrow[dl, "m^{C}"] & & N^{C} \arrow[dl, "n^{C}"] \\
X \arrow[rr, "f"'] & & Y &
\end{tikzcd}$$

commutes.

(3) If we regard $\mathcal{M}$ as a full subcategory of the arrow category of $\mathcal{X}$, with the codomain functor from $\mathcal{M}$ to $\mathcal{X}$ denoted by U, then the above definition can also be stated in the following way: A *closure operator* on $\mathcal{X}$ (with respect to $\mathcal{M}$) is a pair $C = (\gamma, F)$, where F is an endofunctor on $\mathcal{M}$ that satisfies $UF = U$, and γ is a natural transformation from $id_{\mathcal{M}}$ to F that satisfies $(id_U)\gamma = id_U$ (cf. [DG]).

DEFINITION 1.4

Given a closure operator C, we say that $m \in \mathcal{M}$ is *C-closed* if m_{C} is an isomorphism. An $\mathcal{X}$-morphism f is called *C-dense* if for some (and hence every) $(\mathbf{E}, \mathcal{M})$-factorization (e, m) of f we have that m^{C} is an isomorphism. We call C *idempotent* provided that m^{C} is C-closed for every $m \in \mathcal{M}$. C is called *weakly hereditary* if m_{C} is C-dense for every $m \in \mathcal{M}$. Furthermore, if $\mathcal{M}' \subseteq \mathcal{M}$, then C is said to be *hereditary with respect to* $\mathcal{M}'$ if whenever $M \xrightarrow{m} X$, $M \xrightarrow{t} N$ and $N \xrightarrow{n} X$ are morphisms in $\mathcal{M}$ with $n \circ t = m$ and $n \in \mathcal{M}'$, we have that t^{C} is the pullback of m^{C} along n (cf. [CG]). If $\mathcal{M}' = \mathcal{M}$, then C is simply called *hereditary.*

Notice that Definition 1.2(c) implies that pullbacks of C-closed $\mathcal{M}$-subobjects are C-closed.

A special case of an idempotent closure operator arises in the following way. Given any class $\mathcal{A}$ of $\mathcal{X}$-objects and $M \xrightarrow{m} X$ in $\mathcal{M}$, define $m^{\mathcal{A}}$ to be the intersection of all equalizers of pairs of $\mathcal{X}$-morphisms r, s from X to some $\mathcal{A}$-object A that satisfy $r \circ m = s \circ m$, and let $m_{\mathcal{A}} \in \mathcal{M}$ be the unique $\mathcal{X}$-morphism by which m factors through $m^{\mathcal{A}}$. It is easy to see that this gives rise to an idempotent closure operator that we will denote by $S_{\mathcal{A}}$. This generalizes the Salbany construction of closure operators induced by classes of topological spaces, cf. [S].

We denote the collection of all closure operators on $\mathcal{M}$ by $CL(\mathcal{X}, \mathcal{M})$ pre-ordered as follows: $C \sqsubseteq D$ if $m^{C} \leq m^{D}$ for all $m \in \mathcal{M}$ (where $\leq$ is the usual order on subobjects). Notice that arbitrary suprema and infima exist in $CL(\mathcal{X}, \mathcal{M})$, they are formed pointwise in the $\mathcal{M}$-subobject fibers.

For more background on closure operators see, e.g., [C$_1$], [CKS$_1$], [CKS$_2$], [DG], [DGT] and [K]. For a detailed survey on the same topic, one could check [Ho].

DEFINITION 1.5

For pre-ordered classes $\mathcal{X} = (\mathbf{X}, \sqsubseteq)$ and $\mathcal{Y} = (\mathbf{Y}, \sqsubseteq)$, a *Galois connection* $\mathcal{X} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{Y}$ consists of order preserving functions F and G that satisfy $F \dashv G$, i.e., $x \sqsubseteq GF(x)$ for every $x \in \mathbf{X}$ and $FG(y) \sqsubseteq y$ for every $y \in \mathbf{Y}$. (G is adjoint and has F as coadjoint).

If $x \in \mathcal{X}$ and $y \in \mathcal{Y}$ are such that $F(x) = y$ and $G(y) = x$, then x and y are said to be corresponding fixed points of the Galois connection $(\mathcal{X}, F, G, \mathcal{Y})$ (we may use at times the shorter notation (F, G)). To be more precise, we may sometimes make use of the expressions "left fixed point" and "right fixed point" for x and y, respectively.

Properties and many examples of Galois connections can be found in [EKMS].

2 GENERAL RESULTS

Throughout the paper we assume that $\mathcal{X}$ is an $(\mathbf{E}, \mathcal{M})$-category for sinks.

Let $S(\mathcal{X})$ denote the collection of all subcategories of $\mathcal{X}$, ordered by inclusion and let $\mathcal{N}$ be a fixed subclass of $\mathcal{M}$. Throughout, for every $X \in \mathcal{X}$, $\mathcal{M}_X$ ($\mathcal{N}_X$) will denote the "lattice" of all $\mathcal{M}$-subobjects ($\mathcal{N}$-subobjects) of X. We begin by recalling the following definition and results from [C$_2$].

DEFINITION 2.1

Let $\mathcal{N} \subseteq \mathcal{M}$. An $\mathcal{X}$-morphism $X \xrightarrow{f} Y$ is called $\mathcal{N}$-constant if for every $\mathcal{N}$-subobject $N \xrightarrow{n} X$, we have that $n_f \simeq (id_X)_f$.

PROPOSITION 2.2 (cf. [H])

Let $\mathcal{N} \subseteq \mathcal{M}$. Define $S(\mathcal{X}) \xrightarrow{\Delta_{\mathcal{N}}} S(\mathcal{X})^{\mathbf{op}}$ and $S(\mathcal{X})^{\mathbf{op}} \xrightarrow{\nabla_{\mathcal{N}}} S(\mathcal{X})$ as follows:

$$\nabla_{\mathcal{N}}(\mathcal{A}) = \{X \in \mathcal{X} : \forall Y \in \mathcal{A},\ \text{every } X \xrightarrow{f} Y \text{ is } \mathcal{N}\text{-constant}\}$$

$$\Delta_{\mathcal{N}}(\mathcal{B}) = \{Y \in \mathcal{X} : \forall X \in \mathcal{B},\ \text{every } X \xrightarrow{f} Y \text{ is } \mathcal{N}\text{-constant}\}$$

Then, $S(\mathcal{X}) \underset{\nabla_{\mathcal{N}}}{\overset{\Delta_{\mathcal{N}}}{\rightleftarrows}} S(\mathcal{X})^{\mathbf{op}}$ is a Galois connection. □

PROPOSITION 2.3

Let $CL(\mathcal{X},\mathcal{M}) \xrightarrow{D_{\mathcal{N}}} S(\mathcal{X})^{\mathbf{op}}$ and $S(\mathcal{X})^{\mathbf{op}} \xrightarrow{T_{\mathcal{N}}} CL(\mathcal{X},\mathcal{M})$ be defined by:

$$D_{\mathcal{N}}(C) = \{X \in \mathcal{X} :\ \text{every } n \in \mathcal{N}_X \text{ is } C\text{-closed}\}$$

$$T_{\mathcal{N}}(\mathcal{A}) = Sup\{C \in CL(\mathcal{X},\mathcal{M}) : D_{\mathcal{N}}(C) \supseteq \mathcal{A}\}.$$

Then, $CL(\mathcal{X},\mathcal{M}) \underset{T_{\mathcal{N}}}{\overset{D_{\mathcal{N}}}{\rightleftarrows}} S(\mathcal{X})^{\mathbf{op}}$ is a Galois connection. □

PROPOSITION 2.4

Let $CL(\mathcal{X},\mathcal{M}) \xrightarrow{I_{\mathcal{N}}} S(\mathcal{X})$ and $S(\mathcal{X}) \xrightarrow{J_{\mathcal{N}}} CL(\mathcal{X},\mathcal{M})$ be defined by:

$$I_{\mathcal{N}}(C) = \{X \in \mathcal{X} :\ \text{every } n \in \mathcal{N}_X \text{ is } C\text{-dense}\}$$

$$J_{\mathcal{N}}(\mathcal{B}) = Inf\{C \in CL(\mathcal{X},\mathcal{M}) : I_{\mathcal{N}}(C) \supseteq \mathcal{B}\}.$$

Then, $S(\mathcal{X}) \underset{I_{\mathcal{N}}}{\overset{J_{\mathcal{N}}}{\rightleftarrows}} CL(\mathcal{X},\mathcal{M})$ is a Galois connection. □

PROPOSITION 2.5

Let $\mathcal{A} \in S(\mathcal{X})^{\mathbf{op}}$. For every $X \in \mathcal{X}$ and for every $\mathcal{M}$-subobject $M \xrightarrow{m} X$, we define

$$_{\mathcal{A}}m = \cap\{f^{-1}(n) : Y \in \mathcal{A}, X \xrightarrow{f} Y, N \xrightarrow{n} Y \in \mathcal{N}_Y \text{and } m \leq f^{-1}(n)\}.$$

For every $\mathcal{A} \in S(\mathcal{X})^{\mathbf{op}}$ we have that the function $_{\mathcal{A}}(\ )$ that to every $\mathcal{M}$-subobject $M \xrightarrow{m} X$ associates $_{\mathcal{A}}m$ is an idempotent closure operator on $\mathcal{X}$ and $_{\mathcal{A}}m \simeq m^{T_{\mathcal{N}}(\mathcal{A})}$. □

PROPOSITION 2.6

Let $\mathcal{B} \in S(\mathcal{X})$. For every $Y \in \mathcal{X}$ and for every $\mathcal{M}$-subobject $M \xrightarrow{m} Y$, we define

$$C_{\mathcal{B}_{\mathcal{N}}}(m) = sup\Big(\{m\} \cup \{(id_X)_f : X \in \mathcal{B}, X \xrightarrow{f} Y \text{ and } \exists n \in \mathcal{N}_X \text{ with } n_f \leq m\}\Big).$$

For every $\mathcal{B} \in S(\mathcal{X})$, the function $C_{\mathcal{B}_{\mathcal{N}}}$ is a weakly hereditary closure operator on $\mathcal{X}$. Moreover, we have that $C_{\mathcal{B}_{\mathcal{N}}}(m) \simeq m^{J_{\mathcal{N}}(\mathcal{B})}$. □

THEOREM 2.7

Let $\mathcal{N}$ be a subclass of $\mathcal{M}$ closed under the formation of direct images. Then the Galois connection $S(\mathcal{X}) \underset{\nabla_{\mathcal{N}}}{\overset{\Delta_{\mathcal{N}}}{\rightleftarrows}} S(\mathcal{X})^{\mathbf{op}}$ *factors through* $CL(\mathcal{X},\mathcal{M})$ *via the two Galois connections* $S(\mathcal{X}) \underset{I_{\mathcal{N}}}{\overset{J_{\mathcal{N}}}{\rightleftarrows}} CL(\mathcal{X},\mathcal{M})$ and $CL(\mathcal{X},\mathcal{M}) \underset{T_{\mathcal{N}}}{\overset{D_{\mathcal{N}}}{\rightleftarrows}} S(\mathcal{X})^{\mathbf{op}}$. $\square$

PROPOSITION 2.8

Let $\mathcal{N} \subseteq \mathcal{M}$ and let $X \xrightarrow{f} Y$ be an $\mathcal{X}$-morphism. The following are equivalent:

(a) For every $\mathcal{N}$-subobject $N \xrightarrow{n} X$ we have that $n_f \simeq (id_X)_f$;

(b) f factors through n_f, for every $\mathcal{N}$-subobject $N \xrightarrow{n} X$.

Proof:

(a) $\Rightarrow$ (b). Consider the $\mathcal{N}$-subobject $N \xrightarrow{n} X$ and let $(e,(id_X)_f)$ and (e_f,n_f) be $(\mathbf{E},\mathcal{M})$-factorizations of f and $f \circ n$, respectively. From the hypothesis, there is an isomorphism $X_f \xrightarrow{i} N_f$ such that $n_f \circ i = (id_X)_f$. Consequently we have that $f = (id_X)_f \circ e = n_f \circ i \circ e$. Thus (b) holds.

(b) $\Rightarrow$ (a). We will use here the same notation as in the first part of the proof. From the hypothesis we obtain a morphism $X \xrightarrow{d} N_f$ such that $n_f \circ d = f$. Now we have the following two commutative diagrams

$$\begin{array}{ccc} X & \xrightarrow{e} & X_f \\ {\scriptstyle d}\downarrow & & \downarrow{\scriptstyle (id_X)_f} \\ N_f & \xrightarrow[n_f]{} & Y \end{array} \quad \text{and} \quad \begin{array}{ccc} N & \xrightarrow{e_f} & N_f \\ {\scriptstyle e\circ n}\downarrow & & \downarrow{\scriptstyle n_f} \\ X_f & \xrightarrow[(id_X)_f]{} & Y \end{array}$$

The $(\mathbf{E},\mathcal{M})$-diagonalization property yields a morphism r such that $n_f \circ r = (id_X)_f$ and a morphism s such that $(id_X)_f \circ s = n_f$. Therefore we obtain that $(id_X)_f \circ s \circ r = (id_X)_f = (id_X)_f \circ id_{X_f}$. Since $(id_X)_f$ is a monomorphism, we conclude that $s \circ r = id_{X_f}$. This shows that s is a retraction and a monomorphism and consequently is actually an isomorphism. $\square$

REMARK 2.9

Part (b) of Proposition 2.8 provides an alternative formulation of Definition 2.1. Since this formulation of the definition of $\mathcal{N}$-constant morphism simplifies the writing of the proofs, we will be using it throughout the paper.

Next we examine the behavior of the Galois closed classes of $S(\mathcal{X}) \underset{\nabla_{\mathcal{N}}}{\overset{\Delta_{\mathcal{N}}}{\rightleftarrows}} S(\mathcal{X})^{\mathbf{op}}$ with respect to some classical constructions.

PROPOSITION 2.10

For every subcategory $\mathcal{B} \in S(\mathcal{X})$, $\Delta_{\mathcal{N}}(\mathcal{B})$ is closed under $\mathcal{M}$-subobjects.

Proof:

Let $M \xrightarrow{m} Y$ be an $\mathcal{M}$-subobject with $Y \in \Delta_{\mathcal{N}}(\mathcal{B})$, let $X \in \mathcal{B}$, let $N \xrightarrow{n} X$ be an $\mathcal{N}$-subobject of X and let $X \xrightarrow{f} M$ be an $\mathcal{X}$-morphism. If (e_f, n_f) and $(e_{m\circ f}, n_{m\circ f})$, denote the $(\mathbf{E}, \mathcal{M})$-factorizations of $f \circ n$ and $m \circ f \circ n$, respectively, then we obtain the following commutative diagram

$$\begin{tikzcd}
X \arrow[r, "f"] & M \arrow[dr, "m"] & \\
N \arrow[u, "n"] \arrow[r, "e_f"] \arrow[dr, "e_{m\circ f}"'] & N_f \arrow[u, "n_f"'] & Y \\
& N_{m\circ f} \arrow[ur, "n_{m\circ f}"'] &
\end{tikzcd}$$

We first observe that since $Y \in \Delta_{\mathcal{N}}(\mathcal{B})$, from Remark 2.9, there is a morphism $X \xrightarrow{d} N_{m\circ f}$ such that $n_{m\circ f} \circ d = m \circ f$.

Now we have that $m \circ n_f \circ e_f = m \circ f \circ n = n_{m\circ f} \circ e_{m\circ f}$. The $(\mathbf{E}, \mathcal{M})$-diagonalization property implies the existence of a unique morphism $N_{m\circ f} \xrightarrow{t} N_f$ that satisfies $t \circ e_{m\circ f} = e_f$ and $m \circ n_f \circ t = n_{m\circ f}$. So, we have that $m \circ n_f \circ t \circ d = n_{m\circ f} \circ d = m \circ f$. Since m is a monomorphism, we obtain that $n_f \circ t \circ d = f$. Thus f factors through n_f and again from Remark 2.9, $M \in \Delta_{\mathcal{N}}(\mathcal{B})$. $\square$

PROPOSITION 2.11

Let $\mathcal{X}$ have a terminal object T and assume that any morphism with T as domain belongs to $\mathcal{M}$. If $\mathbf{E}$ is contained in the class of episinks and $\mathcal{N}$ consists of all morphisms having T as domain, then for every subclass $\mathcal{B} \in S(\mathcal{X})$, we have that $\Delta_{\mathcal{N}}(\mathcal{B})$ is closed under monosources.

Proof:

Let us consider the following commutative diagram

$$
\begin{tikzcd}
X \arrow[r, "f"] & Y \arrow[r, "p_i"] & Y_i \\
 & T_f \arrow[u, "n_f"'] & \\
T \arrow[uu, "t_X"] \arrow[ur, "e_f"'] \arrow[rr, "e_i"'] & & T_i \arrow[uu, "m_i"']
\end{tikzcd}
$$

where (e_i, m_i) and (e_f, n_f) are the $(\mathbf{E}, \mathcal{M})$-factorizations of $p_i \circ f \circ t_X$ and $f \circ t_X$, respectively and $(Y \xrightarrow{p_i} Y_i)_{i \in I}$ is a monosource with $Y_i \in \Delta_{\mathcal{N}}(\mathcal{B})$ for every $i \in I$. Let $X \xrightarrow{d_X} T$ be the unique morphism with codomain T. By hypothesis, for every $i \in I$, there exists a morphism $X \xrightarrow{t_i} T_i$ such that $p_i \circ f = m_i \circ t_i$. Notice that since epimorphisms with terminal domain are already isomorphisms, we have that e_i is an isomorphism for every $i \in I$. Since T is a terminal object, we have that $d_X = e_i^{-1} \circ t_i$ for every $i \in I$ and so $t_i = e_i \circ d_X$, for every $i \in I$. Now, $p_i \circ f = m_i \circ t_i = m_i \circ e_i \circ d_X = p_i \circ f \circ t_X \circ d_X = p_i \circ n_f \circ e_f \circ d_X$ for every $i \in I$. Since $(p_i)_{i \in I}$ is a monosource, we have that $f = n_f \circ e_f \circ d_X$. Thus, f factors through n_f and so $Y \in \Delta_{\mathcal{N}}(\mathcal{B})$. $\square$

We recall that an $\mathcal{X}$-morphism $X \xrightarrow{q} Q$ is called an $\mathbf{E}$-quotient if the singleton sink q belongs to $\mathbf{E}$.

PROPOSITION 2.12

If $\mathcal{N}$ is closed under pullbacks along $\mathbf{E}$-morphisms, then for every $\mathcal{A} \in S(\mathcal{X})^{\mathbf{op}}$, $\nabla_{\mathcal{N}}(\mathcal{A})$ is closed under $\mathbf{E}$-quotients.

Proof:

Let us consider the following commutative diagram

$$
\begin{tikzcd}
X \arrow[r, "q"] & Q \arrow[r, "f"] & Y \\
q^{-1}(N) \arrow[u, "q^{-1}(n)"] \arrow[r, dashed, "t"'] & N \arrow[u, "n"'] \arrow[r, "e_f"'] & N_f \arrow[u, "n_f"']
\end{tikzcd}
$$

where $q \in \mathbf{E}$, $Y \in \mathcal{A}$, $X \in \nabla_{\mathcal{N}}(\mathcal{A})$ and $q^{-1}(n)$ is the pullback of $n \in \mathcal{N}$ along q. Notice that by hypothesis $q^{-1}(n) \in \mathcal{N}$. Since $f \circ q$ is $\mathcal{N}$-constant, it factors through $(q^{-1}(n))_{f \circ q}$ via a morphism $X \xrightarrow{d} (q^{-1}(N))_{f \circ q}$. Now let us consider the commutative diagram

$$
\begin{tikzcd}
X \arrow[r, "q"] \arrow[d, "d"'] & Q \arrow[d, "f"] \\
(q^{-1}(N))_{f\circ q} \arrow[r, "(q^{-1}(n))_{f\circ q}"'] & Y \\
q^{-1}(N) \arrow[u, "e_{f\circ q}"] \arrow[r, "e_f \circ t"'] & N_f \arrow[u, "n_f"']
\end{tikzcd}
$$

From the $(\mathbf{E},\mathcal{M})$-diagonalization property we obtain a morphism d' such that $d' \circ q = d$ and $(q^{-1}(n))_{f\circ q} \circ d' = f$ and a morphism t' such that $t' \circ e_{f\circ q} = e_f \circ t$ and $n_f \circ t' = (q^{-1}(n))_{f\circ q}$. Therefore we obtain that $f = n_f \circ t' \circ d'$, and so $Q \in \nabla_{\mathcal{N}}(\mathcal{A})$. $\square$

PROPOSITION 2.13

Let $\mathcal{N}$ be closed under pullbacks and let $X_i \in \nabla_{\mathcal{N}}(\mathcal{A})$ for every $i \in I$. If the coproduct $\amalg X_i$ exists, then it also belongs to $\nabla_{\mathcal{N}}(\mathcal{A})$.

Proof:

Let us consider the following commutative diagram

$$
\begin{tikzcd}
X_i \arrow[r, "k_i"] & \amalg X_i \arrow[r, "f"] & Y \\
k_i^{-1}(N) \arrow[u, "k_i^{-1}(n)"] \arrow[r, "h_i"'] & N \arrow[u, "n"] \arrow[r, "e_f"'] & N_f \arrow[u, "n_f"']
\end{tikzcd}
$$

where (e_f, n_f) is an $(\mathbf{E},\mathcal{M})$-factorization of $f \circ n$, $k_i^{-1}(n)$ is the pullback of $n \in \mathcal{N}$ along k_i and $Y \in \mathcal{A}$.

Now, let (e_i, n_i) be an $(\mathbf{E},\mathcal{M})$-factorization of $f \circ k_i \circ k_i^{-1}(n)$. Since, for every $i \in I$, $X_i \in \nabla_{\mathcal{N}}(\mathcal{A})$ and $k_i^{-1}(n) \in \mathcal{N}$, we have that for every $i \in I$ there exists a morphism t_i having X_i as domain such that $n_i \circ t_i = f \circ k_i$. Since for every $i \in I$ we have that $n_i \circ e_i = n_f \circ e_f \circ h_i$, then the $(\mathbf{E},\mathcal{M})$-diagonalization property implies that for every $i \in I$ there exists a morphism d_i with codomain N_f, such that $n_f \circ d_i = n_i$ and $d_i \circ e_i = e_f \circ h_i$. Therefore, there exists a unique $\amalg X_i \xrightarrow{t} N_f$ such that $t \circ k_i = d_i \circ t_i$. Now we have that $f \circ k_i = n_i \circ t_i = n_f \circ d_i \circ t_i = n_f \circ t \circ k_i$, for every $i \in I$. The uniqueness condition in the definition of coproduct implies that $n_f \circ t = f$. Thus, we have that $\amalg X_i \in \nabla_{\mathcal{N}}(\mathcal{A})$. $\square$

REMARK 2.14

Notice that if in the category **TOP** of topological spaces (**Grp** of groups), we choose for instance the (episink, extremal monomorphism)-factorization structure, and $\mathcal{N}$ consists of all inclusions of singleton subobjects, then Propositions 2.10 and 2.11 imply that for any subcategory $\mathcal{B}$ of **TOP** (**Grp**), $\Delta_{\mathcal{N}}(\mathcal{B})$ is an epireflective subcategory of **TOP** (**Grp**). Moreover, if we choose as $\mathcal{N}$ any class of extremal monomorphisms that is closed under pullbacks, then Propositions 2.12 and 2.13 imply that for any subcategory $\mathcal{A}$ of **TOP** (**Grp**), $\nabla_{\mathcal{N}}(\mathcal{A})$ is a coreflective subcategory of **TOP** (**Grp**).

PROPOSITION 2.15

Let $\mathcal{N}$ be a subclass of $\mathcal{M}$ closed under the formation of direct images. Then the following hold:

(a) *Suppose that $\mathcal{A} \in S(\mathcal{X})^{\mathbf{op}}$ is a right fixed point of the Galois connection $(\Delta_{\mathcal{N}}, \nabla_{\mathcal{N}})$. Then, $\mathcal{A}$ is also a right fixed point of the Galois connection $(D_{\mathcal{N}}, T_{\mathcal{N}})$.*

(b) *Suppose that $\mathcal{B} \in S(\mathcal{X})$ is a left fixed point of the Galois connection $(\Delta_{\mathcal{N}}, \nabla_{\mathcal{N}})$. Then, $\mathcal{B}$ is also a left fixed point of the Galois connection $(J_{\mathcal{N}}, I_{\mathcal{N}})$.*

(c) *If $C \in CL(\mathcal{X}, \mathcal{M})$ satisfies $C = T_{\mathcal{N}}(\mathcal{A}) = J_{\mathcal{N}}(\mathcal{B})$, for some $\mathcal{A} \in S(\mathcal{X})^{\mathbf{op}}$ that is a right fixed point of the Galois connection $(D_{\mathcal{N}}, T_{\mathcal{N}})$ and for some $\mathcal{B} \in S(\mathcal{X})$ that is a left fixed point of the Galois connection $(J_{\mathcal{N}}, I_{\mathcal{N}})$, then $\mathcal{A}$ and $\mathcal{B}$ are corresponding fixed points of the Galois connection $(\Delta_{\mathcal{N}}, \nabla_{\mathcal{N}})$.*

Proof:

(a). Suppose that $\mathcal{A} \in S(\mathcal{X})^{\mathbf{op}}$ is a right fixed point of the Galois connection $(\Delta_{\mathcal{N}}, \nabla_{\mathcal{N}})$. In [C$_2$], Proposition 2.10, we proved that under the assumption of $\mathcal{N}$ being closed under direct images, we have that for every $C \in CL(\mathcal{X}, \mathcal{M})$, $D_{\mathcal{N}}(C) \subseteq (\Delta_{\mathcal{N}} \circ I_{\mathcal{N}})(C)$. So, we have that $\mathcal{A} \subseteq D_{\mathcal{N}}(T_{\mathcal{N}}(\mathcal{A})) \subseteq \Delta_{\mathcal{N}}(I_{\mathcal{N}}(T_{\mathcal{N}}(\mathcal{A}))) = (\Delta_{\mathcal{N}} \circ \nabla_{\mathcal{N}})(\mathcal{A}) = \mathcal{A}$. Thus, $\mathcal{A} = (D_{\mathcal{N}} \circ T_{\mathcal{N}})(\mathcal{A})$.

(b). Similarly to (a).

(c). Under our hypotheses we have that $C = T_{\mathcal{N}}(\mathcal{A}) = J_{\mathcal{N}}(\mathcal{B})$. Then we have that $\Delta_{\mathcal{N}}(\mathcal{B}) = D_{\mathcal{N}}(J_{\mathcal{N}}(\mathcal{B})) = D_{\mathcal{N}}(T_{\mathcal{N}}(\mathcal{A})) = \mathcal{A}$. Analagously we obtain that $\nabla_{\mathcal{N}}(\mathcal{A}) = \mathcal{B}$. $\square$

PROPOSITION 2.16

Let $\mathcal{N}$ be a subclass of $\mathcal{M}$ closed under composition and let $\mathcal{M}' \subseteq \mathcal{M}$. If C is a closure operator that is hereditary with respect to $\mathcal{M}'$ and $\mathcal{M}'$ satisfies the condition that for every $n \in \mathcal{N}$ and $m' \in \mathcal{M}'$, the morphism $m' \circ n$ belongs to $\mathcal{N}$, then $D_{\mathcal{N}}(C)$ and $I_{\mathcal{N}}(C)$ are both closed under the formation of $\mathcal{M}'$-subobjects.

Proof:

Let $X \in D_{\mathcal{N}}(C)$ and let $N \xrightarrow{n} X$ be an $\mathcal{M}'$-subobject of X. If $M \xrightarrow{t} N$ is an $\mathcal{N}$-subobject of N then, by our assumption on $\mathcal{M}'$, we have that $m = n \circ t \in \mathcal{N}$. Let us consider the following commutative diagram

$$\begin{tikzcd}
M^C_X \arrow[rr, "m^C"] & & X \\
M \arrow[u, "m_C"] \arrow[urr, "m"] \arrow[rr, "t"'] \arrow[dr, "t_C"'] & & N \arrow[u, "n"'] \\
& M^C_N \arrow[ur, "t^C"'] &
\end{tikzcd}$$

The fact that C is hereditary with respect to $\mathcal{M}'$ implies that t^C is the pullback of m^C along n. However, since $X \in D_{\mathcal{N}}(C)$, we have that m_C is an isomorphism, i.e., m is C-closed. Now, since n is a monomorphism, t is the pullback of m along n and so t is also C-closed. Thus $N \in D_{\mathcal{N}}(C)$.

Let $X \in I_{\mathcal{N}}(C)$, and let $N \xrightarrow{n} X$ be an $\mathcal{M}'$-subobject of X. If $M \xrightarrow{t} N$ is an $\mathcal{N}$-subobject of N then, by our assumption on $\mathcal{M}'$, we have that $m = n \circ t \in \mathcal{N}$. Let us consider again the commutative diagram in the proof of part (a). Since $X \in I_{\mathcal{N}}(C)$, we have that m^C is an isomorphism. Again, the hereditary property of C with respect to $\mathcal{M}'$ implies that t^C is the pullback of m^C along n. Since t^C, as a pullback of an isomorphism, is an isomorphism, we can conclude that $N \in I_{\mathcal{N}}(C)$. $\square$

Next we show that under some additional assumptions on the subclass $\mathcal{A}$, the descriptions of the closure operators $T_{\mathcal{N}}(\mathcal{A})$ and $J_{\mathcal{N}}(\mathcal{A})$ given in Propositions 2.5 and 2.6 can be further simplified. First we need the following

LEMMA 2.17

Let $\mathcal{N} \subseteq \mathcal{M}$ and let $\mathcal{A} \subseteq \mathcal{X}$. Denote by $\mathcal{N}^{pb}_{\mathcal{A}}$ the union of $\mathcal{N}$ with all inverse images (pullbacks) of elements of $\mathcal{N}$ along all $\mathcal{X}$-morphisms having codomain in $\mathcal{A}$. Then, we have that $T_{\mathcal{N}}(\mathcal{A}) \simeq T_{\mathcal{N}^{pb}_{\mathcal{A}}}(\mathcal{A})$.

Proof:

Clearly, since $\mathcal{N} \subseteq \mathcal{N}^{pb}_{\mathcal{A}}$, from Proposition 2.5 we have that $T_{\mathcal{N}^{pb}_{\mathcal{A}}}(\mathcal{A}) \sqsubseteq T_{\mathcal{N}}(\mathcal{A})$. On the other hand, since pullbacks of C-closed subobjects are C-closed, any closure operator C that satisfies $D_{\mathcal{N}}(C) \supseteq \mathcal{A}$ also satisfies $D_{\mathcal{N}^{pb}_{\mathcal{A}}}(C) \supseteq \mathcal{A}$. Thus, $T_{\mathcal{N}}(\mathcal{A}) \sqsubseteq T_{\mathcal{N}^{pb}_{\mathcal{A}}}(\mathcal{A})$. This, together with the previous inequality yields that $T_{\mathcal{N}}(\mathcal{A}) \simeq T_{\mathcal{N}^{pb}_{\mathcal{A}}}(\mathcal{A})$. $\square$

PROPOSITION 2.18

Let $\mathcal{A}$ be a full, reflective subcategory of $\mathcal{X}$ and for $X \in \mathcal{X}$, let $X \xrightarrow{r_X} rX$ denote the reflection morphism. If $\mathcal{N}$ is closed under the formation of pullbacks along $\mathcal{X}$-morphisms with codomain in $\mathcal{A}$ then, for every $\mathcal{M}$-subobject $M \xrightarrow{m} X$, we have that

$$m^{T_{\mathcal{N}}(\mathcal{A})} \simeq \cap\{r_X^{-1}(n) : N \xrightarrow{n} rX \in \mathcal{N} \text{and } m \leq r_X^{-1}(n)\}.$$

Proof:

First of all, it is straightforward to notice that $m^{T_{\mathcal{N}}(\mathcal{A})} \leq \cap\{r_X^{-1}(n) : N \xrightarrow{n} rX \in \mathcal{N} \text{and } m \leq r_X^{-1}(n)\}$ (cf. Proposition 2.5). Now, let $X \xrightarrow{f} Y$ be an $\mathcal{X}$-morphism with $Y \in \mathcal{A}$ and let $N \xrightarrow{n} Y$ be an $\mathcal{N}$-subobject of Y. Since $\mathcal{A}$ is reflective in $\mathcal{X}$, there exists a unique $\mathcal{X}$-morphism $rX \xrightarrow{g} Y$ such that $g \circ r_X = f$. Clearly, if the $\mathcal{M}$-subobject $M \xrightarrow{m} X$ satisfies $m \leq f^{-1}(n)$, then we have that $m \leq f^{-1}(n) \simeq (g \circ r_X)^{-1}(n) \simeq r_X^{-1}(g^{-1}(n))$. From the hypothesis, $g^{-1}(n) \in \mathcal{N}$. Thus we have that $m^{T_{\mathcal{N}}(\mathcal{A})} \geq \cap\{r_X^{-1}(n) : N \xrightarrow{n} rX \in \mathcal{N} \text{and } m \leq r_X^{-1}(n)\}$. This, together with the other inequality proves the result. $\square$

COROLLARY 2.19

Let $\mathcal{A}$ be a full, reflective subcategory of $\mathcal{X}$ and for $X \in \mathcal{X}$, let $X \xrightarrow{r_X} rX$ denote the reflection morphism. Then, for every $\mathcal{N} \subseteq \mathcal{M}$ and for every $\mathcal{M}$-subobject $M \xrightarrow{m} X$, we have that

$$m^{T_{\mathcal{N}}(\mathcal{A})} \simeq \cap\{r_X^{-1}(n) : N \xrightarrow{n} rX \in \mathcal{N} \text{and } m \leq r_X^{-1}(n)\}.$$

Proof:

From Lemma 2.17 we can replace $\mathcal{N}$ by $\mathcal{N}_{\mathcal{A}}^{pb}$ and apply the previous proposition. $\square$

PROPOSITION 2.20

Let $\mathcal{A}$ be a full, coreflective subcategory of $\mathcal{X}$ and for $Y \in \mathcal{X}$, let $cY \xrightarrow{c_Y} Y$ denote the coreflection morphism. If $\mathcal{N}$ is closed under the formation of direct images then, for every $\mathcal{M}$-subobject $M \xrightarrow{m} Y$, we have that

$$m^{J_{\mathcal{N}}(\mathcal{A})} \simeq sup\left(\{m\} \cup \{(id_{cY})_{cY} : \exists n \in \mathcal{N}_{cY} \text{ with } n_{c_Y} \leq m\}\right).$$

Proof:

Clearly we have that $m^{J_{\mathcal{N}}(\mathcal{A})} \geq sup\left(\{m\} \cup \{(id_{cY})_{cY} : \exists n \in \mathcal{N}_{cY} \text{ with } n_{c_Y} \leq m\}\right)$ (cf. Proposition 2.6).

Now, let $M \xrightarrow{m} Y$ be an $\mathcal{M}$-subobject and let $X \xrightarrow{f} Y$ be such that $X \in \mathcal{A}$ and there exists an $\mathcal{N}$-morphism $N \xrightarrow{n} X$ with $n_f \leq m$. Since $\mathcal{A}$ is a coreflective subcategory of $\mathcal{X}$, there exists a unique morphism $X \xrightarrow{g} cY$ such that $c_Y \circ g = f$. Let (e_1, m_1) and (e_2, m_2)

be the $(\mathbf{E}, \mathcal{M})$-factorizations of f and c_Y, respectively. Clearly, $m_2 \circ e_2 \circ g = m_1 \circ e_1$, from the $(\mathbf{E}, \mathcal{M})$-diagonalization property, we obtain a monomorphism $(id_X)_f \xrightarrow{d} (id_{cY})_{c_Y}$ such that $m_2 \circ d = m_1$. Again, using the $(\mathbf{E}, \mathcal{M})$-diagonalization property, from the following commutative diagram

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "g"] & & Y \\
& cY \arrow[ur, "c_Y"] & \\
& N_g \arrow[u, "n_g"] \arrow[d] & \\
N \arrow[uuu, "n"] \arrow[ur, "e_g"'] \arrow[r, "e_f"'] & N_f \arrow[r] & M \arrow[uuu, "m"']
\end{tikzcd}$$

we obtain that $(n_g)_{c_Y} \leq m$. Notice that since $\mathcal{N}$ is closed under direct images, we have that $n_g \in \mathcal{N}$. Therefore every $(id_X)_f$ that occurs in the formation of $m^{J_{\mathcal{N}}(\mathcal{A})}$ is dominated by some $(id_{cY})_{cY}$ that satisfies the condition that there exists a morphism $n \in N_{cY}$ such that $n_{c_Y} \leq m$. Thus, we have that $m^{J_{\mathcal{N}}(\mathcal{A})} \leq sup\,(\{m\} \cup \{(id_{cY})_{cY} : \exists n \in \mathcal{N}_{cY} \text{ with } n_{c_Y} \leq m\})$ This, together with the previous inequality, gives the wanted result. $\square$

Now, following the ideas presented in [CKS$_2$], we would like to show how the two Galois connections $S(\mathcal{X}) \underset{I_{\mathcal{N}}}{\overset{J_{\mathcal{N}}}{\rightleftarrows}} CL(\mathcal{X}, \mathcal{M})$ and $CL(\mathcal{X}, \mathcal{M}) \underset{T_{\mathcal{N}}}{\overset{D_{\mathcal{N}}}{\rightleftarrows}} S(\mathcal{X})^{\mathbf{op}}$ can each be factored via three other Galois connections.

Let $S(\mathcal{M})$ denote the collection of all subclasses of $\mathcal{M}$, ordered by inclusion. We begin with the following result.

PROPOSITION 2.21

Let $\mathcal{N} \subseteq \mathcal{M}$ and let $S(\mathcal{X}) \xrightarrow{H_{\mathcal{N}}} S(\mathcal{M})$ and $S(\mathcal{M}) \xrightarrow{K_{\mathcal{N}}} S(\mathcal{X})$ be defined by:

$$H_{\mathcal{N}}(\mathcal{A}) = \{n \in \mathcal{N}_X : X \in \mathcal{A}\}$$

$$K_{\mathcal{N}}(\mathcal{M}') = \{X \in \mathcal{X} : n \in \mathcal{N}_X \Rightarrow n \in \mathcal{M}'\}.$$

Then $S(\mathcal{X}) \underset{K_{\mathcal{N}}}{\overset{H_{\mathcal{N}}}{\rightleftarrows}} S(\mathcal{M})$ is a Galois connection.

Proof:

Clearly both $H_{\mathcal{N}}$ and $K_{\mathcal{N}}$ are order-preserving.

Now, if $X \in \mathcal{A}$, then every $n \in \mathcal{N}_X$ also belongs to $H_{\mathcal{N}}(\mathcal{A})$. Consequently we have that $X \in (K_{\mathcal{N}} \circ H_{\mathcal{N}})(\mathcal{A})$.

On the other hand, if $n \in (H_{\mathcal{N}} \circ K_{\mathcal{N}})(\mathcal{M}')$, then $n \in \mathcal{N}_X$ with $X \in K_{\mathcal{N}}(\mathcal{M}')$. This implies that $n \in \mathcal{M}'$. □

As a consequence we obtain that $S(\mathcal{M})^{\mathbf{op}} \underset{H_{\mathcal{N}}^{\mathbf{op}}}{\overset{K_{\mathcal{N}}^{\mathbf{op}}}{\rightleftarrows}} S(\mathcal{X})^{\mathbf{op}}$ is also a Galois connection.

We recall the following definition and the next three results from [CKS_2].

DEFINITION 2.22

(1) A subclass $\mathcal{N}$ of $\mathcal{M}$ is called $\mathbf{E}$-*sink stable*, if for every commutative square

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle n} \\ X & \xrightarrow[g]{} & Y \end{array}$$

with $n \in \mathcal{M}$ and the 2-sink $(g, n) \in \mathbf{E}$ we have that $m \in \mathcal{N}$ implies $n \in \mathcal{N}$.

(2) $P_{es}(\mathcal{M})$ denotes the collection of all $\mathbf{E}$-sink stable subclasses of $\mathcal{M}$, ordered by inclusion.

(3) $P_{pb}(\mathcal{M})$ denotes the collection of all pullback-stable subclasses of $\mathcal{M}$, ordered by inclusion.

THEOREM 2.23 [CKS_2, Theorem 2.3]

(1) Let $\mathcal{N} \in P_{pb}(\mathcal{M})$. If for every $\mathcal{M}$-subobject $M \xrightarrow{m} X$, we define:

$$m^{S_{\mathcal{N}}} = \inf\{m' \in \mathcal{N} : M' \xrightarrow{m'} X \text{ and } m \leq m'\}$$

then $S_{\mathcal{N}}$ is an idempotent closure operator with respect to $\mathcal{M}$.

(2) Let $\mathcal{N} \in P_{es}(\mathcal{M})$. If for every $\mathcal{M}$-subobject $M \xrightarrow{m} X$, we define:

$$m^{C_{\mathcal{N}}} = \sup\{(N \xrightarrow{n} X) \in \mathcal{M} : \exists (M \xrightarrow{t} N) \in \mathcal{N} \text{ with } n \circ t = m\}$$

then $C_{\mathcal{N}}$ is a weakly hereditary closure operator with respect to $\mathcal{M}$. □

THEOREM 2.24 [cf. CKS_2, Theorem 2.4]

(1) Let $CL(\mathcal{X}, \mathcal{M}) \xrightarrow{R_*} P_{pb}(\mathcal{M})^{\mathbf{op}}$ and $P_{pb}(\mathcal{M})^{\mathbf{op}} \xrightarrow{R^*} CL(\mathcal{X}, \mathcal{M})$ be defined by:

$R_*(C) = \{m \in \mathcal{M} : m \text{ is } C\text{-closed}\}$

$R^*(\mathcal{N}) = P_{\mathcal{N}}$.

Then, $CL(\mathcal{X}, \mathcal{M}) \underset{R^*}{\overset{R_*}{\rightleftarrows}} P_{pb}(\mathcal{M})^{\mathbf{op}}$ is a Galois connection;

(2) Let $CL(\mathcal{X}, \mathcal{M}) \xrightarrow{K^*} P_{es}(\mathcal{M})$ and $P_{es}(\mathcal{M}) \xrightarrow{K_*} CL(\mathcal{X}, \mathcal{M})$ be defined by:

$K^*(C) = \{m \in \mathcal{M} : m \text{ is } C\text{-dense}\}$

$K_*(\mathcal{N}) = C_{\mathcal{N}}$.

Then, $P_{es}(\mathcal{M}) \underset{K^*}{\overset{K_*}{\rightleftarrows}} CL(\mathcal{X}, \mathcal{M})$ *is a Galois connection.* $\square$

PROPOSITION 2.25 [CKS$_2$, Proposition 2.7]

(1) Let $P_{pb}(\mathcal{M})^{\mathbf{op}} \xrightarrow{Q_*} S(\mathcal{M})^{\mathbf{op}}$ *and* $S(\mathcal{M})^{\mathbf{op}} \xrightarrow{Q^*} P_{pb}(\mathcal{M})^{\mathbf{op}}$ *be defined by:*

$Q_*(\mathcal{M}') = \mathcal{M}'$

$Q^*(\mathcal{M}'') = \{m \in \mathcal{M} : m \text{ is a pullback of some } n \in \mathcal{M}''\}$.

Then, $P_{pb}(\mathcal{M})^{\mathbf{op}} \underset{Q^*}{\overset{Q_*}{\rightleftarrows}} S(\mathcal{M})^{\mathbf{op}}$ *is a Galois connection.*

(2) Let $S(\mathcal{M}) \xrightarrow{L_*} P_{es}(\mathcal{M})$ *and* $P_{es}(\mathcal{M}) \xrightarrow{L^*} S(\mathcal{M})$ *be defined by:*

$L_*(\mathcal{M}') = \{n \in \mathcal{M} : n \circ f = g \circ m' \text{ for some } m' \in \mathcal{M}' \text{ and some } \mathcal{X}\text{-morphisms } f \text{ and } g \text{ with } (g, n) \in \mathbf{E}\}$

$L^*(\mathcal{M}'') = \mathcal{M}''$.

Then, $S(\mathcal{M}) \underset{L^*}{\overset{L_*}{\rightleftarrows}} P_{es}(\mathcal{M})$ *is a Galois connection.* $\square$

Now we can prove the following

THEOREM 2.26

For every $\mathcal{N} \subseteq \mathcal{M}$*, we have the following two commutative diagrams of Galois connections*

$$
\begin{array}{ccc}
S(\mathcal{X}) & \underset{I_{\mathcal{N}}}{\overset{J_{\mathcal{N}}}{\rightleftarrows}} & CL(\mathcal{X}, \mathcal{M}) \\
K_{\mathcal{N}} \uparrow\downarrow H_{\mathcal{N}} & & K_* \uparrow\downarrow K^* \\
S(\mathcal{M}) & \underset{L^*}{\overset{L_*}{\rightleftarrows}} & P_{es}(\mathcal{M})
\end{array}
\qquad
\begin{array}{ccc}
CL(\mathcal{X}, \mathcal{M}) & \underset{T_{\mathcal{N}}}{\overset{D_{\mathcal{N}}}{\rightleftarrows}} & S(\mathcal{X})^{\mathbf{op}} \\
R_* \uparrow\downarrow R^* & & K_{\mathcal{N}}^{\mathbf{op}} \uparrow\downarrow H_{\mathcal{N}}^{\mathbf{op}} \\
P_{pb}(\mathcal{M})^{\mathbf{op}} & \underset{Q^*}{\overset{Q_*}{\rightleftarrows}} & S(\mathcal{M})^{\mathbf{op}}
\end{array}
$$

Proof:

Let us start with the left diagram. Given $C \in CL(\mathcal{X}, \mathcal{M})$, $K^*(C) = \{m \in \mathcal{M} : m \text{ is } C\text{-dense}\}$ and so $L^*(K^*(C)) = K^*(C)$. Thus, $K_{\mathcal{N}}(L^*(K^*(C))) = K_{\mathcal{N}}(K^*(C)) = \{X \in \mathcal{X} : n \in \mathcal{N}_X \Rightarrow n \text{ is } C\text{-dense}\} = I_{\mathcal{N}}(C)$.

Now let $\mathcal{B} \in S(\mathcal{X})$. Then, $H_{\mathcal{N}}(\mathcal{B}) = \{n \in \mathcal{N}_X : X \in \mathcal{B}\}$ and therefore we have that $L_*(H_{\mathcal{N}}(\mathcal{B})) = \{t \in \mathcal{M} : t \circ f = g \circ m' \text{ for some } m' \in \mathcal{N}_X, X \in \mathcal{B} \text{ and } (g, t) \in \mathbf{E}\}$. Now, notice that $I_{\mathcal{N}}(K_*(L_*(H_{\mathcal{N}}(\mathcal{B})))) = (K_{\mathcal{N}} \circ L^* \circ K^*)(K_* \circ L_* \circ H_{\mathcal{N}})(\mathcal{B}) \supseteq \mathcal{B}$. Thus, from Proposition 2.4 we have that $J_{\mathcal{N}}(\mathcal{B}) \sqsubseteq K_*(L_*(H_{\mathcal{N}}(\mathcal{B})))$.

To show that $K_*(l_*(H_{\mathcal{N}}(\mathcal{B}))) \sqsubseteq J_{\mathcal{N}}(\mathcal{B})$, consider the $\mathcal{M}$-subobject $M \xrightarrow{m} Y$. Let $m' \in \mathcal{M}_Y$ be such that there exists $t \in L_*(H_{\mathcal{N}}(\mathcal{B}))$ with $m = m' \circ t$. Therefore we obtain the following commutative diagram

$$\begin{tikzcd}
X \arrow[r, "g"] & M' \arrow[r, "m'"] & Y \\
 & M \arrow[u, "t"'] \arrow[ur, "m"'] & \\
N \arrow[uu, "n"] \arrow[ur, "h"'] \arrow[rr, "e_f"'] & & N_f \arrow[uu, "n_f"']
\end{tikzcd}$$

where $n \in \mathcal{N}$, $(g,t) \in \mathbf{E}$, $X \in \mathcal{B}$ and n_f is the direct image of n along the morphism $f = m' \circ g$. Now, since $m \circ h = n_f \circ e_f$, the $(\mathbf{E}, \mathcal{M})$-diagonalization property yields the existence of a morphism $N_f \xrightarrow{d} M$ such that $m \circ d = n_f$ and $d \circ e_f = h$. This implies that the morphism $X_f \xrightarrow{(id_X)_f} Y$ occurs in the construction of $m^{J_{\mathcal{N}}(\mathcal{B})}$ (cf. Proposition 2.6). Let $X_f \vee M \xrightarrow{\gamma} Y$ be the supremum of $(id_X)_f$ and m, and let i_M and i_{X_f} be the morphisms with codomain $X_f \vee M$ induced by the supremum construction. Notice that $\gamma \leq m^{J_{\mathcal{N}}(\mathcal{B})}$. From the following commutative diagram

$$\begin{tikzcd}
X \arrow[rr, "m' \circ g"] \arrow[dr, "e"'] \arrow[dd, "g"'] & & Y & & X_f \vee M \arrow[ll, "\gamma"'] \\
 & X_f \arrow[ur, "(id_X)_f"] & & M \arrow[ul, "m"'] \arrow[ur, "i_M"'] \arrow[dl, "t"'] & \\
M' \arrow[rr, "id_{M'}"'] & & M' \arrow[uu, "m'"'] & &
\end{tikzcd}$$

it is easily seen that $m' \circ t = \gamma \circ i_M$ and $m' \circ g = (id_X)_f \circ e = \gamma \circ i_{X_f} \circ e$. Again the $(\mathbf{E}, \mathcal{M})$-diagonalization property yields a morphism d' that makes the following diagram commute

$$\begin{tikzcd}
\bullet \arrow[r, "{(g,t)}"] \arrow[d, "{(i_{X_f} \circ e, i_M)}"'] & M' \arrow[d, "m'"] \arrow[dl, "d'"'] \\
X_f \vee M \arrow[r, "\gamma"'] & Y
\end{tikzcd}$$

Notice that the bullet in the above diagram represents the objects X and M. Thus, $m' \leq \gamma \leq m^{J_{\mathcal{N}}(\mathcal{B})}$ and so $K_*(l_*(H_{\mathcal{N}}(\mathcal{B}))) \sqsubseteq J_{\mathcal{N}}(\mathcal{B})$.

Now let us prove the commutativity of the diagram on the right. If $C \in CL(\mathcal{X}, \mathcal{M})$, then $R_*(C) = \{m \in \mathcal{M} : m$ is C-closed$\}$. From the properties of closure operators, $R_*(C)$ is pullback-stable. Consequently, $Q_*(R_*(C)) = R_*(C)$ and $K^{\mathbf{op}}_{\mathcal{N}}(Q_*(R_*(C))) = \{X \in \mathcal{X} : n \in \mathcal{N}_X \Rightarrow n \in R_*(C)\} = \{X \in \mathcal{X} : n \in \mathcal{N}_X \Rightarrow n$ is C-closed$\} = D_{\mathcal{N}}(C)$.

Now let $\mathcal{A} \in S(\mathcal{X})^{\mathbf{op}}$. $H^{\mathbf{op}}_{\mathcal{N}}(\mathcal{A}) = \{n \in \mathcal{N}_Y : Y \in \mathcal{A}\}$. $Q^*(H^{\mathbf{op}}_{\mathcal{N}}(\mathcal{A})) = \{m \in \mathcal{M}_X :$ m is a pullback of some $n \in H^{\mathbf{op}}_{\mathcal{N}}(\mathcal{A})$ along some $\mathcal{X}$-morphism $X \xrightarrow{f} Y\} = \{m \in \mathcal{M}_X :$ m is a pullback of some $n \in \mathcal{N}_Y$ along some $\mathcal{X}$-morphism $X \xrightarrow{f} Y, Y \in \mathcal{A}\}$. Consequently, given the $\mathcal{M}$-subobject $M \xrightarrow{m} X$, we have that $m^{R^*(Q^*(H^{\mathbf{op}}_{\mathcal{N}}(\mathcal{A})))} = inf\{m' \in Q^*(H^{\mathbf{op}}_{\mathcal{N}}(\mathcal{A})) : m \leq m'\} = inf\{m' \in \mathcal{M}_X : m \leq m'$ and $m' = f^{-1}(n), n \in \mathcal{N}_Y, X \xrightarrow{f} Y, Y \in \mathcal{A}\} = m^{T_{\mathcal{N}}(\mathcal{A})}$. This concludes the proof. □

PROPOSITION 2.27

(a) *Let $\mathcal{X}$ have squares and equalizers and let $\mathcal{M}$ contain all regular subobjects. Assume that $\mathcal{N}$ is the class of diagonal morphisms, i.e., morphisms of the form $Y \xrightarrow{\delta_Y} Y \times Y$, with $Y \in \mathcal{X}$, where δ_Y is the equalizer of the projections of the square $Y \times Y$ into Y. Then for every subcategory $\mathcal{A}$ of $\mathcal{X}$ that is closed under squares and $\mathcal{M}$-subobjects, $T_{\mathcal{N}}(\mathcal{A})$ agrees with $S_{\mathcal{A}}$, that is the Salbany closure induced by $\mathcal{A}$.*

(b) *Let $\mathcal{X}$ have squares and equalizers and let $\mathcal{M}$ contain all regular subobjects. If $\mathcal{A}$ is closed under squares and $\mathcal{N}$ consists of all $\mathcal{A}$-regular subobjects, then $T_{\mathcal{N}}(\mathcal{A})$ agrees with $S_{\mathcal{A}}$.*

Proof:

(a). Let $M \xrightarrow{m} X$ be an $\mathcal{M}$-subobject and let $E \xrightarrow{e} X$ be the equalizer of a pair of morphims f, g with codomain $Y \in \mathcal{A}$ such that $f \circ m = g \circ m$. It is easy to see that e is the pullback of the diagonal morphism $Y \xrightarrow{\delta_Y} Y \times Y$ along the morphism $X \xrightarrow{<f,g>} Y \times Y$. Since $m \leq e$, $\mathcal{N}$ contains all diagonal morphisms and $Y \times Y \in \mathcal{A}$, we have that e occurs in the construction of $T_{\mathcal{N}}(\mathcal{A})$. Therefore, $m^{T_{\mathcal{N}}(\mathcal{A})} \leq m^{S_{\mathcal{A}}}$.

Now, let $X \xrightarrow{f} Y \times Y$ be an $\mathcal{X}$-morphism with $Y \times Y \in \mathcal{A}$. Notice that, since $\mathcal{M}$ contains all regular subobjects and $\mathcal{A}$ is closed under $\mathcal{M}$-subobjects, we have that $Y \in \mathcal{A}$. If $m \leq f^{-1}(\delta_Y)$, then since $f^{-1}(\delta_Y) = equ(\pi_1 \circ f, \pi_2 \circ f)$ and $Y \in \mathcal{A}$, we have that $f^{-1}(\delta_Y)$ occurs in the construction of $S_{\mathcal{A}}$. Thus, we have that $m^{S_{\mathcal{A}}} \leq m^{T_{\mathcal{N}}(\mathcal{A})}$ and therefore we can conclude that $T_{\mathcal{N}}(\mathcal{A}) \simeq S_{\mathcal{A}}$.

(b). Let $M \xrightarrow{m} X$ be an $\mathcal{M}$-subobject, let $X \xrightarrow{f} Y$ be an $\mathcal{X}$-morphism with $Y \in \mathcal{A}$ and let $n \in \mathcal{N}_Y$. Since pullbacks of equalizers are equalizers, we have that if $m \leq f^{-1}(n)$ then $f^{-1}(n)$ is the equalizer of a pair of morphisms with codomain in $\mathcal{A}$ that agree on m. Thus, $f^{-1}(n)$ occurs in the construction of $S_{\mathcal{A}}$. This implies that $m^{S_{\mathcal{A}}} \leq m^{T_{\mathcal{N}}(\mathcal{A})}$.

On the other hand, since $\delta_Y \in \mathcal{N}$, exactly as in part a), we can show that $m^{T_{\mathcal{N}}(\mathcal{A})} \leq m^{S_{\mathcal{A}}}$. Consequently $S_{\mathcal{A}} \simeq T_{\mathcal{N}}(\mathcal{A})$. □

3 EXAMPLES

EXAMPLE 3.1

Let $\mathcal{N}$ be the class of all $\mathcal{X}$-isomorphisms.

For every $\mathcal{A} \in S(\mathcal{X})^{\mathbf{op}}$, it follows from Propositions 2.3 and 2.4 that $T_{\mathcal{N}}(\mathcal{A})$ is the indiscrete closure operator and $(I_{\mathcal{N}} \circ T_{\mathcal{N}})(\mathcal{A}) = \mathcal{X}$. Moreover, for every $\mathcal{B} \in S(\mathcal{X})$, $J_{\mathcal{N}}(\mathcal{B})$ is the discrete closure operator and clearly $(D_{\mathcal{N}} \circ J_{\mathcal{N}})(\mathcal{B}) = \mathcal{X}$.

Notice that, although $\mathcal{N}$ does not satisfy the hypotheses of Theorem 2.7, we still have that $\nabla_{\mathcal{N}} = I_{\mathcal{N}} \circ T_{\mathcal{N}}$ and $\Delta_{\mathcal{N}} = D_{\mathcal{N}} \circ J_{\mathcal{N}}$. This is due to the fact that in this case every $\mathcal{X}$-morphism is $\mathcal{N}$-constant.

In what follows, for the category **Top** of topological spaces we will choose as $\mathcal{M}$ the class of all extremal monomorphisms (embeddings). We recall that if **E** is the class of episinks in **Top**, then **Top** is an $(\mathbf{E}, \mathcal{M})$-category. For the category **Grp** of groups and **Ab** of abelian groups we will use the (episink,monomorphism)-factorization structure.

EXAMPLE 3.2 (cf. [CH])

Let $\mathcal{X}$ be the category **Top** and let $\mathcal{N}$ be the class of all extremal monomorphisms with nonempty domain. Notice that since $\mathcal{N}$ contains all singleton monomorphisms (i.e., morphisms with singleton domain), to say that a morphism $X \xrightarrow{f} Y$ is $\mathcal{N}$-constant simply means that $f(X)$ is a singleton.

(a). If C is the closure operator induced by the topology, then the class $D_{\mathcal{N}}(C)$ agrees with the class **Discr** of discrete topological spaces and $\nabla_{\mathcal{N}}(\mathbf{Discr})$ consists of the classical connected topological spaces.

If $M \xrightarrow{m} X$ be an $\mathcal{M}$-subobject of $X \in \mathbf{Top}$, then $M^{T_{\mathcal{N}}(\mathbf{Discr})}$ equals the intersection of all clopen subsets of X containing M. Since $\mathcal{M}$ satisfies the conditions of Theorem 2.7, we have that the class $(I_{\mathcal{N}} \circ T_{\mathcal{N}})(\mathbf{Discr})$ consists of all connected topological spaces.

Now, let $\mathcal{B}$ be the class of all connected topological spaces. From Proposition 2.6, $M^{J_{\mathcal{N}}(\mathcal{B})}$ is the union of M with all connected subsets of X which intersect M and the subcategory of all totally disconnected topological spaces agrees with $(D_{\mathcal{N}} \circ J_{\mathcal{N}})(\mathcal{B})$. Thus, again from Theorem 2.7, connected topological spaces and totally disconnected topological spaces are fixed points of the Galois connection $(\Delta_{\mathcal{N}}, \nabla_{\mathcal{N}})$ of Proposition 2.2.

(b). Let $\mathcal{A} = \mathbf{Top_0} \in S(\mathcal{X})^{\mathbf{op}}$. **Ind** and $\mathbf{Top_0}$ are corresponding fixed points of the Galois connection $(\Delta_{\mathcal{N}}, \nabla_{\mathcal{N}})$ of Proposition 2.2 (cf. [AW]).

Let $M \xrightarrow{m} X$ be an $\mathcal{M}$-subobject of $X \in \mathbf{Top}$ and let $c(M) = \{y \in X : \exists x \in M \textit{ with } \bar{\{x\}} = \bar{\{y\}}\}$. where, $\bar{\{x\}}$ denotes the usual topological closure of $\{x\}$. If $X \xrightarrow{r_0} r_0X$ is the $\mathbf{Top_0}$-

reflection, then $M^{T_{\mathcal{N}}(\mathbf{Top_0})} = c(M) = r_0^{-1} r_0(M)$. Moreover, $M^{T_{\mathcal{N}}(\mathbf{Top_0})} \subseteq b(M)$, where $b(M)$ is the **b**-closure of M (cf. [B], [NW]). $M^{J_{\mathcal{N}}(\mathbf{Ind})}$ is the union of M with all indiscrete subobjects of X which intersect M and $(D_{\mathcal{N}} \circ J_{\mathcal{N}})(\mathbf{Ind}) = \mathbf{Top_0}$.

(c). Let $\mathcal{A} = \mathbf{Top_1} \in S(\mathcal{X})^{\mathbf{op}}$ and let $\mathcal{B}$ be the class of all absolutely connected topological spaces, i.e., $\mathcal{B} = \{X \in \mathbf{Top}$ such that X cannot be decomposed into any disjoint family $\mathcal{L}$ of nonempty closed subsets with $|\mathcal{L}| > 1\}$ (cf. [P_1]). $\mathcal{A}$ and $\mathcal{B}$ are corresponding fixed points of the Galois connection $(\Delta_{\mathcal{N}}, \nabla_{\mathcal{N}})$ of Proposition 2.2. Let $M \xrightarrow{m} X$ be an $\mathcal{M}$-subobject of $X \in \mathbf{Top}$. We have that $M^{S_{\mathbf{Top_1}}} \simeq M^{T_{\mathcal{N}}(\mathbf{Top_1})}$ ([CH, Example 4.3]), i.e., the $T_{\mathcal{N}}(\mathbf{Top_1})$-closure agrees with the Salbany closure induced by $\mathbf{Top_1}$. So, from Theorem 2.7, we have that $\mathcal{B} = I_{\mathcal{N}}(S_{\mathbf{Top_1}})$.

From Proposition 2.6 one can see that $M^{J_{\mathcal{N}}(\mathcal{B})}$ is the union of M with all absolutely connected subsets of X that intersect M. Theorem 2.7 implies that $\mathbf{Top_1} = (D_{\mathcal{N}} \circ J_{\mathcal{N}})(\mathcal{B})$. This can be also easily verified directly.

We observe that, from Proposition 2.15, $\mathbf{Top_0}$, $\mathbf{Top_1}$ and all totally disconnected topological spaces are fixed points of the Galois connection $(D_{\mathcal{N}}, T_{\mathcal{N}})$. Moreover, **Ind**, connected topological spaces and absolutely connected topological spaces are fixed points of the Galois connection $(J_{\mathcal{N}}, I_{\mathcal{N}})$.

EXAMPLE 3.3

Let $\mathcal{X}$ be the category **Top** and let $\mathcal{N}$ be the the class of all singleton monomorphisms.

Let $\mathcal{A} = \mathbf{Top_1} \in S(\mathcal{X})^{\mathbf{op}}$ and let $\mathcal{B}$ be the class of all absolutely connected topological spaces, i.e., $\mathcal{B} = \{X \in \mathbf{Top}$ such that X cannot be decomposed into any disjoint family $\mathcal{L}$ of nonempty closed subsets with $|\mathcal{L}| > 1\}$. Since $\mathcal{N}$-constant in this case simply means constant, $\mathcal{A}$ and $\mathcal{B}$ are corresponding fixed points of the Galois connection $(\Delta_{\mathcal{N}}, \nabla_{\mathcal{N}})$ of Proposition 2.2 (cf. [P_1]). Let $X \in \mathbf{Top_1}$ and let $M \xrightarrow{m} X$ be an $\mathcal{M}$-subobject.First notice that $M^{T_{\mathcal{N}}(\mathcal{A})}$ is closed in the usual topology of X. Now let C be a closed subspace of X and let X_c denote the topological spaces with underlying set $|X|$ endowed with the cofinite topology. Choose $x \in C$ and define $X \xrightarrow{f} X_c$ by $f(C) = \{x\} \subseteq C$ and $f|_{X-C} = id_X|_{X-C}$. Clearly f is continuous and $C = f^{-1}f(C)$. Therefore, $M^{T_{\mathcal{N}}(\mathcal{A})}$ is the intersection of all closed sets containing M, i.e., it is the closure of M in the topology of X. However, notice that if $X \notin \mathbf{Top_1}$, then the $T_{\mathcal{N}}(\mathcal{A})$-closure might be larger then the topological closure of X. For example if $X = \{0, 1\}$ with $\{0\}$ open and $\{1\}$ closed (Sierpinski space) and $M = \{1\}$ then, clearly $M^{T_{\mathcal{N}}(\mathcal{A})} = X$.

Since $\mathcal{N}$ satisfies the conditions of Theorem 2.2, we obtain that $\mathcal{B} = (I_{\mathcal{N}} \circ T_{\mathcal{N}})(\mathbf{Top_1})$. As in Example 3.2(c), $M^{J_{\mathcal{N}}(\mathcal{B})}$ is the union of M with all absolutely connected subsets of X that intersect M.

EXAMPLE 3.4

Let $\mathcal{X}$ be the category **Top** of topological spaces and let $\mathcal{N}$ be the class of all diagonal morphisms. If $\mathcal{A}$ is any epireflective subcategory of **Top**, then it satisfies the hypotheses of Proposition 2.27(a) and so $T_{\mathcal{N}}(\mathcal{A}) = S_{\mathcal{A}}$. Therefore if $\mathcal{A} = \mathbf{Top_0}$, then $T_{\mathcal{N}}(\mathcal{A})$ is the b-closure ([B], [NW]). If $\mathcal{A} = \mathbf{Top_1}$ or any bireflective subcategory of **Top**, then $T_{\mathcal{N}}(\mathcal{A})$ is discrete inside $\mathcal{A}$ ([G]). If $\mathcal{A} = \mathbf{Top_2}$, then $T_{\mathcal{N}}(\mathcal{A})$ agrees with the usual topological closure inside $\mathbf{Top_2}$.

EXAMPLE 3.5

Let $\mathcal{X}$ be the category **Grp** and let $\mathcal{N} = \mathcal{M}$ be the class of all monomorphisms in **Grp**. Clearly, to say that a **Grp**-morphism $X \xrightarrow{f} Y$ is $\mathcal{N}$-constant simply means that the image of X under f is a singleton.

(a). If $\mathcal{A} = \mathbf{Grp} \in S(\mathbf{Grp})^{\mathbf{op}}$, the category **Sng** of singleton groups and **Grp** are corresponding fixed points of the Galois connection $(\Delta_{\mathcal{N}}, \nabla_{\mathcal{N}})$. Now, from Proposition 2.27 b) $T_{\mathcal{N}}(\mathbf{Grp}) = S_{\mathbf{Grp}}$, which is the discrete operator. Actually, in this case, $J_{\mathcal{N}}(\mathbf{Sng}) = T_{\mathcal{N}}(\mathbf{Grp})$. On the other side, if we take $\mathcal{A} = \mathbf{Grp} \in S(\mathbf{Grp})$, then again **Grp** and **Sng** are corresponding fixed points of the same Galois connection. Moreover, $J_{\mathcal{N}}(\mathbf{Grp})$ and $T_{\mathcal{N}}(\mathbf{Sng})$ both agree with the indiscrete operator.

(b). Let $\mathcal{A}$ be the subcategory **Ab** of abelian groups. We have that $S_{\mathbf{Ab}} \simeq T_{\mathcal{N}}(\mathbf{Ab})$ (cf. [CH, Example 4.4]). $\mathcal{N}$ satisfies the hypotheses of Theorem 2.7 and consequently, $\nabla_{\mathcal{N}}(\mathbf{Ab})$ agrees with $I_{\mathcal{N}}(S_{\mathbf{Ab}})$ which is equal to the class of perfect groups, i.e., $X \in \nabla_{\mathcal{N}}(\mathbf{Ab})$ iff $X = X'$, where X' denotes the subgroup generated by the commutators. Thus $M^{J_{\mathcal{N}}(\nabla(\mathbf{Ab}))}$ is the subgroup generated by M and all perfect subgroups of X. and $(D_{\mathcal{N}} \circ J_{\mathcal{N}})(\nabla_{\mathcal{N}}(\mathbf{Ab}))$ is the class of all groups which do not have any non-trivial perfect subgroup.

EXAMPLE 3.6

Let $\mathcal{X}$=**Grp**, let $\mathcal{N}$ be the class of all singleton monomorphisms. Clearly in this case $\mathcal{N}$-constant simply means constant.

(a). $(\mathbf{Sng}, \mathbf{Grp})$ is a pair of corresponding fixed points of $(\Delta_{\mathcal{N}}, \nabla_{\mathcal{N}})$. It is easy to see that $T_{\mathcal{N}}(\mathbf{Grp})$ is the normal closure and $J_{\mathcal{N}}(\mathbf{Sng})$ is the discrete operator. On the other side, if we consider the pair of corresponding fixed points $(\mathbf{Grp}, \mathbf{Sng})$, then both $J_{\mathcal{N}}(\mathbf{Grp})$ and $T_{\mathcal{N}}(\mathbf{Sng})$ agree with the indiscrete operator.

(b). As in Example 3.5(b), the class $\mathcal{B}$ of perfect groups and the class $\mathcal{A}$ that consists of all groups that do not have any non-trivial perfect subgroup form a pair $(\mathcal{B}, \mathcal{A})$ of corresponding fixed points of $(\Delta_{\mathcal{N}}, \nabla_{\mathcal{N}})$. For every $M \leq X$, $m^{T_{\mathcal{N}}(\mathcal{A})}$ is the intersection of all normal subgroups of X containing M such that $X/M \in \mathcal{A}$. Moreover, $M^{J_{\mathcal{N}}(\mathcal{B})}$ is the subgroup generated by M and all perfect subgroups of X.

(c). Part b) can be generalized as follows. Let $\mathcal{A} \in S(\mathbf{Grp})$. If $\mathcal{A}$ is closed under subgroups, then $T_{\mathcal{N}}(\mathcal{A})$ agrees with the $\mathcal{A}$-normal closure (cf. [FJ], [FW]). $\mathcal{B} = (I_{\mathcal{N}} \circ T_{\mathcal{N}})(\mathcal{A})$ consists of all those groups X that do not have any proper normal subgroup N such that $X/N \in \mathcal{A}$. For every subgroup M of Y, $M^{J_{\mathcal{N}}(\mathcal{B})}$ is the subgroup generated by M and by those subgroups S of Y which do not have any proper normal subgroup N such that $S/N \in \mathcal{A}$.

EXAMPLE 3.7

(a). Let $\mathcal{X}$ be the category $\mathbf{Ab}$ and let $\mathcal{N} = \mathcal{M}$ be the class of al monomorphisms in $\mathbf{Ab}$. Let $(\mathcal{T}, \mathcal{F})$ be a torsion theory. Clearly, $\mathcal{T}$ and $\mathcal{F}$ are corresponding fixed points of the Galois connection $(\Delta_{\mathcal{N}}, \nabla_{\mathcal{N}})$ of Proposition 2.2. Let $X \in \mathbf{Ab}$ and let $X \xrightarrow{r_X} rX$ be its $\mathcal{F}$-reflection. For every subobject $M \xrightarrow{m} X$ we have that $M^{T_{\mathcal{N}}(\mathcal{F})} \simeq r_X^{-1}(r_X(M)) \simeq M + Ker(r_X)$. Since $\mathcal{T}$ is closed under quotients, $M^{J_{\mathcal{N}}(\mathcal{T})}$ is the subgroup generated by M and all subgroups $S \leq X$ such that $S \in \mathcal{T}$. In particular, if $(\mathcal{T}, \mathcal{F}) =$ (Torsion,Torsion-free), then $M^{T_{\mathcal{N}}(\mathcal{F})} \simeq M + Tor(X)$, where $Tor(X)$ denotes the torsion subgroup of X. If $(\mathcal{T}, \mathcal{F}) =$ (Divisible,Reduced), then $M^{T_{\mathcal{N}}(\mathcal{F})} \simeq M + Div(X)$, where $Div(X)$ denotes the largest divisible subgroup of X. It is interesting to notice that in both cases, $M^{J_{\mathcal{N}}(\mathcal{T})} = M^{T_{\mathcal{N}}(\mathcal{F})}$ (cf. [CH]).

(b). Now let $\mathcal{N}$ be the class of all inclusions of divisible subgroups. Again $\mathcal{N}$-constant means constant. As above, if $(\mathcal{T}, \mathcal{F})$ is a torsion theory, then $\mathcal{T}$ and $\mathcal{F}$ are corresponding fixed points of the Galois connection $(\Delta_{\mathcal{N}}, \nabla_{\mathcal{N}})$ of Proposition 2.2. If $\mathbf{Red}$ is the subcategory of reduced abelian groups, then for every subgroup $M \xrightarrow{m} X$, $M^{T_{\mathcal{N}}(\mathbf{Red})}$ is the intersection of all subgroups of $\mathcal{X}$ containing M such that X/M is reduced. As it is easily seen, this agrees with the Salbany closure $S_{\mathbf{Red}}$. Moreover, if $\mathbf{Div}$ is the subcategory of divisible abelian groups, then for every subgroup $M \xrightarrow{m} X$, $M^{J_{\mathcal{N}}(\mathbf{Div})} \simeq M + Div(X)$.

(c). If $\mathcal{N}$ is the class of all inclusions of torsion subgroups, then also in this case $\mathcal{N}$-constant means constant. If we consider the torsion theory $(\mathbf{T}, \mathbf{Tf})$ where $\mathbf{T}$ is the subcategory of all torsion abelian groups and $\mathbf{Tf}$ is the subcategory of all torsion free abelian groups, then for every subgroup $M \xrightarrow{m} X$, $M^{T_{\mathcal{N}}(\mathbf{Tf})}$ is the intersection of all subgroups of $\mathcal{X}$ containing M such that X/M is torsion free. As it is easily seen, this agrees with the Salbany closure $S_{\mathbf{Tf}}$. Moreover, for every subgroup $M \xrightarrow{m} X$, $M^{J_{\mathcal{N}}(\mathbf{T})} \simeq M + Tor(X)$.

(d). Let $\mathcal{N}$ consist of all singleton monomorphisms and let $(\mathcal{T}, \mathcal{F})$ be a torsion theory. Then, for every subgroup $M \leq X$, $M^{T_{\mathcal{N}}(\mathcal{F})} = S_{\mathcal{F}}$ and $M^{J_{\mathcal{N}}(\mathcal{T})}$ is the subgroup generated by M and all subgroups $S \leq X$ such that $S \in \mathcal{T}$.

(e). Notice that if $\mathcal{N}$ consists of all diagonal morphisms and $(\mathcal{T}, \mathcal{F})$ is a torsion theory then, from Proposition 2.27(a), for every subgroup $M \leq X$, $M^{T_{\mathcal{N}}(\mathcal{F})} = S_{\mathcal{F}}$.

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Department of Mathematics
University of Puerto Rico
Mayagüez campus
P.O. Box 5000
Mayagüez, PR 00681-5000
U.S.A.