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Connectedness with respect to a Closure Operator

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To Horst Herrlich on his 60th birthday

Abstract. A notion of connectedness with respect to a closure operator C and a class of monomorphisms $\mathcal{N}$ is introduced in an arbitrary category $\mathcal{X}$. It is shown that under appropriate hypotheses, most classical results about topological connectedness can be generalized to this setting. Examples that illustrate this new concept are provided.

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0 INTRODUCTION

The development of a general theory of topological connectedness was started by Preuß ([21]) and by Herrlich ([15]). Afterwards, a considerable number of papers have been publised on this subject and on possible generalizations of it (e.g. [2], [3], [11], [17], [19], [20], [22], [23], [25] and [26]). However, most of these papers used the common approach of first defining a notion of constant morphism and then use it to introduce the notions of connectedness and disconnectedness, accordingly. So did we in [8], [5] and [6]. More precisely, let $\mathcal{X}$ be an arbitrary category with an $(\mathbf{E}, \mathcal{M})$-factorization structure for sinks and let $\mathcal{N} \subseteq \mathcal{M}$. In [5] an $\mathcal{X}$-morphism $X \xrightarrow{f} Y$ was called $\mathcal{N}$-constant if the direct image of X under f was isomorphic to the direct image under f of every $\mathcal{N}$-subobject of X. If $S(\mathcal{X})$ denotes the collection of all subclasses of objects of $\mathcal{X}$, ordered by inclusion, for every $\mathcal{N} \subseteq \mathcal{M}$, the relation: $X \mathcal{R}_{\mathcal{N}} Y$ if and only if every $\mathcal{X}$-morphism $X \xrightarrow{f} Y$ is $\mathcal{N}$-constant yields a Galois connection $S(\mathcal{X}) \underset{\nabla_{\mathcal{N}}}{\overset{\Delta_{\mathcal{N}}}{\rightleftarrows}} S(\mathcal{X})^{\mathbf{op}}$. Again in [5] it was proved that if $\mathcal{N}$ is closed under direct images, we have that this Galois connection factors through $CL(\mathcal{X}, \mathcal{M})$, i.e., the collection of all closure

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operators on $\mathcal{X}$ with respect to $\mathcal{M}$, via two Galois connections $S(\mathcal{X}) \underset{I_{\mathcal{N}}}{\overset{J_{\mathcal{N}}}{\rightleftarrows}} CL(\mathcal{X},\mathcal{M})$ and $CL(\mathcal{X},\mathcal{M}) \underset{T_{\mathcal{N}}}{\overset{D_{\mathcal{N}}}{\rightleftarrows}} S(\mathcal{X})^{\mathbf{op}}$.

Recently in [7] we introduced the following definition: an $\mathcal{X}$-morphism $X \xrightarrow{f} Y$ is called $\mathcal{N}$-fixed if for every $n \in \mathcal{N}_Y$ we have that $f^{-1}(n) \simeq id_Y$. The relation: $X\mathcal{R}'_{\mathcal{N}}Y$ if and only if every $\mathcal{X}$-morphism $X \xrightarrow{f} Y$ is $\mathcal{N}$-fixed yields a Galois connection $S(\mathcal{X}) \underset{\hat{\nabla}_{\mathcal{N}}}{\overset{\hat{\Delta}_{\mathcal{N}}}{\rightleftarrows}} S(\mathcal{X})^{\mathbf{op}}$ that under the assumption of $\mathcal{N}$ being closed under the formation of pullbacks, was shown to factor via the same Galois connections $S(\mathcal{X}) \underset{I_{\mathcal{N}}}{\overset{J_{\mathcal{N}}}{\rightleftarrows}} CL(\mathcal{X},\mathcal{M})$ and $CL(\mathcal{X},\mathcal{M}) \underset{T_{\mathcal{N}}}{\overset{D_{\mathcal{N}}}{\rightleftarrows}} S(\mathcal{X})^{\mathbf{op}}$.

This new discovery lead us to think that this factorization could be the appropriate tool to introduce a notion of connectedness in an arbitrary category and that the two notions of $\mathcal{N}$-constant and $\mathcal{N}$-fixed morphism could be used to provide additional descriptions of it depending on whether the class $\mathcal{N}$ is closed under the formation of direct images or under the formation of pullbacks.

Therefore, in Section 2 we use this idea to introduce a notion of connectedness in an arbitrary category. This notion is given in terms of a closure operator C and a class of monomorphisms $\mathcal{N}$. Some general results about this notion are presented.

In Section 3 we give up part of the generality by assuming the existence of a terminal object and that $\mathcal{N}$ is closed under the formation of direct images. This allows us to obtain results that closely resemble classical properties of topological connectedness.

Finally, Section 4 concludes the paper with a number of examples that illustrate the theory.

We use the terminology of [1] throughout.

1 PRELIMINARIES

Throughout we consider a category $\mathcal{X}$ and a fixed class $\mathcal{M}$ of $\mathcal{X}$-monomorphisms, which contains all $\mathcal{X}$-isomorphisms. It is assumed that $\mathcal{X}$ is $\mathcal{M}$-complete; i.e.,

(1) $\mathcal{M}$ is closed under composition

(2) Pullbacks of $\mathcal{M}$-morphisms exist and belong to $\mathcal{M}$, and multiple pullbacks of (possibly large) families of $\mathcal{M}$-morphisms with common codomain exist and belong to $\mathcal{M}$.

One of the consequences of the above assumptions is that there is a uniquely determined class $\mathbf{E}$ of sinks in $\mathcal{X}$ such that $\mathcal{X}$ is an $(\mathbf{E},\mathcal{M})$-*category for sinks*, that is:

(a) each of $\mathbf{E}$ and $\mathcal{M}$ is closed under compositions with isomorphisms;

(b) $\mathcal{X}$ has $(\mathbf{E},\mathcal{M})$-factorizations (of sinks); i.e., each sink $\mathbf{s}$ in $\mathcal{X}$ has a factorization $\mathbf{s} = m \circ \mathbf{e}$ with $\mathbf{e} \in \mathbf{E}$ and $m \in \mathcal{M}$, and

(c) $\mathcal{X}$ has the unique $(\mathbf{E},\mathcal{M})$-diagonalization property; i.e., if $B \xrightarrow{g} D$ and $C \xrightarrow{m} D$ are $\mathcal{X}$-morphisms with $m \in \mathcal{M}$, and $\mathbf{e} = (A_i \xrightarrow{e_i} B)_I$ and $\mathbf{s} = (A_i \xrightarrow{s_i} C)_I$ are sinks in $\mathcal{X}$ with $\mathbf{e} \in \mathbf{E}$, such that $m \circ \mathbf{s} = g \circ \mathbf{e}$, then there exists a unique diagonal $B \xrightarrow{d} C$ such that for every $i \in I$ the following diagrams commute:

$$\begin{array}{ccc} A_i & \xrightarrow{e_i} & B \\ {\scriptstyle s_i}\downarrow & \swarrow {\scriptstyle d} & \\ C & & \end{array} \qquad \text{and} \qquad \begin{array}{ccc} & & B \\ & {\scriptstyle d}\swarrow & \downarrow{\scriptstyle g} \\ C & \xrightarrow[m]{} & D \end{array}$$

That $\mathcal{X}$ is an $(\mathbf{E}, \mathcal{M})$-category implies the following features of $\mathcal{M}$ and $\mathbf{E}$ (cf. [1] for the dual case):

PROPOSITION 1.1

(0) Every isomorphism is in both $\mathcal{M}$ and $\mathbf{E}$ (as a singleton sink). Moreover, every morphism that is in both $\mathcal{M}$ and $\mathbf{E}$ is an isomorphism.

(1) Every m in $\mathcal{M}$ is a monomorphism.

(2) $\mathcal{M}$ is closed under $\mathcal{M}$-relative first factors, i.e., if $n \circ m \in \mathcal{M}$, and $n \in \mathcal{M}$, then $m \in \mathcal{M}$.

(3) $\mathcal{M}$ is closed under composition.

(4) Pullbacks of $\mathcal{X}$-morphisms in $\mathcal{M}$ exist and belong to $\mathcal{M}$.

(5) The $\mathcal{M}$-subobjects of every $\mathcal{X}$-object form a (possibly large) complete lattice; suprema are formed via $(\mathbf{E}, \mathcal{M})$-factorizations and infima are formed via intersections. □

If $X \xrightarrow{f} Y$ is an $\mathcal{X}$-morphism and $M \xrightarrow{m} X$ is an $\mathcal{M}$-subobject, then $X \xrightarrow{e_{f \circ m}} M_f \xrightarrow{m_f} Y$ will denote the $(\mathbf{E}, \mathcal{M})$-factorization of $f \circ m$. $M_f \xrightarrow{m_f} Y$ will be called the direct image of m along f. If $N \xrightarrow{n} Y$ is an $\mathcal{M}$-subobject, then the pullback $f^{-1}(N) \xrightarrow{f^{-1}(n)} X$ of n along f will be called the inverse image of n along f. Whenever no confusion is likely to arise, to simplify the notation we will denote the morphism $e_{f \circ m}$ simply e_f.

DEFINITION 1.2

A *closure operator* C on $\mathcal{X}$ (with respect to $\mathcal{M}$) is a family $\{(\;)^C_X\}_{X \in \mathcal{X}}$ of functions on the $\mathcal{M}$-subobject lattices of $\mathcal{X}$ with the following properties that hold for each $X \in \mathcal{X}$:

(a) [*expansiveness*] $m \leq (m)^C_X$, for every $\mathcal{M}$-subobject $M \xrightarrow{m} X$;

(b) [*order-preservation*] $m \leq n \Rightarrow (m)^C_X \leq (n)^C_X$ for every pair of $\mathcal{M}$-subobjects of X;

(c) [*morphism-consistency*] If p is the pullback of the $\mathcal{M}$-subobject $M \xrightarrow{m} Y$ along some $\mathcal{X}$-morphism $X \xrightarrow{f} Y$ and q is the pullback of $(m)^C_Y$ along f, then $(p)^C_X \leq q$, i.e., the closure of the inverse image of m is less than or equal to the inverse image of the closure of m.

Condition (a) implies that for every closure operator C on $\mathcal{X}$, every $\mathcal{M}$-subobject $M \xrightarrow{m} X$ has a canonical factorization

$$\begin{array}{ccc} M & \xrightarrow{t} & (M)^C_X \\ & m \searrow & \downarrow (m)^C_X \\ & & X \end{array}$$

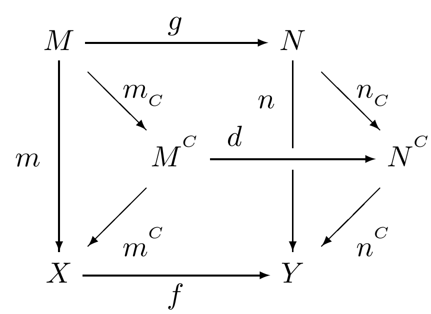
where $((M)^C_X, (m)^C_X)$ is called the *C-closure* of the subobject (M, m).

When no confusion is likely we will write m^C rather than $(m)^C_X$ and for notational symmetry we will denote the morphism t by m_C.

REMARK 1.3

(1) Notice that in the above definition, under condition (b), the morphism-consist-ency condition (c) is equivalent to the following statement concerning direct images: if $M \xrightarrow{m} X$ is an $\mathcal{M}$-subobject and $X \xrightarrow{f} Y$ is a morphism, then $((m)^C_Y)_f \leq (m_f)^C_Y$, i.e., the direct image of the closure of m is less than or equal to the closure of the direct image of m; (cf. [12]).

(2) Under condition (a), both order-preservation and morphism-consistency, i.e., conditions (b) and (c) together are equivalent to the following: given (M, m) and (N, n) $\mathcal{M}$-subobjects of X and Y, respectively, if f and g are morphisms such that $n \circ g = f \circ m$, then there exists a unique morphism d such that the following diagram



commutes.

DEFINITION 1.4

Given a closure operator C, we say that $m \in \mathcal{M}$ is *C-closed* if m_C is an isomorphism. An $\mathcal{X}$-morphism f is called *C-dense* if for every $(\mathbf{E}, \mathcal{M})$-factorization (e, m) of f we have that m^C is an isomorphism. We call C *idempotent* provided that m^C is C-closed for every $m \in \mathcal{M}$. C is called *weakly hereditary* if m_C is C-dense for every $m \in \mathcal{M}$.

Notice that Definition 1.2(c) implies that pullbacks of C-closed $\mathcal{M}$-subobjects are C-closed.

A special case of an idempotent closure operator arises in the following way. Given any class $\mathcal{A}$ of $\mathcal{X}$-objects and $M \xrightarrow{m} X$ in $\mathcal{M}$, define $m^{\mathcal{A}}$ to be the intersection of all equalizers of pairs of $\mathcal{X}$-morphisms r, s from X to some $\mathcal{A}$-object A that satisfy $r \circ m = s \circ m$, and let $m_{\mathcal{A}} \in \mathcal{M}$ be the unique $\mathcal{X}$-morphism by which m factors through $m^{\mathcal{A}}$. It is easy to see that this gives rise to an idempotent closure operator that we will denote by $S_{\mathcal{A}}$. This generalizes the Salbany construction of closure operators induced by classes of topological spaces, cf. [24].

We denote the collection of all closure operators on $\mathcal{M}$ by $\mathbf{CL}(\mathcal{X}, \mathcal{M})$ pre-ordered as follows: $C \sqsubseteq D$ if $m^{C} \leq m^{D}$ for all $m \in \mathcal{M}$ (where $\leq$ is the usual order on subobjects). Notice that arbitrary suprema and infima exist in $\mathbf{CL}(\mathcal{X}, \mathcal{M})$, they are formed pointwise in the $\mathcal{M}$-subobject fibers.

For more background on closure operators see, e.g., [4], [9], [10], [12], [13] and [18]. For a detailed survey on the same topic, one could check [16].

DEFINITION 1.5

For pre-ordered classes $\mathcal{X} = (\mathbf{X}, \sqsubseteq)$ and $\mathcal{Y} = (\mathbf{Y}, \sqsubseteq)$, a *Galois connection* $\mathcal{X} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{Y}$ consists of order preserving functions F and G that satisfy $F \dashv G$, i.e., $x \sqsubseteq GF(x)$ for every $x \in \mathbf{X}$ and $FG(y) \sqsubseteq y$ for every $y \in \mathbf{Y}$. (G is adjoint and has F as coadjoint).

If $x \in \mathcal{X}$ and $y \in \mathcal{Y}$ are such that $F(x) = y$ and $G(y) = x$, then x and y are said to be corresponding fixed points of the Galois connection $(\mathcal{X}, F, G, \mathcal{Y})$.

Properties and many examples of Galois connections can be found in [14].

2 CONNECTEDNESS IN A CATEGORY

The main purpose of this section is to use some already available theory (cf. [5], [6] and [7]) to introduce a notion of connectedness in a category $\mathcal{X}$. This notion is given with respect to a closure operator C on $\mathcal{X}$ and a class of $\mathcal{X}$-monomorphisms $\mathcal{N}$. Moreover, we present some general properties of this new concept, some of which can be directly obtained as a consequence of previous results.

Throughout the paper we will assume that $\mathcal{X}$ is an $(\mathbf{E}, \mathcal{M})$-category for sinks. Unless otherwise specified, C will always denote a closure operator on $\mathcal{X}$ with respect to the given class $\mathcal{M}$ of $\mathcal{X}$-monomorphisms and $\mathcal{N}$ will be a subclass of $\mathcal{M}$.

We begin by recalling the following two propositions from [5].

PROPOSITION 2.1

Let $CL(\mathcal{X}, \mathcal{M}) \xrightarrow{D_{\mathcal{N}}} S(\mathcal{X})^{\mathbf{op}}$ *and* $S(\mathcal{X})^{\mathbf{op}} \xrightarrow{T_{\mathcal{N}}} CL(\mathcal{X}, \mathcal{M})$ *be defined by:*

$$D_{\mathcal{N}}(C) = \{X \in \mathcal{X} : \text{ every } n \in \mathcal{N}_X \text{ is } C\text{-closed}\}$$
$$T_{\mathcal{N}}(\mathcal{A}) = Sup\{C \in CL(\mathcal{X}, \mathcal{M}) : D_{\mathcal{N}}(C) \supseteq \mathcal{A}\}.$$

Then, $CL(\mathcal{X}, \mathcal{M}) \underset{T_{\mathcal{N}}}{\overset{D_{\mathcal{N}}}{\rightleftarrows}} S(\mathcal{X})^{\mathbf{op}}$ *is a Galois connection.* □

PROPOSITION 2.2

Let $CL(\mathcal{X}, \mathcal{M}) \xrightarrow{I_{\mathcal{N}}} S(\mathcal{X})$ *and* $S(\mathcal{X}) \xrightarrow{J_{\mathcal{N}}} CL(\mathcal{X}, \mathcal{M})$ *be defined by:*

$$I_{\mathcal{N}}(C) = \{X \in \mathcal{X} : \text{ every } n \in \mathcal{N}_X \text{ is } C\text{-dense}\}$$
$$J_{\mathcal{N}}(\mathcal{B}) = Inf\{C \in CL(\mathcal{X}, \mathcal{M}) : I_{\mathcal{N}}(C) \supseteq \mathcal{B}\}.$$

Then, $S(\mathcal{X}) \underset{I_{\mathcal{N}}}{\overset{J_{\mathcal{N}}}{\rightleftarrows}} CL(\mathcal{X},\mathcal{M})$ *is a Galois connection.* □

As a consequence of the above propositions, in [7] we gave the following:

DEFINITION 2.3

The Galois connection $S(\mathcal{X}) \underset{I_{\mathcal{N}}\circ T_{\mathcal{N}}}{\overset{D_{\mathcal{N}}\circ J_{\mathcal{N}}}{\rightleftarrows}} S(\mathcal{X})^{\mathbf{op}}$ is called the connectedness-disconnectedness Galois connection.

In [5] we presented some characterizations of the functions $T_{\mathcal{N}}$ and $J_{\mathcal{N}}$. For reference purposes we collect them under the following:

PROPOSITION 2.4

For every $\mathcal{A} \in S(\mathcal{X})^{\mathbf{op}}$ *and* $\mathcal{M}$*-subobject* $M \xrightarrow{m} X$*, with* $X \in \mathcal{X}$*, we have that*

$$m^{T_{\mathcal{N}}(\mathcal{A})} = \cap\{f^{-1}(n) : Y \in \mathcal{A}, X \xrightarrow{f} Y, N \xrightarrow{n} Y \in \mathcal{N}_Y \text{and } m \leq f^{-1}(n)\}.$$

Moreover, for every $\mathcal{B} \in S(\mathcal{X})$ *and* $\mathcal{M}$*-subobject* $M \xrightarrow{m} Y$*, with* $Y \in \mathcal{X}$*, we have that*

$$m^{J_{\mathcal{N}}(\mathcal{B})} = sup\left(\{m\} \cup \{(id_X)_f : X \in \mathcal{B}, X \xrightarrow{f} Y \text{ and } \exists n \in \mathcal{N}_X \text{ with } n_f \leq m\}\right).$$

□

As we already mentioned in [7], the Galois connection $S(\mathcal{X}) \underset{I_{\mathcal{N}}\circ T_{\mathcal{N}}}{\overset{D_{\mathcal{N}}\circ J_{\mathcal{N}}}{\rightleftarrows}} S(\mathcal{X})^{\mathbf{op}}$ can be used to provide a notion of connectedness that does not require a notion of constant morphism. More precisely we can give the following:

DEFINITION 2.5

An $\mathcal{X}$-object X is called $(C,\mathcal{N})$-connected if $X \in I_{\mathcal{N}}(T_{\mathcal{N}}(D_{\mathcal{N}}(C)))$.

In [5, Theorem 2.8] an appropriate description of the connectedness-disconnectedness Galois connection was provided under the hypothesis of $\mathcal{N}$ being closed under the formation of direct images. In [7, Theorem 2.13] a further description of the same Galois connection was presented under the assumption of $\mathcal{N}$ closed under the formation of pullbacks. These theorems can be used to provide alternative descriptions of the notion of $(C,\mathcal{N})$-connectedness under the appropriate closedness conditions as follows.

PROPOSITION 2.6

(a) *If* $\mathcal{N}$ *is closed under the formation of direct images, then an* $\mathcal{X}$*-object* X *is* $(C,\mathcal{N})$*-connected if every morphism* $X \xrightarrow{f} A$ *with* $A \in D_{\mathcal{N}}(C)$ *is* $\mathcal{N}$*-constant; i.e.,* f *factors through* n_f *for every* $n \in \mathcal{N}_X$*.*

(b) *If* $\mathcal{N}$ *is closed under the formation of pullbacks, then an* $\mathcal{X}$*-object* X *is* $(C,\mathcal{N})$*-connected if every morphism* $X \xrightarrow{f} A$ *with* $A \in D_{\mathcal{N}}(C)$ *is* $\mathcal{N}$*-fixed; i.e.,* $f^{-1}(n) \simeq id_X$ *for every* $n \in \mathcal{N}_A$*.* □

As a consequence of Proposition 2.5 of [7], we obtain the following:

PROPOSITION 2.7

Let $\mathcal{N}$ *be closed under the formation of pullbacks along morphisms in* $\mathbf{E}$*. If* $X \xrightarrow{f} Y$ *belongs to* $\mathbf{E}$ *and* X *is* $(C,\mathcal{N})$*-connected, then so is* Y*.* □

COROLLARY 2.8

Let $\mathcal{N}$ be closed under the formation of pullbacks along morphisms in **E** *and let $X \xrightarrow{f} Y$ be an $\mathcal{X}$-morphism. If X is $(C,\mathcal{N})$-connected, so is X_f.* □

REMARK 2.9

Suppose that the category $\mathcal{X}$ has products and that each projection belongs to **E**. Moreover, assume that $\mathcal{N}$ is closed under the formation of pullbacks along morphisms in **E**. Then from Proposition 2.7 we obtain that if the product of a family of $\mathcal{X}$-objects is $(C,\mathcal{N})$-connected, so is each of its factors. However, the converse is not true. As a counterexample, it is enough to consider in the category **Ab** of abelian groups, the subcategory **T** consisting of all torsion abelian groups. Clearly, **Ab** satisfies our assumptions. As Example 4.3(a) shows, this subcategory is the connectedness class of a certain closure operator, but it is not closed under products.

Proposition 2.6 of [7] yields the following:

PROPOSITION 2.10

Let $\mathcal{N}$ be closed under the formation of pullbacks and let $(X_i)_{i\in I}$ be a family of $(C,\mathcal{N})$-connected $\mathcal{X}$-objects. If the coproduct $\amalg X_i$ exists, then it is also $(C,\mathcal{N})$-connected. □

REMARK 2.11

It may be interesting to observe that in the case that $\mathcal{X}$ is well-powered and has coproducts, if $\mathcal{N}$ is closed under the formation of pullbacks, Corollary 2.8 and Proposition 2.10 imply that for any closure operator C, the $(C,\mathcal{N})$-connected objects form an $\mathcal{M}$-coreflective subcategory of $\mathcal{X}$ (cf. [1, Theorem 16.8], dual).

PROPOSITION 2.12

(a) *Let $M \xrightarrow{m} X$ be a C-dense $\mathcal{M}$-subobject of $X \in \mathcal{X}$ and let $\mathcal{N}$ be closed under the formation of pullbacks along morphisms in $\mathcal{M}$. If M is $(C,\mathcal{N})$-connected, then so is X.*

(b) *Let C be weakly hereditary, let $\mathcal{N}$ be closed under the formation of pullbacks along morphisms in $\mathcal{M}$ and let $M \xrightarrow{m} X$ be an $\mathcal{M}$-subobject. If M is $(C,\mathcal{N})$-connected then so is $\overline{M}^{C}$.*

Proof:

(a). We recall from Proposition 2.4 that for $\mathcal{A} \in S(\mathcal{X})^{\mathbf{op}}$ and any $\mathcal{N}$-subobject $N \xrightarrow{n} X$, we have that $n^{T_\mathcal{N}(\mathcal{A})} = \cap\{f^{-1}(p) : A \in \mathcal{A}, X \xrightarrow{f} A, P \xrightarrow{p} A \in \mathcal{N}_A \text{and } n \leq f^{-1}(p)\}$. Now, set $\mathcal{A} = D_\mathcal{N}(C)$. We need to show that $n^{T_\mathcal{N}(\mathcal{A})} \simeq id_X$. From our hypothesis, $m^{-1}(n) \in \mathcal{N}$ and since M is $(C,\mathcal{N})$-connected, we have that $(m^{-1}(n))^{T_\mathcal{N}(\mathcal{A})} \simeq id_M$. By considering the morphisms f and the $\mathcal{N}$-subobjects p that occur in the construction of $n^{T_\mathcal{N}(\mathcal{A})}$, we obtain that $(m^{-1}(n))^{T_\mathcal{N}(\mathcal{A})} \leq \cap\{(f \circ m)^{-1}(p)\} \simeq \cap\{m^{-1}(f^{-1}(p))\} \simeq m^{-1}(\cap\{f^{-1}(p)\})$. Clearly we conclude that $id_M \simeq m^{-1}(\cap\{f^{-1}(p)\})$. Consequently we have that $m \simeq (id_M)_m \simeq (m^{-1}(\cap\{f^{-1}(p)\}))_m \leq \cap\{f^{-1}(p)\} \simeq n^{T_\mathcal{N}(\mathcal{A})}$. Now, by taking the C-closure we obtain $id_X \simeq \overline{m}^{C} \leq \overline{(\cap\{f^{-1}(p)\})}^{C} \simeq \cap\{f^{-1}(p))\} \simeq n^{T_\mathcal{N}(\mathcal{A})}$. Notice that here we have used the fact that each $\mathcal{N}$-subobject of $A \in D_\mathcal{N}(C)$ is C-closed and that the intersection of C-closed subobjects is C-closed. Thus, we conclude that $n^{T_\mathcal{N}(\mathcal{A})} \simeq id_X$, and so X is $(C,\mathcal{N})$-connected.

(b). Just observe that since C is weakly hereditary, m_C is C-dense and apply part (a). $\square$

PROPOSITION 2.13

Assume that $\mathcal{N}$ is closed under the formation of pullbacks along morphisms in $\mathcal{M}$ and let $(M_i \xrightarrow{m_i} X)_{i\in I}$ be a family of $\mathcal{M}$-subobjects of $X \in \mathcal{X}$. If each M_i is $(C,\mathcal{N})$-connected then so is their supremum $\vee M_i$.

Proof:

Again we recall from Proposition 2.4 that for $\mathcal{A} \in S(\mathcal{X})$ and any $\mathcal{N}$-subobject $N \xrightarrow{n} \vee M_i$, we have that $n^{T_{\mathcal{N}}(\mathcal{A})} = \cap\{f^{-1}(p) : A \in \mathcal{A}, \vee M_i \xrightarrow{f} A, P \xrightarrow{p} A \in \mathcal{N}_A \text{and } n \leq f^{-1}(p)\}$. Now, set $\mathcal{A} = D_{\mathcal{N}}(C)$. For each $i \in I$, let $M_i \xrightarrow{t_i} \vee M_i$ be such that $\vee m_i \circ t_i = m_i$. Since $\vee m_i$ and m_i both belong to $\mathcal{M}$, so does t_i, for each $i \in I$ (cf. Proposition 1.1(2)). Consequently from our hypothesis, $t_i^{-1}(n) \in \mathcal{N}$. Since M_i is $(C,\mathcal{N})$-connected for every $i \in I$, by considering the morphisms f and the $\mathcal{N}$-subobjects p that appear in the expression of $n^{T_{\mathcal{N}}(\mathcal{A})}$, we obtain that $id_{M_i} \simeq (t_i^{-1}(n))^{T_{\mathcal{N}}(\mathcal{A})} \leq \cap\{(f \circ t_i)^{-1}(p)\} \simeq \cap\{t_i^{-1}(f^{-1}(p))\} \simeq t_i^{-1}(\cap\{f^{-1}(p)\}) \simeq t_i^{-1}(n^{T_{\mathcal{N}}(\mathcal{A})})$. Consequently we have that $t_i \simeq (id_{M_i})_{t_i} \simeq (t_i^{-1}(n^{T_{\mathcal{N}}(\mathcal{A})}))_{t_i} \leq n^{T_{\mathcal{N}}(\mathcal{A})}$. Thus we have that $t_i \leq n^{T_{\mathcal{N}}(\mathcal{A})}$, for every $i \in I$ and consequently $m_i = \vee m_i \circ t_i \leq \vee m_i \circ n^{T_{\mathcal{N}}(\mathcal{A})}$. The universal property of suprema implies that $\vee m_i \leq \vee m_i \circ n^{T_{\mathcal{N}}(\mathcal{A})}$. Since $\vee m_i \circ n^{T_{\mathcal{N}}(\mathcal{A})} \leq \vee m_i$, we obtain that $\vee m_i \circ n^{T_{\mathcal{N}}(\mathcal{A})} \simeq \vee m_i \simeq \vee m_i \circ id_{\vee M_i}$. The fact that $\vee m_i$ is a monomorphism implies that $n^{T_{\mathcal{N}}(\mathcal{A})} \simeq id_{\vee M_i}$. Thus, $\vee M_i$ is $(C,\mathcal{N})$-connected. $\square$

We conclude this section by observing that Clementino and Tholen ([11]) recently introduced a notion of connectedness with respect to a closure operator on an arbitrary category $\mathcal{X}$. They call an object $X \in \mathcal{X}$ connected with respect to a closure operator C if the diagonal morphism $X \xrightarrow{\delta_X} X \times X$ is C-dense. This definition is quite different from ours and actually we could not find any relationship between the two of them.

3 CONNECTEDNESS IN CATEGORIES WITH A TERMINAL OBJECT

In Example 4.1(a) we obtain the usual notion of connectedness in the category of topological spaces by choosing as closure operator C, the closure induced by the topology and as $\mathcal{N}$, the class of all extremal monomorphisms (embeddings) with nonempty domain. Clearly $\mathcal{N}$ is closed under the formation of direct images but not under the formation of pullbacks along morphisms in $\mathcal{M}$, so Propositions 2.10 and 2.12 do not apply to this case. However, independently of this fact, we know that the results in Proposition 2.12 hold for topological connectedness. Therefore in this section we present some results that hold in the case that $\mathcal{N}$ is closed under the formation of direct images. These results are less general than the ones in the previous section, since here we assume the existence of a terminal object. However, this allows us to generalize most classical results on topological connectedness to arbitrary categories with a terminal object.

Throughout this section we will make the following assumptions:

(a) $\mathcal{X}$ has a terminal object T;

(b) $\mathcal{N}$ is closed under the formation of direct images and any morphism with domain T belongs to $\mathcal{N}$;

(c) T is an $\mathcal{M}$-subobject of every element of $\mathcal{N}$.

Notice that at times we will use the expression X contains T to mean that T is a subobject of X.

LEMMA 3.1

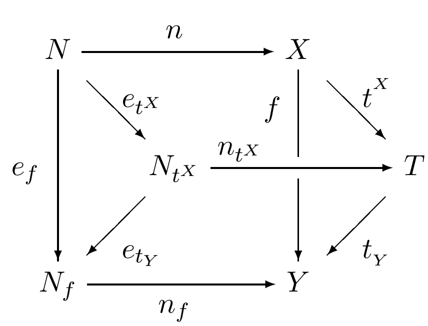
(a) Let X be an $\mathcal{X}$-object that has T as subobject. Then, for every $\mathcal{X}$-object Y, we have that a morphism $X \xrightarrow{f} Y$ is $\mathcal{N}$-constant if and only if f factors through T.

(b) Let Y be an $\mathcal{X}$-object that has T as subobject. Then, for every $\mathcal{X}$-object X, we have that a morphism $X \xrightarrow{f} Y$ is $\mathcal{N}$-fixed if and only if f factors through T.

Proof:

(a). Let $T \xrightarrow{t_X} X$ be an $\mathcal{N}$-subobject of X and let (e_f, n_f) be the $(\mathbf{E}, \mathcal{M})$-factorization of $f \circ t_X$. If f is $\mathcal{N}$-constant, then f factors through n_f for every $n \in \mathcal{N}_X$. Since $t_X \in \mathcal{N}$, there exists a morphism $X \xrightarrow{d} T_f$ such that $n_f \circ d = f$. Now, from the hypothesis on T, $n_f \circ e_f \in \mathcal{M}$, and since $n_f \in \mathcal{M}$, from Proposition 1.1(2), so does e_f. Thus $e_f \in \mathbf{E} \cap \mathcal{M}$ and therefore is an isomorphism. Consequently f factors through T.

Viceversa, suppose that f factors through T, i.e., there exist morphisms $X \xrightarrow{t^X} T$ and $T \xrightarrow{t_Y} Y$ such that $f = t_Y \circ t^X$. For every $n \in \mathcal{N}_X$ let (e_{t^X}, n_{t^X}) and (e_{t_Y}, n_{t_Y}) be the $(\mathbf{E}, \mathcal{M})$-factorizations of $t^X \circ n$ and $t_Y \circ n_{t^X}$, respectively. Since (e_f, n_f) and $(e_{t_Y} \circ e_{t^X}, n_{t_Y})$ are two $(\mathbf{E}, \mathcal{M})$-factorizations of $f \circ n$ and the fact that $(\mathbf{E}, \mathcal{M})$-factorizations of the same morphism are unique up to isomorphism, we obtain the following commutative diagram:



where we have omitted the isomorphism between n_f and n_{t_Y}. Now, the fact that T is an $\mathcal{M}$-subobject of N_{t^X} implies that there is an $\mathcal{M}$-morphism $T \xrightarrow{t} N_{t^X}$. Since T is a terminal object, we have that $n_{t^X} \circ t = id_T$. So, n_{t^X} is a monomorphism and a retraction and consequently an isomorphism. Thus we have that $n_f \circ e_{t_Y} \circ (n_{t^X})^{-1} \circ t^X = t_Y \circ t^X = f$. Hence f factors through n_f, that is, f is $\mathcal{N}$-constant.

(b). Consider the $\mathcal{N}$-subobject $T \xrightarrow{t} Y$ and a morphism h such that $f \circ f^{-1}(t) = t \circ h$. Since f is $\mathcal{N}$-fixed, $f^{-1}(t)$ is an isomorphism and consequently f factors through T.

Viceversa, let k be a morphism such that $f = t_Y \circ k$ with $T \xrightarrow{t_Y} Y$. The universal property of pullbacks implies the existence of a morphism d such that $f^{-1}(t_Y) \circ d = id_X$.

Thus, $f^{-1}(t_Y)$ is a monomorphism and a retraction and so it is actually an isomorphism. Now, if $n \in \mathcal{N}_Y$, then $t_Y \leq n$ implies $id_X \simeq f^{-1}(t_Y) \leq f^{-1}(n)$. Therefore we conclude that $f^{-1}(n) \simeq id_X$; that is, f is $\mathcal{N}$-fixed. □

PROPOSITION 3.2

Let $X \in \mathcal{X}$ and let $M \xrightarrow{m} X$ be an $\mathcal{M}$-subobject of X that contains T.

(a) If m is C-dense and M is $(C,\mathcal{N})$-connected, then so is X.

(b) If C is weakly hereditary and M is $(C,\mathcal{N})$-connected, then so is its C-closure M^C.

Proof:

(a). Consider the morphism $X \xrightarrow{f} A$ with $A \in D_{\mathcal{N}}(C)$. From Proposition 2.6(a), $f \circ m$ is $\mathcal{N}$-constant and from Lemma 3.1(a), $f \circ m$ factors through T, i.e., $f \circ m = t_A \circ t^M$ with $M \xrightarrow{t^M} T$ and $T \xrightarrow{t_A} A$ $\mathcal{A}$-morphisms. By hypothesis, $t_A \in \mathcal{N}_A$ and so is C-closed. Since m is C-dense, the diagonalization property between C-dense and C-closed morphisms (cf. [12, Proposition 3.1]) implies the existence of a morphism $X \xrightarrow{d} T$ such that $d \circ m = t^M$ and $t_A \circ d = f$. Thus, f factors through T and again from Lemma 3.1(a) and Proposition 2.6(a), X is $(C,\mathcal{N})$-connected.

(b). If C is weakly hereditary, then m_C is C-dense and part (a) applies. □

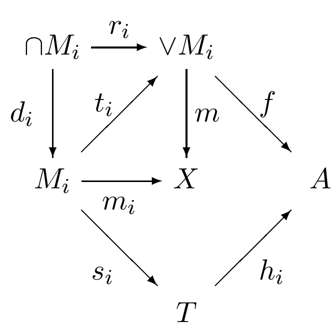
PROPOSITION 3.3

(a) If $(M_i \xrightarrow{m_i} X)_{i \in I}$ is a family of $(C,\mathcal{N})$-connected $\mathcal{M}$-subobjects of $X \in \mathcal{X}$ and $\cap M_i$ has T as subobject, then its supremum $\vee M_i$ is also $(C,\mathcal{N})$-connected.

(b) If $(M_n \xrightarrow{m_n} X)_{n \in \mathbf{N}}$ is a sequence of $(C,\mathcal{N})$-connected subobjects of $X \in \mathcal{X}$ such that $M_{n-1} \cap M_n$ contains the terminal object T for every $n \in \mathbf{N}$, then $\vee M_n$ is $(C,\mathcal{N})$-connected.

Proof:

(a). Let $\vee M_i \xrightarrow{f} A$ be a morphism with $A \in D_{\mathcal{N}}(C)$. Let us consider the following commutative diagram



where d_i, t_i and m are the appropriate subobject morphisms, $r_i = t_i \circ d_i$ for every $i \in I$ and $h_i \circ s_i = f \circ t_i$ is a factorization through T, since M_i is $(C,\mathcal{N})$-connected for every $i \in I$ (cf. Lemma 3.1(a)). Let $\cap M_i \xrightarrow{t} X$ be the morphism that satisfies $m_i \circ d_i = t$ for every $i \in I$. Notice that $m \circ r_i = m \circ t_i \circ d_i = m_i \circ d_i = t$, for every

$i \in I$. This implies that $m \circ r_i = m \circ r_j$ for every $i, j \in I$. Thus $r_i = r_j$, since m is a monomorphism.

By hypothesis there exists a morphism $T \xrightarrow{d} \cap M_i$. Notice that $s_i \circ d_i \circ d = id_T$ and $f \circ r_i = f \circ t_i \circ d_i = h_i \circ s_i \circ d_i$ for all $i \in I$. Since $r_i = r_j$ for all $i, j \in I$, we have that $h_i \circ s_i \circ d_i = h_j \circ s_j \circ d_j$ for all $i, j \in I$, and so $h_i \circ s_i \circ d_i \circ d = h_j \circ s_j \circ d_j \circ d$. Consequently, $h_i \circ id_T = h_j \circ id_T$, for all $i, j \in I$, and so $h_i = h_j$ for all $i, j \in I$. Call this morphism h. Since $(M_i \xrightarrow{t_i} \vee M_i)_{i \in I} \in \mathbf{E}$ and $h \in \mathcal{M}$, from the $(\mathbf{E}, \mathcal{M})$-diagonalization property the unique morphism $\vee M_i \xrightarrow{r} T$ satisfies $h \circ r = f$ and $r \circ t_i = s_i$, for every $i \in I$. Thus, f factors through T and consequently from Lemma 3.1(a) and Proposition 2.6(a), $\vee M_i$ is $(C, \mathcal{N})$-connected.

(b). Consider the sequence $(U_n \xrightarrow{u_n} X)_{n \in \mathbf{N}}$ of $\mathcal{M}$-subobjects of X defined as follows. $U_0 = M_0$ and for $n \geq 1$, $U_n = U_{n-1} \vee M_n$. From part (a), each U_n is $(C, \mathcal{N})$-connected and since $\cap U_n$ contains T, again from part (a) we have that also $\vee U_n$ is $(C, \mathcal{N})$-connected. The fact that $\vee M_n \simeq \vee U_n$ concludes the proof. □

Since the terminal object T is $(C, \mathcal{N})$-connected for any choice of C and $\mathcal{N}$ satisfying our current hypotheses, the result in Proposition 3.3(a) allows us to give the following:

DEFINITION 3.4

If $T \xrightarrow{t} X$ is an $\mathcal{X}$-morphism, then the largest $(C, \mathcal{N})$-connected $\mathcal{M}$-subobject of X that has t as subobject will be called the $(C, \mathcal{N})$-*component* of t in X.

LEMMA 3.5

Let $\mathcal{A}$ be a reflective subcategory of $\mathcal{X}$ and for every $X \in \mathcal{X}$, let $X \xrightarrow{r_X} rX$ denote the corresponding reflection morphism. The following are equivalent:

(a) Any morphism with domain X and codomain in $\mathcal{A}$ factors through T;

(b) r_X factors through T.

Proof:

(a)⇒(b) follows immediately from the fact that $rX \in \mathcal{A}$.

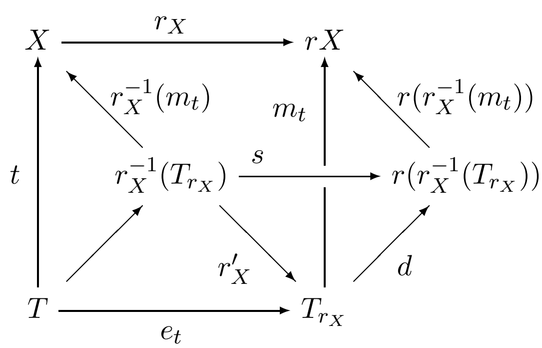
(b)⇒(a). Consider the morphism $X \xrightarrow{f} Y$, with $Y \in \mathcal{A}$. Since $\mathcal{A}$ is reflective in $\mathcal{X}$, we have that there is a morphism g such that $g \circ r_X = f$. By hypothesis r_X factors through T and consequently so does f. □

THEOREM 3.6

Let C be such that $D_{\mathcal{N}}(C)$ is $\mathbf{E}$-reflective in $\mathcal{X}$ such that for every $m \in \mathcal{M}$, $r(m)$ belongs to $\mathcal{M}$. Also assume that $\mathbf{E}$ is closed under the formation of pullbacks along elements of $\mathcal{M}$. Then for any morphism $T \xrightarrow{t} X$, $r_X^{-1}(T_{r_X})$ is the $(C, \mathcal{N})$-component of t.

Proof:

Let us consider the following commutative diagram



where (e_t, m_t) is the $(\mathbf{E}, \mathcal{M})$-factorization of $r_X \circ t$, r'_X is the pullback of r_X along m_t and s denotes the reflection morphism $r_{r_X^{-1}(T_{r_X})}$. Moreover, since our assumptions imply that $r'_X \in \mathbf{E}$ and $r(r_X^{-1}(m_t)) \in \mathcal{M}$, the morphism d is the one induced by the $(\mathbf{E}, \mathcal{M})$-diagonalization property. Consequently d satisfies $r(r_X^{-1}(m_t)) \circ d = m_t$ and $d \circ r'_X = s$. From our hypotheses, $e_t \in \mathbf{E} \cap \mathcal{M}$ and consequently it is an isomorphism. Consequently s factors through T and so, from Lemma 3.5, Lemma 3.1(a) and Proposition 2.6(a), $r_X^{-1}(T_{r_X})$ is $(C, \mathcal{N})$-connected.

Now let us show that $r_X^{-1}(T_{r_X})$ is the largest $(C, \mathcal{N})$-connected subobject of X that contains t. Let $M \xrightarrow{m} X$ be a $(C, \mathcal{N})$-connected subobject of X that contains t, i.e., there is a morphism $T \xrightarrow{\alpha} M$ such that $m \circ \alpha = t$. Then we have that $r_X \circ m = r(m) \circ r_M$. By hypothesis r_M factors through T, i.e., $r_M = t_{rM} \circ t^M$ with morphisms $M \xrightarrow{t^M} T$ and $T \xrightarrow{t_{rM}} rM$. Since $m_t \circ e_t = r_X \circ t = r_X \circ m \circ \alpha = r(m) \circ r_M \circ \alpha$, the $(\mathbf{E}, \mathcal{M})$-diagonalization property yields a morphism h such that $r(m) \circ h = m_t$ and $r_M \circ \alpha = h \circ e_t$. Now we have that $r_M \circ \alpha = t_{r_M} \circ t^M \circ \alpha = t_{r_M} \circ id_T = t_{r_M} \circ e_t^{-1} \circ e_t$. Thus, we have that $h \circ e_t = t_{r_M} \circ e_t^{-1} \circ e_t$. Since e_t is an isomorphism, we conclude that $h = t_{r_M} \circ e_t^{-1}$. Hence we obtain that $m_t \circ e_t \circ t^M = r(m) \circ h \circ e_t \circ t^M = r(m) \circ t_{r_M} \circ e_t^{-1} \circ e_t \circ t^M = r(m) \circ r_M = r_X \circ m$. The universal property of pullbacks implies the existence of a morphism k that in particular satisfies $r_X^{-1}(m_t) \circ k = m$. This concludes the proof. □

REMARK 3.7

Notice that the hypotheses of Theorem 3.6 are often satisfied in concrete categories. In particular, if we look at Example 4.3(a) we see that in the case of the pair (torsion, torsion-free), for every abelian group X, the $(C, \mathcal{N})$-component of $0 \in X$ is the torsion subgroup $Tor(X)$. In the case of the pair (divisible, reduced), the $(C, \mathcal{N})$-component of 0 in X is the largest divisible subgroup of X, $div(X)$. Also notice that if M is a $(C, \mathcal{N})$-connected subgroup of X, since $0 \in M$, we must have that M is contained in the $(C, \mathcal{N})$-component of 0. Consequently, we have that the $(C, \mathcal{N})$-component of 0 in X is the largest $(C, \mathcal{N})$-connected subgroup of X.

PROPOSITION 3.8

If C is weakly hereditary then each $(C, \mathcal{N})$-component is C-closed.

Proof:

Let $X \in \mathcal{X}$ and let $D \xrightarrow{m} X$ be a $(C, \mathcal{N})$-component in X. Let us consider the canonical factorization $m = m^C \circ m_C$ induced by the C-closure. D is $(C, \mathcal{N})$-connected and from Proposition 3.2(b) so is D^C. By the maximality of $(C, \mathcal{N})$-components, we have that $D \simeq D^C$. Thus D is C-closed. □

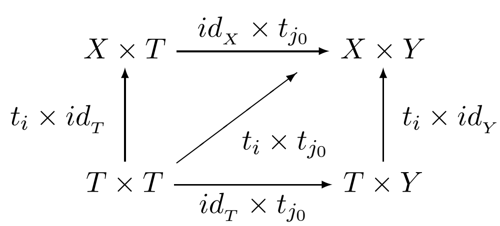
PROPOSITION 3.9

Let X and Y be two $(C,\mathcal{N})$-connected $\mathcal{X}$-objects that contain the terminal object T. Consider the family $(T \xrightarrow{t_i} X)_{i\in I}$ that consists of all morphisms with domain T and codomain X and the family $(T \xrightarrow{t_j} Y)_{j\in J}$ of all morphisms with domain T and codomain Y. Then $\vee\{(X \times T \xrightarrow{id_X \times t_i} X \times Y)_{i\in I} \cup (T \times Y \xrightarrow{t_j \times id_Y} X \times Y)_{j\in J}\}$ is $(C,\mathcal{N})$-connected.

Proof:

First of all we observe that since $X \simeq X \times T$ and $Y \simeq T \times Y$, we have that both $X \times T$ and $T \times Y$ are $(C,\mathcal{N})$-connected.

By assumption on X and Y there exist morphisms $T \xrightarrow{t_{i_0}} X$ and $T \xrightarrow{t_{j_0}} Y$. For every morphism $T \xrightarrow{t_i} X$, the following commutative diagram



shows that $T \simeq T \times T \xrightarrow{t_i \times t_{j_0}} X \times Y$ is a subobject of both $X \times T \xrightarrow{id_X \times t_{j_0}} X \times Y$ and $T \times Y \xrightarrow{t_i \times id_Y} X \times Y$. Therefore, form Proposition 3.3(a), their supremum is $(C,\mathcal{N})$-connected. Let us denote this supremum by $M_i \xrightarrow{m_i} X \times Y$. Now, $X \times T \xrightarrow{id_X \times t_{j_0}} X \times Y$ contains the terminal object $T \times T \xrightarrow{t_{i_0} \times t_{j_0}} X \times Y$ and is a subobject of each M_i. Thus, again from Proposition 3.3(a), we have that $\vee M_i$ is $(C,\mathcal{N})$-connected.

Similarly we can start with a morphism $T \xrightarrow{t_{i_0}} X$ and for every morphism $T \xrightarrow{t_j} Y$ construct $N_j = (T \times Y \xrightarrow{t_{i_0} \times id_Y} X \times Y) \vee (X \times T \xrightarrow{id_X \times t_j} X \times Y)$. As above we obtain that $\vee N_j$ is $(C,\mathcal{N})$-connected. Since $T \times T \xrightarrow{t_{i_0} \times t_{j_0}} X \times Y$ is a subobject of both $\vee M_i$ and $\vee N_j$ we obtain that $\vee\{\vee M_i, \vee N_j\}$ is $(C,\mathcal{N})$-connected. The fact that $\vee\{\vee M_i, \vee N_j\} \simeq \vee\{(X \times T \xrightarrow{id_X \times t_i} X \times Y)_{i\in I} \cup (T \times Y \xrightarrow{t_j \times id_Y} X \times Y)_{j\in J}\}$ concludes the proof. □

REMARK 3.10

Notice that in many concrete categories such as **Top** and **Grp** for instance, the supremum in the above proposition actually agrees with $X \times Y$. Therefore in such cases we obtain that the finite product of non-empty $(C,\mathcal{N})$-connected objects is $(C,\mathcal{N})$-connected. However, we have already observed in Remark 2.9 that this result does not hold for the product of an arbitrary family of $(C,\mathcal{N})$-connected objects.

We conclude by observing that if $X \xrightarrow{f} Y$ is an $\mathcal{X}$-morphism and X contains T, then from our assumptions, so does Y. Consequently, by using Lemma 3.1(b) and Proposition 2.6(b), all the results presented in this section can be proved under the assumption of $\mathcal{N}$ closed under the formation of pullbacks. This allows us to apply these results to a wider variety of cases.

4 EXAMPLES

In this section, we will present some examples to illustrate the above theory.

In what follows, for the category **Top** of topological spaces and continuous functions, we will choose as $\mathcal{M}$ the class of all extremal monomorphisms (embeddings). We recall that if **E** is the class of episinks in **Top**, then **Top** is an $(\mathbf{E}, \mathcal{M})$-category. For the category **Grp** of groups and **Ab** of abelian groups and corresponding homomorphisms we will use the (episink,monomorphism)-factorization structure.

EXAMPLE 4.1 (cf. [8])

Let $\mathcal{X}$ be the category **Top** and let $\mathcal{N}$ be the class of all extremal monomorphisms with nonempty domain. Notice that the class of morphisms $\mathcal{N}$ is closed under the formation of direct images, therefore in this case $I_{\mathcal{N}}(T_{\mathcal{N}}(D_{\mathcal{N}}(C)))$ can be described using Proposition 2.6(a). It is also easy to verify that since $\mathcal{N}$ contains all singleton monomorphisms (i.e., morphisms with singleton domain), then the $\mathcal{N}$-constant morphisms are simply the constant functions.

(a). If C is the closure operator induced by the topology, then the class $D_{\mathcal{N}}(C)$ agrees with the class **Discr** of discrete topological spaces and consequently from the above observation the $(C,\mathcal{N})$-connected objects are exactly the classical connected topological spaces.

(b). Let C be the closure operator that to each $\mathcal{M}$-subobject $M \xrightarrow{m} X$ of $X \in \mathbf{Top}$ associates the intersection of all clopen subsets of X containing M. Notice that $C = T_{\mathcal{N}}(\mathbf{Discr})$. It is easy to see directly that $D_{\mathcal{N}}(C)$ consists of all discrete topological spaces and therefore also in this case the $(C,\mathcal{N})$-connected objects turn out to be exactly the classical connected topological spaces.

(c). Let C be the closure operator that to each $\mathcal{M}$-subobject $M \xrightarrow{m} X$ of $X \in \mathbf{Top}$ associates the union of M with all connected subsets of X that intersect M. We have that $D_{\mathcal{N}}(C)$ agrees with the subcategory of all totally disconnected topological spaces. Notice that from Proposition 2.6(a) we obtain that the $(C,\mathcal{N})$-connected objects are exactly the classical connected topological spaces.

(d). Let $\mathcal{B} = \mathbf{Top_0}$, the subcategory of T_0 topological spaces and let C be the closure operator that to each $\mathcal{M}$-subobject $M \xrightarrow{m} X$ of $X \in \mathbf{Top}$ associates $c(M) = \{y \in X : \exists x \in M \textit{ with } \bar{\{x\}} = \bar{\{y\}}\}$, where $\bar{\{x\}}$ denotes the usual topological closure of $\{x\}$. Notice that if $X \xrightarrow{r_0} r_0X$ is the $\mathbf{Top_0}$-reflection, then $M^{T_{\mathcal{N}}(\mathbf{Top_0})} = c(M) = r_0^{-1}r_0(M)$. In [2] it was shown that the class **IND** of indiscrete topological spaces and $\mathbf{Top_0}$ are corresponding fixed points of the connectedness-disconnectedness Galois connection (cf. [7, Definition 2.3]). Consequently, from [6, Proposition 2.15], we obtain that $\mathbf{Top_0}$ is also a fixed point of the Galois connection in Proposition 2.1. Consequently, $\mathbf{Top_0} = D_{\mathcal{N}}(C)$ and using again Proposition 2.6(a), we obtain that the $(C,\mathcal{N})$-connected objects are exactly the indiscrete topological spaces.

(e). Let C be the closure operator that to each $\mathcal{M}$-subobject $M \xrightarrow{m} X$ of $X \in \mathbf{Top}$ associates the union of M with all indiscrete subobjects of X that intersect M. We have that $D_{\mathcal{N}}(C) = \mathbf{Top_0}$. Again from the observations in part (d) we obtain that the $(C,\mathcal{N})$-connected objects are exactly the indiscrete topological spaces.

(f). Let $\mathcal{B} = \mathbf{Top_1}$ be the subcategory of all T_1 topological spaces. Let $M \xrightarrow{m} X$ be an $\mathcal{M}$-subobject of $X \in \mathbf{Top}$. We have that $M^{S_{\mathbf{Top_1}}} \simeq M^{T_{\mathcal{N}}(\mathbf{Top_1})}$ ([8, Example 4.3]), i.e., the $T_{\mathcal{N}}(\mathbf{Top_1})$-closure agrees with the Salbany closure induced by $\mathbf{Top_1}$. Again from [6, Proposition 2.15] we have that $\mathbf{Top_1}$ is also a fixed point of the Galois connection in Proposition 2.1. Using again Proposition 2.6(a), we have that if $C =$

$S_{\mathbf{TOP_1}}$, then the class of $(C,\mathcal{N})$-connected objects consists of all absolutely connected topological spaces, i.e., $\mathcal{B} = \{X \in \mathbf{Top}$ such that X cannot be decomposed into any disjoint family $\mathcal{L}$ of nonempty closed subsets with $|\mathcal{L}| > 1\}$ (cf. [21]).

(g). Let C be the closure operator that to each $\mathcal{M}$-subobject $M \xrightarrow{m} X$ of $X \in \mathbf{Top}$ associates the union of M with all absolutely connected subobjects of X that intersect M. We have that $D_{\mathcal{N}}(C) = \mathbf{Top_1}$. Again from [21] we obtain that the $(C,\mathcal{N})$-connected objects are the absolutely connected topological spaces.

EXAMPLE 4.2

Let $\mathcal{X}$ be the category $\mathbf{Grp}$ and let $\mathcal{N} = \mathcal{M}$ be the class of all monomorphisms in $\mathbf{Grp}$. Clearly, $\mathcal{N}$ is closed under the formation of direct images and to say that a $\mathbf{Grp}$-morphism $X \xrightarrow{f} Y$ is $\mathcal{N}$-constant simply means that the image of X under f is a singleton.

(a). Let $\mathcal{A}$ be the subcategory $\mathbf{Ab}$ of abelian groups. We have that $S_{\mathbf{Ab}} \simeq T_{\mathcal{N}}(\mathbf{Ab})$ (cf. [8, Example 4.4]). Moreover, if $X \in D_{\mathcal{N}}(S_{\mathbf{Ab}})$, then its subobject $\{0\}$ is $S_{\mathbf{Ab}}$-closed. This means that there exist two homomorphisms $X \overset{f}{\underset{g}{\rightrightarrows}} A$ with $A \in \mathbf{Ab}$ such that $\{0\} = equ(f,g) = Ker(f-g)$ (cf. [4, Proposition 1.6]). Consequently the morphism $X \xrightarrow{f-g} A$ is a monomorphism and therefore $X \in \mathbf{Ab}$. Thus we conclude that $\mathbf{Ab} = D_{\mathcal{N}}(S_{\mathbf{Ab}})$. From Proposition 2.6(a) we obtain that the $(S_{\mathbf{Ab}},\mathcal{N})$-connected objects are exactly the perfect groups. We recall that a group X is perfect iff $X = X'$, where X' denotes the subgroup generated by the commutators.

(b). Let C be the closure operator that to each $\mathcal{M}$-subobject $M \xrightarrow{m} X$ of $X \in \mathbf{Grp}$ associates the subgroup generated by M and all perfect subgroups of X. We have that $D_{\mathcal{N}}(C)$ is the class of all groups which do not have any non-trivial perfect subgroup. Since this subcategory and the one of perfect groups are corrresponding fixed points of the connectedness-disconnectedness Galois connection (cf. [8, Example 4.4]), we obtain that perfect groups are exactly the $(C,\mathcal{N})$-connected objects.

Let $\mathcal{N}$ be the class of all singleton monomorphisms. Clearly, $\mathcal{N}$ is closed under the formation of direct images and also in this case $\mathcal{N}$-constant simply means constant.

(c). As in part (b), the class $\mathcal{B}$ of perfect groups and the class $\mathcal{A}$ that consists of all groups that do not have any non-trivial perfect subgroup form a pair $(\mathcal{B},\mathcal{A})$ of corresponding fixed points of the connectedness-disconnectedness Galois connection. For every $M \leq X$, $m^{T_{\mathcal{N}}(\mathcal{A})}$ is the intersection of all normal subgroups of X containing M such that $X/M \in \mathcal{A}$. Since $\mathcal{A} = D_{\mathcal{N}}(T_{\mathcal{N}}(\mathcal{A}))$, we have that the $(T_{\mathcal{N}}(\mathcal{A}),\mathcal{N})$-connected groups are precisely the perfect groups.

Let $\mathcal{N}$ consist of all inclusions of normal subgroups. Clearly $\mathcal{N}$ is closed under the formation of pullbacks but not under the formation of direct images. Now, Proposition 2.6(b) can be used. Notice that in this case, $X \xrightarrow{f} Y$ is $\mathcal{N}$-fixed if and only if f is constant in the classical sense.

(d). Let $\mathbf{Sim}$ denote the subcategory of simple groups, i.e., all those groups that have no nontrivial normal subgroups and let $\mathbf{Simfree}$ denote the subcategory of all groups that have no simple subgroup different from zero. Using Proposition 2.4 it is easy to see that for every subgroup $M \leq Y$, $M^{J_{\mathcal{N}}(\mathbf{Sim})}$ is the subgroup generated by M and all simple subgroups of Y. It was proved in [7, Example 2.24(a)] that $D_{\mathcal{N}}(J_{\mathcal{N}}(\mathbf{Sim})) = \mathbf{Simfree}$.

Let $\mathbf{Simquo}$ consist of all groups X such that if K is a normal subgroup of X, then X/K has a simple subgroup different from zero. It was also shown in [7] that

Simquo $= I_{\mathcal{N}}(T_{\mathcal{N}}(\textbf{Simfree}))$. Consequently if $C = J_{\mathcal{N}}(\textbf{Sim})$, then **Simquo** is the class of all $(C,\mathcal{N})$-connected objects.

EXAMPLE 4.3

Let $\mathcal{X}$ be the category **Ab** and let $\mathcal{N} = \mathcal{M}$ be the class of all monomorphisms in **Ab**. $\mathcal{N}$ is closed under the formation of direct images and $\mathcal{N}$-constant means constant.

(a). Let $(\mathcal{T},\mathcal{F})$ be a torsion theory in **Ab**. Clearly, $\mathcal{T}$ and $\mathcal{F}$ are corresponding fixed points of the connectedness-disconnectedness Galois connection. Let $X \in \textbf{Ab}$ and let $X \xrightarrow{r_X} rX$ be its $\mathcal{F}$-reflection. For every subobject $M \xrightarrow{m} X$ we have that $M^{T_{\mathcal{N}}(\mathcal{F})} \simeq r_X^{-1}(r_X(M)) \simeq M + Ker(r_X)$. In particular, if $(\mathcal{T},\mathcal{F}) =$ (Torsion,Torsion-free), then $M^{T_{\mathcal{N}}(\mathcal{F})} \simeq M + Tor(X)$, where $Tor(X)$ denotes the torsion subgroup of X. Since $\mathcal{F} = D_{\mathcal{N}}(T_{\mathcal{N}}(\mathcal{F}))$, we have that the $(T_{\mathcal{N}}(\mathcal{F}),\mathcal{N})$-connected objects are precisely the torsion abelian groups. If $(\mathcal{T},\mathcal{F}) =$ (Divisible,Reduced), then $M^{T_{\mathcal{N}}(\mathcal{F})} \simeq M + Div(X)$, where $Div(X)$ denotes the largest divisible subgroup of X. Since also in this case $\mathcal{F} = D_{\mathcal{N}}(T_{\mathcal{N}}(\mathcal{F}))$, we have that the $(T_{\mathcal{N}}(\mathcal{F}),\mathcal{N})$-connected objects are precisely the divisible abelian groups.

Let $\mathcal{N}$ be the class of all inclusions of divisible subgroups. $\mathcal{N}$ is closed under the formation of direct images and again $\mathcal{N}$-constant means constant.

(b). As above, if $(\mathcal{T},\mathcal{F})$ is a torsion theory, then $\mathcal{T}$ and $\mathcal{F}$ are corresponding fixed points of the connectedness-disconnectedness Galois connection. If **Red** is the subcategory of reduced abelian groups, then for every subgroup $M \xrightarrow{m} X$, $M^{T_{\mathcal{N}}(\textbf{Red})}$ is the intersection of all subgroups of $\mathcal{X}$ containing M such that X/M is reduced. As it is easily seen, this agrees with the Salbany closure $S_{\textbf{Red}}$. Obviously we have that $\textbf{Red} \subseteq D_{\mathcal{N}}(T_{\mathcal{N}}(\textbf{Red})) = D_{\mathcal{N}}(S_{\textbf{Red}})$. Now let $X \in D_{\mathcal{N}}(S_{\textbf{Red}})$. The fact that $\{0\}$ is $S_{\textbf{Red}}$-closed in X implies that there exist two morphisms $X \overset{f}{\underset{g}{\rightrightarrows}} Y$ with $Y \in \textbf{Red}$ such that $\{0\} = equ(f,g) = Ker(f-g)$ (cf. [4, Proposition 1.6]). Consequently $f-g$ is a monomorphism and so X is reduced since it is isomorphic to a subgroup of Y. Therefore, $\textbf{Red} = D_{\mathcal{N}}(T_{\mathcal{N}}(\textbf{Red}))$ and consequently **Div** is the class of $(S_{\textbf{Red}},\mathcal{N})$-connected objects.

Let $\mathcal{N}$ is the class of all inclusions of torsion subgroups. $\mathcal{N}$ is closed under the formation of direct images and again $\mathcal{N}$-constant means constant.

(c). If we consider the torsion theory $(\textbf{T},\textbf{Tf})$ where **T** is the subcategory of all torsion abelian groups and **Tf** is the subcategory of all torsion free abelian groups, then for every subgroup $M \xrightarrow{m} X$, $M^{T_{\mathcal{N}}(\textbf{Tf})}$ is the intersection of all subgroups of $\mathcal{X}$ containing M such that X/M is torsion free. As it is easily seen, this agrees with the Salbany closure $S_{\textbf{Tf}}$. A similar arguement used in part (b) shows that $\textbf{Tf} = D_{\mathcal{N}}(T_{\mathcal{N}}(\textbf{Tf}))$. Consequently **T** is the class of $(S_{\textbf{Tf}},\mathcal{N})$-connected objects.

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