CONNECTEDNESS CLASSES

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ABSTRACT: Let $\mathcal{X}$ be an $(\mathbf{E},\mathcal{M})$-category for sinks. A recently introduced notion of connectedness with respect to a closure operator C on $\mathcal{X}$ and to a class of $\mathcal{X}$-monomorphisms $\mathcal{N}$ is further analyzed. The notion of $\mathcal{N}$-connectedness hull of a class of $\mathcal{X}$-objects is introduced and a characterization of it is presented under the assumption of $\mathcal{N}$ being closed under the formation of pullbacks. Moreover, a characterization of the related notion of $\mathcal{N}$-connectedness class is presented under the assumption that $\mathcal{X}$ contains a terminal object. Some examples are provided.

KEY WORDS: Closure operator, connectedness, Galois connection, terminal object.

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0 INTRODUCTION

The development of a general theory about topological connectedness was started by Preuß (cf. [Pr_1]) and by Herrlich ([H]). Further literature on this topic can be found in [AW], [CC], [Cl], [CT], [HP], [L], [Pr_2], [Pr_3], [T] and [SV].

Let $\mathcal{X}$ be an arbitrary category with an $(\mathbf{E},\mathcal{M})$-factorization structure for sinks and let $\mathcal{N}\subseteq\mathcal{M}$. An $\mathcal{X}$-morphism $X\xrightarrow{f}Y$ is called $\mathcal{N}$-dependent if for any $\mathcal{N}$-subobject n of X and any $\mathcal{N}$-subobject p of Y, $n_f\leq p$ implies $f^{-1}(p)\simeq id_X$ (where n_f is the direct image of n along f and $f^{-1}(p)$ is the pullback of p along f [see §1]). Let $S(\mathcal{X})$ denote the collection of all subclasses of objects of $\mathcal{X}$, ordered by inclusion. For every $\mathcal{N}\subseteq\mathcal{M}$, the relation: $X\mathcal{R}_{\mathcal{N}}Y$ if and only if every $\mathcal{X}$-morphism $X\xrightarrow{f}Y$ is $\mathcal{N}$-dependent yields a Galois connection $S(\mathcal{X})\underset{\nabla'_{\mathcal{N}}}{\overset{\Delta'_{\mathcal{N}}}{\rightleftarrows}}S(\mathcal{X})^{\mathbf{op}}$. It is proved that this Galois connection factors through $CL(\mathcal{X},\mathcal{M})$, i.e., the collection of all closure operators on $\mathcal{X}$ with respect to $\mathcal{M}$, via two previously introduced Galois connections $S(\mathcal{X})\underset{I_{\mathcal{N}}}{\overset{J_{\mathcal{N}}}{\rightleftarrows}}CL(\mathcal{X},\mathcal{M})$ and $CL(\mathcal{X},\mathcal{M})\underset{T_{\mathcal{N}}}{\overset{D_{\mathcal{N}}}{\rightleftarrows}}S(\mathcal{X})^{\mathbf{op}}$.

Notice that the notion of $\mathcal{N}$-constant morphism introduced in [C_2] and the notion of $\mathcal{N}$-fixed morphism introduced in [C_4] yielded similar factorizations under the assumption of $\mathcal{N}$ being closed under the formation of pullbacks in the first case or direct images in the second case. The advantage of the notion of $\mathcal{N}$-dependent morphism is that neither of these two closure assumptions on $\mathcal{N}$ are needed to obtain the above factorization.

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The notion of $(C,\mathcal{N})$-connectedness presented in [C_5] is used to introduce the concept of $\mathcal{N}$-connectedness class and the one of $\mathcal{N}$-connectedness hull of a subcategory $\mathcal{A}$ of $\mathcal{X}$. A characterization of this last notion is presented in section 2, under the assumption of $\mathcal{N}$ being closed under the formation of pullbacks.

In section 3, the assumption of the existence of a terminal object in $\mathcal{X}$ is added. This allowed us to obtain a characterization of $\mathcal{N}$-connectedness classes that in the category of topological spaces yields as a special case the one given by Arhangel'skii and Wiegandt, [AW, Theorem 3.10].

The paper ends with some examples that illustrate the Galois conection $S(\mathcal{X}) \underset{\nabla'_{\mathcal{N}}}{\overset{\Delta'_{\mathcal{N}}}{\rightleftarrows}} S(\mathcal{X})^{\mathbf{op}}$ in familiar categories. The examples are based on some choices of $\mathcal{N}$ for which $\mathcal{N}$-dependent does not mean constant in the classical sense.

We use the terminology of [AHS] throughout the paper.

1 PRELIMINARIES

Throughout we consider a category $\mathcal{X}$ and a fixed class $\mathcal{M}$ of $\mathcal{X}$-monomorphisms, which contains all $\mathcal{X}$-isomorphisms. It is assumed that $\mathcal{X}$ is $\mathcal{M}$-complete; i.e.,

(1) $\mathcal{M}$ is closed under composition

(2) Pullbacks of $\mathcal{M}$-morphisms exist and belong to $\mathcal{M}$, and multiple pullbacks of (possibly large) families of $\mathcal{M}$-morphisms with common codomain exist and belong to $\mathcal{M}$.

One of the consequences of the above assumptions is that there is a uniquely determined class $\mathbf{E}$ of sinks in $\mathcal{X}$ such that $\mathcal{X}$ is an $(\mathbf{E},\mathcal{M})$-*category for sinks*. This implies the following features of $\mathcal{M}$ and $\mathbf{E}$ (cf. [AHS] for the dual case):

PROPOSITION 1.1

(0) Every isomorphism is in both $\mathcal{M}$ and $\mathbf{E}$ (as a singleton sink).

(1) Every m in $\mathcal{M}$ is a monomorphism.

(2) $\mathcal{M}$ is closed under $\mathcal{M}$-relative first factors, i.e., if $n \circ m \in \mathcal{M}$, and $n \in \mathcal{M}$, then $m \in \mathcal{M}$.

(3) $\mathcal{M}$ is closed under composition.

(4) Pullbacks of $\mathcal{X}$-morphisms in $\mathcal{M}$ exist and belong to $\mathcal{M}$.

(5) The $\mathcal{M}$-subobjects of every $\mathcal{X}$-object form a (possibly large) complete lattice; suprema are formed via $(\mathbf{E},\mathcal{M})$-factorizations and infima are formed via intersections. □

If $X \xrightarrow{f} Y$ is an $\mathcal{X}$-morphism and $M \xrightarrow{m} X$ is an $\mathcal{M}$-subobject, then $M \xrightarrow{e_{f\circ m}} M_f \xrightarrow{m_f} Y$ will denote the $(\mathbf{E},\mathcal{M})$-factorization of $f \circ m$. $M_f \xrightarrow{m_f} Y$ will be called the direct image of m

along f. If $N \xrightarrow{n} Y$ is an $\mathcal{M}$-subobject, then the pullback $f^{-1}(N) \xrightarrow{f^{-1}(n)} X$ of n along f will be called the inverse image of n along f. Whenever no confusion is likely to arise, to simplify the notation we will denote the morphism $e_{f \circ m}$ simply e_f.

DEFINITION 1.2

A *closure operator* C on $\mathcal{X}$ (with respect to $\mathcal{M}$) is a family $\{(\ )_X^C\}_{X \in \mathcal{X}}$ of functions on the $\mathcal{M}$-subobject lattices of $\mathcal{X}$ with the following properties that hold for each $X \in \mathcal{X}$:

(a) [*expansiveness*] $m \leq (m)_X^C$, for every $\mathcal{M}$-subobject $M \xrightarrow{m} X$;

(b) [*order-preservation*] $m \leq n \Rightarrow (m)_X^C \leq (n)_X^C$ for every pair of $\mathcal{M}$-subobjects of X;

(c) [*morphism-consistency*] If p is the pullback of the $\mathcal{M}$-subobject $M \xrightarrow{m} Y$ along some $\mathcal{X}$-morphism $X \xrightarrow{f} Y$ and q is the pullback of $(m)_Y^C$ along f, then $(p)_X^C \leq q$, i.e., the closure of the inverse image of m is less than or equal to the inverse image of the closure of m.

Condition (a) implies that for every closure operator C on $\mathcal{X}$, every $\mathcal{M}$-subobject $M \xrightarrow{m} X$ has a canonical factorization

$$
\begin{array}{ccc}
M & \xrightarrow{t} & (M)_X^C \\
 & {\scriptstyle m} \searrow & \Big\downarrow {\scriptstyle (m)_X^C} \\
 & & X
\end{array}
$$

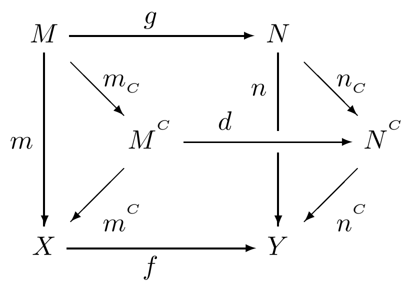
where $((M)_X^C, (m)_X^C)$ is called the *C-closure* of the subobject (M, m).

When no confusion is likely we will write m^C rather than $(m)_X^C$ and for notational symmetry we will denote the morphism t by m_C.

REMARK 1.3

(1) Notice that in the above definition, under condition (b), the morphism-consistency condition (c) is equivalent to the following statement concerning direct images: if $M \xrightarrow{m} X$ is an $\mathcal{M}$-subobject and $X \xrightarrow{f} Y$ is a morphism, then $((m)_Y^C)_f \leq (m_f)_Y^C$, i.e., the direct image of the closure of m is less than or equal to the closure of the direct image of m; (cf. [DG]).

(2) Under condition (a), both order-preservation and morphism-consistency, i.e., conditions (b) and (c) together are equivalent to the following: given (M, m) and (N, n) $\mathcal{M}$-subobjects of X and Y, respectively, if f and g are morphisms such that $n \circ g = f \circ m$, then there exists a unique morphism d such that the following diagram



commutes.

DEFINITION 1.4

Given a closure operator C, we say that $m \in \mathcal{M}$ is *C-closed* if m_C is an isomorphism. An $\mathcal{X}$-morphism f is called *C-dense* if for every $(\mathbf{E}, \mathcal{M})$-factorization (e, m) of f we have that m^C is an isomorphism. We call C *idempotent* provided that m^C is C-closed for every $m \in \mathcal{M}$. C is called *weakly hereditary* if m_C is C-dense for every $m \in \mathcal{M}$.

Notice that Definition 1.2(c) implies that pullbacks of C-closed $\mathcal{M}$-subobjects are C-closed.

We denote the collection of all closure operators on $\mathcal{M}$ by $\mathbf{CL}(\mathcal{X}, \mathcal{M})$ pre-ordered as follows: $C \sqsubseteq D$ if $m^C \leq m^D$ for all $m \in \mathcal{M}$ (where $\leq$ is the usual order on subobjects). Notice that arbitrary suprema and infima exist in $\mathbf{CL}(\mathcal{X}, \mathcal{M})$, they are formed pointwise in the $\mathcal{M}$-subobject fibers.

For more background on closure operators see, e.g., [C_1], [CKS_1], [CKS_2], [DG], [DGT] and [K]. For a detailed survey on the same topic, one could check [Ho].

DEFINITION 1.5

For pre-ordered classes $\mathcal{X} = (\mathbf{X}, \sqsubseteq)$ and $\mathcal{Y} = (\mathbf{Y}, \sqsubseteq)$, a *Galois connection* $\mathcal{X} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{Y}$ consists of order preserving functions F and G that satisfy $F \dashv G$, i.e., $x \sqsubseteq GF(x)$ for every $x \in \mathbf{X}$ and $FG(y) \sqsubseteq y$ for every $y \in \mathbf{Y}$. (G is adjoint and has F as coadjoint).

If $x \in \mathcal{X}$ and $y \in \mathcal{Y}$ are such that $F(x) = y$ and $G(y) = x$, then x and y are said to be corresponding fixed points of the Galois connection $(\mathcal{X}, F, G, \mathcal{Y})$.

Properties and many examples of Galois connections can be found in [EKMS].

2 GENERAL RESULTS ABOUT C-CONNECTEDNESS

Throughout the paper we will assume that $\mathcal{X}$ is an $(\mathbf{E}, \mathcal{M})$-category for sinks.

Unless otherwise specified, C will always denote a closure operator on $\mathcal{X}$ with respect to the given class $\mathcal{M}$ of $\mathcal{X}$-monomorphisms and $\mathcal{N}$ will be a subclass of $\mathcal{M}$. If X is an $\mathcal{X}$-object, $\mathcal{N}_X$ denotes the set of morphisms in $\mathcal{N}$ with codomain X.

We begin by recalling the following two propositions from [C$_2$].

PROPOSITION 2.1

Let $CL(\mathcal{X},\mathcal{M}) \xrightarrow{D_\mathcal{N}} S(\mathcal{X})^{\mathbf{op}}$ *and* $S(\mathcal{X})^{\mathbf{op}} \xrightarrow{T_\mathcal{N}} CL(\mathcal{X},\mathcal{M})$ *be defined by:*

$D_\mathcal{N}(C) = \{X \in \mathcal{X} : \text{ every } n \in \mathcal{N}_X \text{ is } C\text{-closed}\}$

$T_\mathcal{N}(\mathcal{A}) = Sup\{C \in CL(\mathcal{X},\mathcal{M}) : D_\mathcal{N}(C) \supseteq \mathcal{A}\}$.

Then, $CL(\mathcal{X},\mathcal{M}) \underset{T_\mathcal{N}}{\overset{D_\mathcal{N}}{\rightleftarrows}} S(\mathcal{X})^{\mathbf{op}}$ *is a Galois connection.* □

PROPOSITION 2.2

Let $CL(\mathcal{X},\mathcal{M}) \xrightarrow{I_\mathcal{N}} S(\mathcal{X})$ *and* $S(\mathcal{X}) \xrightarrow{J_\mathcal{N}} CL(\mathcal{X},\mathcal{M})$ *be defined by:*

$I_\mathcal{N}(C) = \{X \in \mathcal{X} : \text{ every } n \in \mathcal{N}_X \text{ is } C\text{-dense}\}$

$J_\mathcal{N}(\mathcal{B}) = Inf\{C \in CL(\mathcal{X},\mathcal{M}) : I_\mathcal{N}(C) \supseteq \mathcal{B}\}$.

Then, $S(\mathcal{X}) \underset{I_\mathcal{N}}{\overset{J_\mathcal{N}}{\rightleftarrows}} CL(\mathcal{X},\mathcal{M})$ *is a Galois connection.* □

In [C$_2$] we presented some characterizations of the functions $T_\mathcal{N}$ and $J_\mathcal{N}$ that will be often used throughout the paper. For reference purposes we collect them under the following:

PROPOSITION 2.3

For every $\mathcal{A} \in S(\mathcal{X})^{\mathbf{op}}$ *and* $\mathcal{M}$*-subobject* $M \xrightarrow{m} X$*, with* $X \in \mathcal{X}$*, we have that*

$$m^{T_\mathcal{N}(\mathcal{A})} = \cap\{f^{-1}(n) : Y \in \mathcal{A}, X \xrightarrow{f} Y, N \xrightarrow{n} Y \in \mathcal{N}_Y \text{and } m \le f^{-1}(n)\}.$$

Moreover, for every $\mathcal{B} \in S(\mathcal{X})$ *and* $\mathcal{M}$*-subobject* $M \xrightarrow{m} Y$*, with* $Y \in \mathcal{X}$*, we have that*

$$m^{J_\mathcal{N}(\mathcal{B})} = sup\Big(\{m\} \cup \{(id_X)_f : X \in \mathcal{B}, X \xrightarrow{f} Y \text{ and } \exists n \in \mathcal{N}_X \text{ with } n_f \le m\}\Big).$$

□

DEFINITION 2.4 (cf. [C$_2$], [C$_4$])

(a) A morphism $X \xrightarrow{f} A$ is $\mathcal{N}$-*constant* if f factors through n_f for every $n \in \mathcal{N}_X$.

(b) A morphism $X \xrightarrow{f} A$ is $\mathcal{N}$-*fixed* if $f^{-1}(n) \simeq id_X$ for every $n \in \mathcal{N}_A$.

REMARK 2.5

Notice that Definition 2.4(a) yields a Galois connection $S(\mathcal{X}) \underset{\nabla_\mathcal{N}}{\overset{\Delta_\mathcal{N}}{\rightleftarrows}} S(\mathcal{X})^{\mathbf{op}}$ where for $\mathcal{A} \in$

$S(\mathcal{X})$, $\Delta_{\mathcal{N}}(\mathcal{A}) = \{Y \in \mathcal{X} : \forall X \in \mathcal{A}$, each $X \xrightarrow{f} Y$ is $\mathcal{N}$-constant$\}$ and for $\mathcal{B} \in S(\mathcal{X})^{\mathbf{op}}$, $\nabla_{\mathcal{N}}(\mathcal{B}) = \{X \in \mathcal{X} : \forall Y \in \mathcal{B}$, each $X \xrightarrow{f} Y$ is $\mathcal{N}$-constant$\}$. In [C$_2$, Theorem 2.8] it was shown that under the assumption of $\mathcal{N}$ being closed under the formation of direct images, one has that $D_{\mathcal{N}} \circ J_{\mathcal{N}} = \Delta_{\mathcal{N}}$ and $I_{\mathcal{N}} \circ T_{\mathcal{N}} = \nabla_{\mathcal{N}}$.

Similarly Definition 2.4(b) yields a Galois connection $S(\mathcal{X}) \underset{\hat{\nabla}_{\mathcal{N}}}{\overset{\hat{\Delta}_{\mathcal{N}}}{\rightleftarrows}} S(\mathcal{X})^{\mathbf{op}}$ where for $\mathcal{A} \in S(\mathcal{X})$, $\hat{\Delta}_{\mathcal{N}}(\mathcal{A}) = \{Y \in \mathcal{X} : \forall X \in \mathcal{A}$, each $X \xrightarrow{f} Y$ is $\mathcal{N}$-fixed$\}$ and for $\mathcal{B} \in S(\mathcal{X})^{\mathbf{op}}$, $\hat{\nabla}_{\mathcal{N}}(\mathcal{B}) = \{X \in \mathcal{X} : \forall Y \in \mathcal{B}$, each $X \xrightarrow{f} Y$ is $\mathcal{N}$-fixed$\}$. In [C$_4$, Theorem 2.13] it was shown that under the assumption of $\mathcal{N}$ being closed under the formation of pullbacks, one has that $D_{\mathcal{N}} \circ J_{\mathcal{N}} = \hat{\Delta}_{\mathcal{N}}$ and $I_{\mathcal{N}} \circ T_{\mathcal{N}} = \hat{\nabla}_{\mathcal{N}}$.

Next we introduce a new notion that will allow us to prove a result similar to the two theorems mentioned in the previous remark without any closedness condition on $\mathcal{N}$.

DEFINITION 2.6

A morphism $X \xrightarrow{f} Y$ is called $\mathcal{N}$-*dependent* if for every $n \in \mathcal{N}_X$ and every $p \in \mathcal{N}_Y$, $n_f \leq p$ implies $f^{-1}(p) \simeq id_X$.

The above notion is strongly related to those in Definition 2.4 as the following proposition shows:

PROPOSITION 2.7

For a morphism $X \xrightarrow{f} Y$ consider the statements:

(a) f is $\mathcal{N}$-dependent;

(b) f is $\mathcal{N}$-constant;

(c) f is $\mathcal{N}$-fixed.

We always have that (b) $\Rightarrow$ (a) and (c) $\Rightarrow$ (a). If $\mathcal{N}$ is closed under the formation of direct images, then (a) $\Leftrightarrow$ (b) $\Leftarrow$ (c). If $\mathcal{N}$ is closed under the formation of pullbacks, then (a) $\Leftrightarrow$ (c) $\Leftarrow$ (b). As a consequence, if $\mathcal{N}$ is closed under the formation of both pullbacks and direct images then the three concepts are equivalent.

Proof:

(b) $\Rightarrow$ (a). Consider $n \in \mathcal{N}_X$ and $p \in \mathcal{N}_Y$ such that $n_f \leq p$. From (b), $(id_X)_f \leq p$ and so $id_X \leq f^{-1}(p)$.

(c) $\Rightarrow$ (a). Straightforward, since $f^{-1}(p) \simeq id_X$ for any $p \in \mathcal{N}_Y$.

Now assume that $\mathcal{N}$ is closed under the formation of direct images.

(a) $\Rightarrow$ (b). Let $n \in \mathcal{N}_X$ and let (e_f, n_f) be the $(\mathbf{E}, \mathcal{M})$-factorization of $f \circ n$. By assumption on $\mathcal{N}$, $n_f \in \mathcal{N}$. Since $n_f \leq n_f$, from (a) we have that $f^{-1}(n_f) \simeq id_X$.

(c) $\Rightarrow$ (b). It follows from the fact that (a) and (b) are equivalent and (c) $\Rightarrow$ (a) [See also C_4, Proposition 2.14].

Now let $\mathcal{N}$ be closed under the formation of pullbacks.

(a) $\Rightarrow$ (c). Consider $p \in \mathcal{N}_Y$. Then by assumption on $\mathcal{N}$, $f^{-1}(p) \in \mathcal{N}_X$. Since $(f^{-1}(p))_f \leq p$, from (a) we have that $f^{-1}(p) \simeq id_X$.

(b) $\Rightarrow$ (c). It follows from the fact that (a) and (c) are equivalent and (b) $\Rightarrow$ (a) [See also C_4, Proposition 2.14]. □

Clearly, Definition 2.6 yields a Galois connection $S(\mathcal{X}) \underset{\nabla'_{\mathcal{N}}}{\overset{\Delta'_{\mathcal{N}}}{\rightleftarrows}} S(\mathcal{X})^{\mathbf{op}}$ where for $\mathcal{A} \in S(\mathcal{X})$, $\Delta'_{\mathcal{N}}(\mathcal{A}) = \{Y \in \mathcal{X} : \forall X \in \mathcal{A},$ each $X \xrightarrow{f} Y$ is $\mathcal{N}$-dependent$\}$ and for $\mathcal{B} \in S(\mathcal{X})^{\mathbf{op}}$, $\nabla'_{\mathcal{N}}(\mathcal{B}) = \{X \in \mathcal{X} : \forall Y \in \mathcal{B},$ each $X \xrightarrow{f} Y$ is $\mathcal{N}$-dependent$\}$. Thus we obtain the following:

THEOREM 2.8

Let $\mathcal{N}$ be a subclass of $\mathcal{M}$. Then the Galois connection $S(\mathcal{X}) \underset{\nabla'_{\mathcal{N}}}{\overset{\Delta'_{\mathcal{N}}}{\rightleftarrows}} S(\mathcal{X})^{\mathbf{op}}$ factors through $CL(\mathcal{X}, \mathcal{M})$ via the Galois connections $S(\mathcal{X}) \underset{I_{\mathcal{N}}}{\overset{J_{\mathcal{N}}}{\rightleftarrows}} CL(\mathcal{X}, \mathcal{M})$ and $CL(\mathcal{X}, \mathcal{M}) \underset{T_{\mathcal{N}}}{\overset{D_{\mathcal{N}}}{\rightleftarrows}} S(\mathcal{X})^{\mathbf{op}}$.

Proof:

Let $\mathcal{A} \in S(\mathcal{X})^{\mathbf{op}}$ and let $X \in I_{\mathcal{N}}(T_{\mathcal{N}}(\mathcal{A}))$. Consider a morphism $X \xrightarrow{f} Y$ with $Y \in \mathcal{A}$ and $\mathcal{M}$-subobjects $n \in \mathcal{N}_X$ and $p \in \mathcal{N}_Y$ such that $n_f \leq p$ or equivalently $n \leq f^{-1}(p)$. Since n is $T_{\mathcal{N}}(\mathcal{A})$-dense, i.e., $n^{T_{\mathcal{N}}(\mathcal{A})} \simeq id_X$, then from Proposition 2.3 we immediately obtain that $f^{-1}(p) \simeq id_X$. Hence f is $\mathcal{N}$-dependent and consequently $X \in \nabla'_{\mathcal{N}}(\mathcal{A})$. Thus, $I_{\mathcal{N}}(T_{\mathcal{N}}(\mathcal{A})) \subseteq \nabla'_{\mathcal{N}}(\mathcal{A})$.

Viceversa, let $X \in \nabla'_{\mathcal{N}}(\mathcal{A})$. Consider $n \in \mathcal{N}_X$. For every morphism $X \xrightarrow{f} Y$ with $Y \in \mathcal{A}$ and $\mathcal{M}$-subobjects $n \in \mathcal{N}_X$ and $p \in \mathcal{N}_Y$ such that $n \leq f^{-1}(p)$ or equivalently $n_f \leq p$ we have that $f^{-1}(p) \simeq id_X$. Again from Proposition 2.3 we immediately obtain that $n^{T_{\mathcal{N}}(\mathcal{A})} \simeq id_X$, i.e., $X \in I_{\mathcal{N}}(T_{\mathcal{N}}(\mathcal{A}))$. Thus $\nabla'_{\mathcal{N}}(\mathcal{A}) \subseteq I_{\mathcal{N}}(T_{\mathcal{N}}(\mathcal{A}))$.

Let $\mathcal{B} \in S(\mathcal{X})$ and let $Y \in D_{\mathcal{N}}(J_{\mathcal{N}}(\mathcal{B}))$. Consider a morphism $X \xrightarrow{f} Y$ with $X \in \mathcal{B}$ and $\mathcal{M}$-subobjects $n \in \mathcal{N}_X$ and $p \in \mathcal{N}_Y$ such that $n_f \leq p$ or equivalently $n \leq f^{-1}(p)$. By assumption on Y, p is $J_{\mathcal{N}}(\mathcal{A})$-closed, i.e., $p \simeq p^{J_{\mathcal{N}}(\mathcal{A})}$. Then, from Proposition 2.3 we can easily conclude that $(id_X)_f \leq p$ or equivalently $f^{-1}(p) \simeq id_X$. Hence f is $\mathcal{N}$-dependent and consequently $Y \in \Delta'_{\mathcal{N}}(\mathcal{B})$. Thus, $D_{\mathcal{N}}(J_{\mathcal{N}}(\mathcal{B})) \subseteq \Delta'_{\mathcal{N}}(\mathcal{B})$.

Now let $Y \in \Delta'_{\mathcal{N}}(\mathcal{B})$. Consider $p \in \mathcal{N}_Y$. By assumption on Y, for every morphism $X \xrightarrow{f} Y$ with $X \in \mathcal{B}$ and $\mathcal{M}$-subobjects $n \in \mathcal{N}_X$ and $p \in \mathcal{N}_Y$ such that $n \leq f^{-1}(p)$ or equivalently $n_f \leq p$ we have that $f^{-1}(p) \simeq id_X$, or equivalently $(id_X)_f \leq p$. Again from Proposition 2.3 we obtain that $p^{J_{\mathcal{N}}(\mathcal{B})} \simeq p$, i.e., $Y \in D_{\mathcal{N}}(J_{\mathcal{N}}(\mathcal{B}))$. Thus $\Delta'_{\mathcal{N}}(\mathcal{B}) \subseteq D_{\mathcal{N}}(J_{\mathcal{N}}(\mathcal{B}))$. □

We recall the following from [C_5]:

DEFINITION 2.9

An $\mathcal{X}$-object X is called $(C,\mathcal{N})$-connected if $X \in I_{\mathcal{N}}(T_{\mathcal{N}}(D_{\mathcal{N}}(C)))$.

As a consequence of Remark 2.5 and Theorem 2.8, we obtain the following alternative descriptions of the notion of $(C,\mathcal{N})$-connectedness under appropriate closedness conditions:

PROPOSITION 2.10

(a) An $\mathcal{X}$-object X is $(C,\mathcal{N})$-connected if every morphism $X \xrightarrow{f} A$ with $A \in D_{\mathcal{N}}(C)$ is $\mathcal{N}$-dependent; i.e., for every $n \in \mathcal{N}_X$ and every $p \in \mathcal{N}_A$, $n_f \leq p$ implies $f^{-1}(p) \simeq id_X$.

(b) If $\mathcal{N}$ is closed under the formation of direct images, then an $\mathcal{X}$-object X is $(C,\mathcal{N})$-connected if every morphism $X \xrightarrow{f} A$ with $A \in D_{\mathcal{N}}(C)$ is $\mathcal{N}$-constant; i.e., f factors through n_f for every $n \in \mathcal{N}_X$.

(c) If $\mathcal{N}$ is closed under the formation of pullbacks, then an $\mathcal{X}$-object X is $(C,\mathcal{N})$-connected if every morphism $X \xrightarrow{f} A$ with $A \in D_{\mathcal{N}}(C)$ is $\mathcal{N}$-fixed; i.e., $f^{-1}(n) \simeq id_X$ for every $n \in \mathcal{N}_A$. □

We are now ready to give the following:

DEFINITION 2.11

(a) Let $\mathcal{A} \in S(\mathcal{X})$. $\mathcal{A}$ is said to be a *connectedness class* if there is a subclass of morphisms $\mathcal{N} \subseteq \mathcal{M}$ and a closure operator C such that $\mathcal{A} = I_{\mathcal{N}}(T_{\mathcal{N}}(D_{\mathcal{N}}(C)))$.

(b) Let $\mathcal{A} \in S(\mathcal{X})$ and $\mathcal{N} \subseteq \mathcal{M}$. $\mathcal{A}$ is said to be an $\mathcal{N}$-*connectedness class* if there is a closure operator C such that $\mathcal{A} = I_{\mathcal{N}}(T_{\mathcal{N}}(D_{\mathcal{N}}(C)))$.

REMARK 2.12

Notice that if $\mathcal{A} = I_{\mathcal{N}}(T_{\mathcal{N}}(D_{\mathcal{N}}(C)))$, then from the properties of Galois connections we have that $\mathcal{A} = I_{\mathcal{N}}(T_{\mathcal{N}}(D_{\mathcal{N}}(J_{\mathcal{N}}(\mathcal{A}))))$. Consequently, part (b) of Definition 2.11 can be also restated as follows: $\mathcal{A}$ is an $\mathcal{N}$-connectedness class if and only if $\mathcal{A} = I_{\mathcal{N}}(T_{\mathcal{N}}(D_{\mathcal{N}}(J_{\mathcal{N}}(\mathcal{A}))))$.

PROPOSITION 2.13

$\mathcal{A} \in S(\mathcal{X})$ is a left fixed point of $S(\mathcal{X}) \underset{\nabla'_{\mathcal{N}}}{\overset{\Delta'_{\mathcal{N}}}{\rightleftarrows}} S(\mathcal{X})^{\mathbf{op}}$ if and only if $\mathcal{A}$ is an $\mathcal{N}$-connectedness class.

Proof:

Let $\mathcal{A} = \nabla'_{\mathcal{N}}(\mathcal{B})$ with $\mathcal{B} \in S(\mathcal{X})^{\mathbf{op}}$. Without loss of generality we can also assume that $\mathcal{B} = \Delta'_{\mathcal{N}}(\mathcal{A})$. Thus, from Theorem 2.8, we have that $\mathcal{A} = \nabla'_{\mathcal{N}}(\Delta'_{\mathcal{N}}(\mathcal{A})) = I_{\mathcal{N}}(T_{\mathcal{N}}(D_{\mathcal{N}}(J_{\mathcal{N}}(\mathcal{A}))))$. Therefore $\mathcal{A}$ is an $\mathcal{N}$-connectedness class. Conversely, let $\mathcal{A} = I_{\mathcal{N}}(T_{\mathcal{N}}(D_{\mathcal{N}}(C)))$, for $C \in CL(\mathcal{X},\mathcal{M})$. Again from Theorem 2.8 we obtain that $\mathcal{A} = \nabla'_{\mathcal{N}}(D_{\mathcal{N}}(C))$. □

REMARK 2.14

Notice that using Remark 2.5, the result of Proposition 2.13 holds for the Galois connection $S(\mathcal{X}) \underset{\nabla_{\mathcal{N}}}{\overset{\Delta_{\mathcal{N}}}{\rightleftarrows}} S(\mathcal{X})^{\mathbf{op}}$ ($S(\mathcal{X}) \underset{\hat{\nabla}_{\mathcal{N}}}{\overset{\hat{\Delta}_{\mathcal{N}}}{\rightleftarrows}} S(\mathcal{X})^{\mathbf{op}}$) under the assumption of $\mathcal{N}$ being closed under the formation of direct images (pullbacks).

The following few results will lead to a characterization of connectedness classes, under the assumption of $\mathcal{N}$ being closed under the formation of pullbacks.

PROPOSITION 2.15

(a) $X \in I_{\mathcal{N}}(C)$ *if and only if for every* $n \in \mathcal{N}_X$, $X \xrightarrow{f} Y$, $Y \in \mathcal{X}$ *we have that* $f^{-1}(n_f^{C}) \simeq id_X$.

(b) $Y \in D_{\mathcal{N}}(C)$ *if and only if for every* $n \in \mathcal{N}_Y$, $X \xrightarrow{f} Y$, $X \in \mathcal{X}$ *we have that* $((f^{-1}(n))^{C})_f \leq n$.

Proof:

(a). Let $X \in I_{\mathcal{N}}(C)$ and let $n \in \mathcal{N}_X$. Notice that $n_f \in \mathcal{M}$ and so we can consider n_f^{C}. From the general properties of closure operators we have that $id_X \geq f^{-1}(n_f^{C}) \geq (f^{-1}(n_f))^{C} \geq n^{C} \simeq id_X$. The last isomorphism is a consequence of the fact that $X \in I_{\mathcal{N}}(C)$. Therefore we conclude that $f^{-1}(n_f^{C}) \simeq id_X$.

Conversely, let $X \in \mathcal{X}$ and $n \in \mathcal{N}_X$. Suppose that for every $X \xrightarrow{f} Y$, $Y \in \mathcal{X}$, $f^{-1}(n_f^{C}) \simeq id_X$. In particular, if $Y = X$ and $f = id_X$, we have that $id_X \simeq f^{-1}(n_f^{C}) \simeq n^{C}$. Thus $X \in I_{\mathcal{N}}(C)$.

(b). Let $Y \in D_{\mathcal{N}}(C)$ and let $n \in \mathcal{N}_Y$. Notice that $f^{-1}(n) \in \mathcal{M}$ and so we can consider $(f^{-1}(n))^{C}$. From the general properties of closure operators we have that $((f^{-1}(n))^{C})_f \leq ((f^{-1}(n))_f)^{C} \leq n^{C} \simeq n$. The last isomorphism is a consequence of the fact that $Y \in D_{\mathcal{N}}(C)$. Therefore we conclude that $((f^{-1}(n))^{C})_f \leq n$.

Conversely, let $Y \in \mathcal{X}$ and let $n \in \mathcal{N}_Y$. Suppose that for every $X \xrightarrow{f} Y$, $X \in \mathcal{X}$, $((f^{-1}(n))^{C})_f \leq n$. In particular, if we take $Y = X$ and $f = id_X$, we obtain that $n \geq ((f^{-1}(n))^{C})_f \simeq n^{C}$. Since it is always true that $n \leq n^{C}$, we can conclude that $n \simeq n^{C}$ and so $Y \in D_{\mathcal{N}}(C)$. □

In the special cases that $C = T_{\mathcal{N}}(\mathcal{B})$ or $C = J_{\mathcal{N}}(\mathcal{A})$, under a further assumption on $\mathcal{N}$, we obtain the following:

PROPOSITION 2.16

Let $\mathcal{N}$ *be closed under the formation of pullbacks and let* X *and* Y *be two* $\mathcal{X}$*-objects.*

(a) $X \in I_{\mathcal{N}}(T_{\mathcal{N}}(\mathcal{B}))$ *if and only if for every* $Y \in \mathcal{B}$, $X \xrightarrow{f} Y$ *and* $n \in \mathcal{N}_Y$, *we have that* $(id_X)_f \leq n$.

(b) $Y \in D_{\mathcal{N}}(J_{\mathcal{N}}(\mathcal{A}))$ *if and only if for every* $X \in \mathcal{A}$, $X \xrightarrow{f} Y$ *and* $n \in \mathcal{N}_Y$, *we have that*

$(id_X)_f \leq n$.

Proof:

(a). Let $X \in I_{\mathcal{N}}(T_{\mathcal{N}}(\mathcal{B}))$ and let $X \xrightarrow{f} Y$ be an $\mathcal{X}$-morphism with $Y \in \mathcal{B}$. The hypothesis on $\mathcal{N}$ implies that f is $\mathcal{N}$-fixed (cf. Remark 2.5). Consequently, $f^{-1}(n) \simeq id_X$ for every $n \in \mathcal{N}_Y$. Thus we have that $(id_X)_f \leq n$, for every $n \in \mathcal{N}_Y$.

Conversely, let $X \in \mathcal{X}$ and $X \xrightarrow{f} Y$ with $Y \in \mathcal{B}$. $(id_X)_f \leq n$ for every $n \in \mathcal{N}_Y$ implies that $f^{-1}(n) \simeq id_X$. So, f is $\mathcal{N}$-fixed and consequently $X \in I_{\mathcal{N}}(T_{\mathcal{N}}(\mathcal{B}))$.

(b). Let $Y \in D_{\mathcal{N}}(J_{\mathcal{N}}(\mathcal{A}))$. Then, from Remark 2.5, every morphism $X \xrightarrow{f} Y$ with $X \in \mathcal{A}$ is $\mathcal{N}$-fixed. Thus, for every $n \in \mathcal{N}_Y$, $f^{-1}(n) \simeq id_X$ and so $(id_X)_f \simeq (f^{-1}(n))_f \leq n$.

Conversely, consider a morphism $X \xrightarrow{f} Y$ with $X \in \mathcal{A}$. $(id_X)_f \leq n$, for every $n \in \mathcal{N}_Y$ implies that $id_X \simeq f^{-1}((id_X)_f) \leq f^{-1}(n)$, for every $n \in \mathcal{N}_Y$. Thus, f is $\mathcal{N}$-fixed and from Remark 2.5, $Y \in D_{\mathcal{N}}(J_{\mathcal{N}}(\mathcal{A}))$. □

The following result is well-known from the theory of $(\mathbf{E}, \mathcal{M})$-categories:

LEMMA 2.17

Let $(X_i \xrightarrow{f_i} Y)$ be a sink and let (e_i, m) be its $(\mathbf{E}, \mathcal{M})$-factorization. If (g_i, m_i) is the $(\mathbf{E}, \mathcal{M})$-factorization of each f_i and (h_i, n) is the $(\mathbf{E}, \mathcal{M})$-factorization of the sink (m_i), then we have that $m \simeq n$. □

PROPOSITION 2.18

Let $\mathcal{N}$ be closed under the formation of pullbacks and let $\mathcal{A} \in S(\mathcal{X})$. Let $(X_i \xrightarrow{f_i} Y)_{i \in I}$ be the total sink from $\mathcal{A}$ into $Y \in \mathcal{X}$ and let $X_i \xrightarrow{e_i} M \xrightarrow{m} Y$ be its $(\mathbf{E}, \mathcal{M})$-factorization. Then, the following are equivalent:

(a) $Y \in I_{\mathcal{N}}(J_{\mathcal{N}}(\mathcal{A}))$

(b) the 2-sink (m, n) belongs to $\mathbf{E}$, for every $n \in \mathcal{N}_Y$.

Proof:

(a)⇒(b). From our hypothesis, for every $n \in \mathcal{N}_Y$, $n^{J_{\mathcal{N}}(\mathcal{A})} \simeq id_Y$. Notice that since $(f^{-1}(n))_f \leq n$ and $f^{-1}(n) \in \mathcal{N}_X$ for every $X \xrightarrow{f} Y$ with $X \in \mathcal{A}$, then any such f occurs in the construction of $n^{J_{\mathcal{N}}(\mathcal{A})}$ (cf. Proposition 2.3). Consequently the total sink $(X_i \xrightarrow{f_i} Y)_{i \in I}$ is used in the construction of $n^{J_{\mathcal{N}}(\mathcal{A})}$. From Lemma 2.17 and from the universal property of suprema we obtain that $m \vee n \simeq n^{J_{\mathcal{N}}(\mathcal{A})} \simeq id_Y$. Since suprema are obtained via $(\mathbf{E}, \mathcal{M})$-factorizations, we obtain that the $\mathcal{M}$-part of the $(\mathbf{E}, \mathcal{M})$-factorization of the 2-sink (m, n) is an isomorphism and so $(m, n) \in \mathbf{E}$.

(b)⇒(a). Now, let $n \in \mathcal{N}_Y$ with $Y \in \mathcal{X}$. As observed in the first part of the proof, the fact that $\mathcal{N}$ is closed under the formation of pullbacks implies that the total sink $(X_i \xrightarrow{f_i} Y)_{i \in I}$

from $\mathcal{A}$ into Y is used in the construction of $n^{J_{\mathcal{N}}(\mathcal{A})}$. Again from Lemma 2.17 and the general property of suprema we have that $n^{J_{\mathcal{N}}(\mathcal{A})} \simeq m \vee n$. Since $(m, n) \in \mathbf{E}$, its supremum (taken via its $(\mathbf{E}, \mathcal{M})$-factorization) is an isomorphism. Thus, $n^{J_{\mathcal{N}}(\mathcal{A})} \simeq id_Y$ and so $Y \in I_{\mathcal{N}}(J_{\mathcal{N}}(\mathcal{A}))$. □

COROLLARY 2.19

If $\mathcal{N}$ is closed under the formation of pullbacks, then for every $\mathcal{A} \in S(\mathcal{X})$, $I_{\mathcal{N}}(J_{\mathcal{N}}(\mathcal{A}))$ is closed under **E**-*sinks.*

Proof:

Let $(Y_i \xrightarrow{g_i} Y)_{i \in I}$ be an **E**-sink with $Y_i \in \mathcal{A}$, for every $i \in I$ and let $n \in \mathcal{N}_Y$. Since $\mathcal{N}$ is closed under the formation of pullbacks, as in the proof of the previous proposition, all the g_i's occur in the construction of $n^{J_{\mathcal{N}}(\mathcal{A})}$. Clearly the $\mathcal{M}$-part of the $(\mathbf{E}, \mathcal{M})$-factorization of the sink $(Y_i \xrightarrow{g_i} Y)_{i \in I}$ is an isomorphism i and consequently, for every $n \in \mathcal{N}_Y$ so is the $\mathcal{M}$-part of the $(\mathbf{E}, \mathcal{M})$-factorization of the 2-sink (i, n). Thus, $n^{J_{\mathcal{N}}(\mathcal{A})} \simeq id_Y$, i.e., $Y \in I_{\mathcal{N}}(J_{\mathcal{N}}(\mathcal{A}))$. □

PROPOSITION 2.20

Let $\mathcal{N}$ be closed under the formation of pullbacks. $Y \in D_{\mathcal{N}}(J_{\mathcal{N}}(\mathcal{A}))$ if and only if the total sink from $\mathcal{A}$ into Y factors through n, for every $n \in \mathcal{N}_Y$.

Proof:

$(\Rightarrow)$. Consider the total sink from $\mathcal{A}$ into Y, $(A_i \xrightarrow{f_i} Y)_{i \in I}$. If $Y \in D_{\mathcal{N}}(J_{\mathcal{N}}(\mathcal{A}))$, then from Proposition 2.16(b), for every $n \in \mathcal{N}_Y$, $(id_{A_i})_{f_i} \leq n$. From Lemma 2.17 we obtain that the sink $(f_i)_{i \in I}$ factors through n.

$(\Leftarrow)$. Consider the morphism $A \xrightarrow{f} Y$ with $A \in \mathcal{A}$ and $n \in \mathcal{N}_Y$. By hypothesis f, as a member of the total sink from $\mathcal{A}$ into Y, factors through n, that is, there is a morphism r such that $f = n \circ r$. If we consider the $(\mathbf{E}, \mathcal{M})$-factorization of f, $(e_f, (id_X)_f)$, we obtain that $(id_X)_f \circ e_f = n \circ r$. The $(\mathbf{E}, \mathcal{M})$-diagonalization property implies the existence of a unique morphism d such that, in particular, $n \circ d = (id_X)_f$. Thus, we have that $(id_X)_f \leq n$ and from Proposition 2.16(b) we conclude that $Y \in D_{\mathcal{N}}(J_{\mathcal{N}}(\mathcal{A}))$. □

DEFINITION 2.21

For every $\mathcal{A} \in S(\mathcal{X})$ and $\mathcal{N} \subseteq \mathcal{M}$, $I_{\mathcal{N}}(T_{\mathcal{N}}(D_{\mathcal{N}}(J_{\mathcal{N}}(\mathcal{A}))))$ is called the $\mathcal{N}$-*connectedness hull* of $\mathcal{A}$.

REMARK 2.22

From Remark 2.12 we obtain that $\mathcal{A} \in S(\mathcal{X})$ is an $\mathcal{N}$-connectedness class if and only if $\mathcal{A}$ agrees with its $\mathcal{N}$-connectedness hull.

We conclude this section with the following:

THEOREM 2.23

Let $\mathcal{N}$ be closed under the formation of pullbacks, let $X \in \mathcal{X}$ and let $\mathcal{A} \in S(\mathcal{X})$. X belongs to the $\mathcal{N}$-connectedness hull of $\mathcal{A}$ if and only if for every morphism $X \xrightarrow{f} Y$, if the total sink $(A_i \xrightarrow{f_i} Y)$ from $\mathcal{A}$ into Y factors through n, for every $n \in \mathcal{N}_Y$, then so does f.

Proof:

($\Rightarrow$). Consider the morphism $X \xrightarrow{f} Y$. If the total sink from $\mathcal{A}$ into Y factors through every $n \in \mathcal{N}_Y$, then from Proposition 2.20, $Y \in D_{\mathcal{N}}(J_{\mathcal{N}}(\mathcal{A}))$. Thus, from Remark 2.5, f is $\mathcal{N}$-fixed, that is, for every $n \in \mathcal{N}_Y$, $f^{-1}(n) \simeq id_X$. Consequently, $(id_X)_f \simeq (f^{-1}(n))_f \leq n$ and so f factors through n, for every $n \in \mathcal{N}_Y$.

($\Leftarrow$). Let $Y \in D_{\mathcal{N}}(J_{\mathcal{N}}(\mathcal{A}))$ and let $X \xrightarrow{f} Y$ be an $\mathcal{X}$-morphism. Then, from Proposition 2.20, the total sink from $\mathcal{A}$ into Y factors through n, for every $n \in \mathcal{N}_Y$. Then, by our hypothesis, f factors through n, for every $n \in \mathcal{N}_Y$. This clearly implies that $(id_X)_f \leq n$ and consequently $f^{-1}(n) \simeq id_X$. Thus f is $\mathcal{N}$-fixed and from Remark 2.5, $X \in I_{\mathcal{N}}(T_{\mathcal{N}}(D_{\mathcal{N}}(J_{\mathcal{N}}(\mathcal{A}))))$. □

3 CONNECTEDNESS IN CATEGORIES WITH A TERMINAL OBJECT

In this section we provide a characterization of connectedness classes in the special case of a category with a terminal object. So, from now on we assume that the category $\mathcal{X}$ has a terminal object T.

DEFINITION 3.1

An $\mathcal{X}$-object X is called *empty* (*non-empty*) if it does not have (it has) T as a subobject. An $\mathcal{X}$-object that is either empty or isomorphic to T is called *trivial*, otherwise it is called *non-trivial*.

Now we make the following

ASSUMPTIONS 3.2

(a) If X is a trivial object, then any morphism with domain X belongs to $\mathcal{M}$;

(b) any morphism with domain a terminal object T belongs to $\mathcal{N}$;

(c) Whenever $T \xrightarrow{t} X$ and $M \xrightarrow{m} X$ are $\mathcal{M}$-subobjects such that $m \leq t$, then M non-empty implies $m \simeq t$;

(d) $\mathcal{N}$ is a class of non-empty $\mathcal{M}$-subobjects.

Notice that at times we will use the expression "X contains T" to mean that T is a subobject of X.

REMARK 3.3

It is important to observe that if X is empty, then as a consequence of Assumption 3.2(d) it cannot have any $\mathcal{N}$-subobject.

The following result will be very useful.

LEMMA 3.4

Let X be non-empty and let $X \xrightarrow{f} Y$ be a morphism. The following are equivalent:

(a) f is $\mathcal{N}$-dependent;

(b) f is $\mathcal{N}$-constant;

(c) f factors through T;

(d) X_f is isomorphic to T.

Proof:

(b) $\Rightarrow$ (a) was shown in Proposition 2.7.

(a) $\Rightarrow$ (c). Consider the $\mathcal{N}$-subobject $T \xrightarrow{t} X$ and let (e_f, t_f) be its $(\mathbf{E}, \mathcal{M})$-factorization, with $T \xrightarrow{e_f} T_f$. Since by Assumption 3.2(a) $e_f \in \mathbf{E} \cap \mathcal{M}$ then it is an isomorphism. So $T \simeq T_f$ together with Assumption 3.2(b) implies that $t_f \in \mathcal{N}$. Clearly $t_f \leq t_f$ implies from (a) that $f^{-1}(t_f) \simeq id_X$. Thus f factors through T.

(c) $\Rightarrow$ (d). Let (e_f, m_f) be the $(\mathbf{E}, \mathcal{M})$-factorization of f and let $X \xrightarrow{t^X} T$ and $T \xrightarrow{t_Y} Y$ be two morphisms such that $f = t_Y \circ t^X$. Since by Assumption 3.2(a) $t_Y \in \mathcal{M}$, the $(\mathbf{E}, \mathcal{M})$-diagonalization property gives a unique morphism $X_f \xrightarrow{d} T$ such that, in particular, $t_Y \circ d = m_f$. Notice that since m_f and t_Y both belong to $\mathcal{M}$, we also have that $d \in \mathcal{M}$. Consequently d is a monomorphism. Since X is non-empty, there is a morphism $T \xrightarrow{t_X} X$. Since T is a terminal object, the morphism d satisfies: $d \circ e_f \circ t_X = id_T$. So, d is an isomorphism since it is a monomorphism and a retraction.

(d) $\Rightarrow$ (b). Let $n \in \mathcal{N}_X$ and let $X \xrightarrow{e_1} X_f \xrightarrow{n_1} Y$ and $N \xrightarrow{e_f} N_f \xrightarrow{n_f} Y$ be the $(\mathbf{E}, \mathcal{M})$-factorizations of f and $f \circ n$, respectively. The $(\mathbf{E}, \mathcal{M})$-diagonalization property yields a morphism d such that, in particular, $n_1 \circ d = n_f$. Since $X_f \simeq T$ and (by 3.2(d)) N_f is non-empty, from assumption 3.2(c) we obtain that d is an isomorphism, i.e., f factors through n_f and so we have (b). □

REMARK 3.5

Notice that under Assumption 3.2 the concepts in 3.4 are not equivalent to f being $\mathcal{N}$-fixed.

As a matter of fact, in the category **Top** of topological spaces with the usual (episink,embedding) factorization structure and $\mathcal{N}$ consisting of all non-empty embeddings, any constant function $X \xrightarrow{f} Y$ with X, Y topological spaces with at least two points is $\mathcal{N}$-constant and $\mathcal{N}$-dependent but not $\mathcal{N}$-fixed. Moreover, these three concepts are generally distinct. Again, in the category **Top** with the (episink,embedding) factorization structure and $\mathcal{N}$ consisting of all non-empty clopen subsets, the morphism $D_2 \xrightarrow{i} X$ where D_2 is the two-point discrete space, $X = \{0,1,2\}$ with non-trivial open sets $\{0,1\}$ and $\{2\}$, and $i(0) = 0, i(1) = 1$, is $\mathcal{N}$-dependent but neither $\mathcal{N}$-constant nor $\mathcal{N}$-fixed. Clearly Assumption 3.2(b) is not satisfied in this case.

However, under the additional assumption that the terminal object T is an $\mathcal{M}$-subobject of every element of $\mathcal{N}$ (this is always the case in the category **Grp** of groups, for instance) then the three notions are equivalent.

PROPOSITION 3.6

Any connectedness class contains all trivial objects.

Proof:

We recall that from Proposition 2.10(a), connectedness classes can be described via $\mathcal{N}$-dependent morphisms. If $X \simeq T$, then clearly any morphism $X \xrightarrow{f} Y$ factors through T, so we can apply the previous lemma. If X is empty, then from the previous remark (3.3) X does not have any $\mathcal{N}$-subobject. Consequently any morphism $X \xrightarrow{f} Y$ is vacuously $\mathcal{N}$-dependent. □

PROPOSITION 3.7

Let $X \xrightarrow{g} Y$ be an **E***-morphism with X non-empty and $(C, \mathcal{N})$-connected. Then Y is $(C, \mathcal{N})$-connected.*

Proof:

From Proposition 2.10(a) and Lemma 3.4, it is enough to show that any morphism $Y \xrightarrow{f} A$ with $A \in D_{\mathcal{N}}(C)$ factors through T. Since X is $(C, \mathcal{N})$-connected and contains T, then $f \circ g$ factors through T, i.e., $f \circ g = t_A \circ t^X$ with $X \xrightarrow{t^X} T$ and $T \xrightarrow{t_A} A$. By Assumption 3.2(b), $t_A \in \mathcal{M}$ and so the $(\mathbf{E}, \mathcal{M})$-diagonalization property implies the existence of a morphism $Y \xrightarrow{d} T$ such that, in particular, $t_A \circ d = f$. This concludes the proof. □

THEOREM 3.8

Let $\mathcal{N}$ be closed under the formation of direct images along elements of $\mathcal{M}$ and let $\mathcal{A}$ be a class that is closed under **E***-quotients and contains all trivial objects in $\mathcal{X}$. $\mathcal{A}$ is an $\mathcal{N}$-connectedness class if and only if $\mathcal{A}$ satisfies the following condition:*

a non-trivial object X belongs to $\mathcal{A}$ if and only if every non-trivial image of X has a non-

trivial $\mathcal{M}$-subobject that belongs to $\mathcal{A}$.

Proof:

($\Rightarrow$). Assume that $\mathcal{A}$ is an $\mathcal{N}$-connectedness class. Let $X \in \mathcal{A}$ and let $X \xrightarrow{f} Y$ be a morphism. Since X is $(C,\mathcal{N})$-connected, from Proposition 3.7, so is X_f. So, if X_f is non-trivial then it has itself as $\mathcal{M}$-subobject belonging to $\mathcal{A}$.

Conversely, suppose that X is non-trivial and $X \notin \mathcal{A}$. Then there is a morphism $X \xrightarrow{f} Y$ with $Y \in D_{\mathcal{N}}(J_{\mathcal{N}}(\mathcal{A}))$ that does not factor through T. Thus X_f is non-trivial and it belongs to $D_{\mathcal{N}}(J_{\mathcal{N}}(\mathcal{A}))$, since by our assumptions on $\mathcal{N}$ we have that $D_{\mathcal{N}}(J_{\mathcal{N}}(\mathcal{A}))$ is closed under $\mathcal{M}$-subobjects (cf. [C$_4$, Proposition 2.7]). Let $M \xrightarrow{m} X_f$ be a non-trivial $\mathcal{M}$-subobject such that $M \in \mathcal{A}$. Again from [C$_4$, Proposition 2.7] we have that $M \in D_{\mathcal{N}}(J_{\mathcal{N}}(\mathcal{A}))$. This implies that $M \simeq T$. Thus some non-trivial image of X has no non-trivial $\mathcal{M}$-subobject in $\mathcal{A}$.

($\Leftarrow$). Suppose that $\mathcal{A}$ satisfies the condition and let $X \in I_{\mathcal{N}}(T_{\mathcal{N}}(D_{\mathcal{N}}(J_{\mathcal{N}}(\mathcal{A}))))$. We just need to show that $X \in \mathcal{A}$. If X is trivial, then by assumption it belongs to $\mathcal{A}$. So, suppose that X is non-trivial and that $X \notin \mathcal{A}$. Then there is a morphism $X \xrightarrow{f} Y$ such that X_f is non-trivial and it does not have any non-trivial $\mathcal{M}$-subobject belonging to $\mathcal{A}$. Now let $A \in \mathcal{A}$ and let $A \xrightarrow{g} X_f$ be a morphism. If A does not contain T, then from Remark 3.3 g is $\mathcal{N}$-constant. So, let A contain T. Consider the $(\mathbf{E},\mathcal{M})$-factorization $m_g \circ e_g = g$. Since $A_g \xrightarrow{m_g} X_f$ is an $\mathcal{M}$-subobject of X_f and $A_g \in \mathcal{A}$ then $A_g \simeq T$. So, $X_f \in D_{\mathcal{N}}(J_{\mathcal{N}}(\mathcal{A}))$. From Proposition 3.7, $X_f \in I_{\mathcal{N}}(T_{\mathcal{N}}(D_{\mathcal{N}}(J_{\mathcal{N}}(\mathcal{A}))))$. Consequently we obtain that $X_f \simeq T$ which contradicts the fact that X_f is non-trivial. □

PROPOSITION 3.9

Let $D_{\mathcal{N}}(C)$ be $\mathbf{E}$-reflective in $\mathcal{X}$. Then a non-trivial object X is $(C,\mathcal{N})$-connected iff its $\mathbf{E}$-reflection $X \xrightarrow{r_X} rX$ satisfies $rX \simeq T$.

Proof:

($\Leftarrow$). For any $Y \in D_{\mathcal{N}}(C)$ and $X \xrightarrow{f} Y$ there is a morphism $rX \xrightarrow{g} Y$ such that $g \circ r_X = f$. Since $rX \simeq T$, we have that f factors through T, i.e., from Lemma 3.4 and Proposition 2.10(a), X is $(C,\mathcal{N})$-connected.

($\Rightarrow$). Since X is $(C,\mathcal{N})$-connected and $rX \in D_{\mathcal{N}}(C)$, again from Lemma 3.4 and Proposition 2.10(a), r_X factors through T, i.e., there exist morphisms $X \xrightarrow{t^X} T$ and $T \xrightarrow{t_{rX}} rX$ such that $t_{rX} \circ t^X = r_X = id_{rX} \circ r_X$. By assumptions $r_X \in \mathbf{E}$ and $t_{rX} \in \mathcal{M}$. So, the $(\mathbf{E},\mathcal{M})$-diagonalization property yields a morphism $rX \xrightarrow{d} T$ such that in particular $t_{rX} \circ d = id_{rX}$. Thus t_{rX} is a monomorphism and a retraction and so an isomorphism. □

PROPOSITION 3.10

The class $\mathcal{A}$ of all trivial objects in $\mathcal{X}$ forms a connectedness class.

Proof:

We just need to show that $I_{\mathcal{N}}(T_{\mathcal{N}}(D_{\mathcal{N}}(J_{\mathcal{N}}(\mathcal{A})))) \subseteq \mathcal{A}$. It is easy to see directly from its definition in Proposition 2.2 that $J_{\mathcal{N}}(\mathcal{A})$ is the discrete closure operator. Consequently, $D_{\mathcal{N}}(J_{\mathcal{N}}(\mathcal{A})) = \mathcal{X}$. Now if $X \in I_{\mathcal{N}}(T_{\mathcal{N}}(D_{\mathcal{N}}(J_{\mathcal{N}}(\mathcal{A}))))$, then every $n \in \mathcal{N}_X$ is $T_{\mathcal{N}}(D_{\mathcal{N}}(J_{\mathcal{N}}(\mathcal{A})))$-dense. Since $X \in D_{\mathcal{N}}(J_{\mathcal{N}}(\mathcal{A})) = \mathcal{X}$, then every $n \in \mathcal{N}_X$ is $T_{\mathcal{N}}(D_{\mathcal{N}}(J_{\mathcal{N}}(\mathcal{A})))$-closed. Now, if X does not contain T, then by assumption $X \in \mathcal{A}$. If X contains T, then from the above we easily obtain that the $\mathcal{N}$-morphism $T \xrightarrow{t_X} X$ is an isomorphism. Thus $X \in \mathcal{A}$. □

PROPOSITION 3.11

For every $\mathcal{B} \in S(\mathcal{X})$, $\Delta_{\mathcal{N}}(\mathcal{B})$ is closed under the formation of products.

Proof:

Let X be a nonempty object in $\mathcal{B}$ and consider a morphism $X \xrightarrow{f} \prod Y_i$ with $Y_i \in \Delta_{\mathcal{N}}(\mathcal{B})$ for every $i \in I$. Clearly, if $\prod Y_i \xrightarrow{p_i} Y_i$ is the usual projection, then the morphism $p_i \circ f$ is $\mathcal{N}$-constant and so factors through T for every $i \in I$. Hence for every $i \in I$ we obtain the factorization $p_i \circ f = m_i \circ e_i$ with $T \xrightarrow{m_i} Y_i$. By the universal property of products we obtain a morphism $T \xrightarrow{t} \prod Y_i$ such that $p_i \circ t = m_i$ for every $i \in I$. Also, since T is the terminal object we have that $e_i = e_j = e$ for every $i, j \in I$. Now, $p_i \circ f = m_i \circ e = p_i \circ t \circ e$ for every $i \in I$. The fact that $(p_i)_{i \in I}$ is a monosource implies that $f = t \circ e$, that is f factors through T and so from Lemma 3.4, f is $\mathcal{N}$-constant. Notice that if X is empty, then from Remark 3.3, any morphism with domain X is $\mathcal{N}$-constant. □

REMARK 3.12

We have already proved in [C$_3$, Proposition 2.10] that $\Delta_{\mathcal{N}}(\mathcal{B})$ is closed under $\mathcal{M}$-subobjects. This, together with the above proposition, under the appropriate hypotheses implies that $\Delta_{\mathcal{N}}(\mathcal{B})$ is an **E**-reflective subcategory of $\mathcal{X}$ (cf. [HS, Theorem 37.1] or [AHS, theorem 16.8]).

DEFINITION 3.13

A non-empty family $(M_i \xrightarrow{m_i} X)_{i \in I}$ of $\mathcal{M}$-subobjects of X is said to be disjoint if $\cap M_i$ is empty or $|I| = 1$.

DEFINITION 3.14

(a) We say that a non-empty disjoint family $(M_i \xrightarrow{m_i} X)_{i \in I}$ of non-empty $\mathcal{M}$-subobjects of X has a *strong* **E**-*quotient* if there is an **E**-morphism $X \xrightarrow{q} Q$ such that:

i) $q \circ m_i$ factors through T for every $i \in I$;

ii) for every morphism $T \xrightarrow{t_Q} Q$ we have that either $q^{-1}(t_Q) \simeq T$ or there is an element

$i_0 \in I$ such that $m_{i_0} = q^{-1}(t_Q)$;

iii) for any **E**-morphism $X \xrightarrow{g} Y$ such that $g \circ m_i$ factors through T for every $i \in I$, there exists a morphism $Q \xrightarrow{h} Y$ such that $h \circ q = g$.

(b) An $\mathcal{X}$-morphism $X \xrightarrow{q} Q$ is called a *strong* **E**-*quotient* if there is a non-empty disjoint family $(M_i \xrightarrow{m_i} X)_{i \in I}$ of non-empty $\mathcal{M}$-subobjects of X, that has q as a strong **E**-quotient.

(c) We say that $\mathcal{X}$ has strong **E**-quotients if for any $X \in \mathcal{X}$, any non-empty disjoint family of non-empty $\mathcal{M}$-subobjects $(M_i \xrightarrow{m_i} X)_{i \in I}$ has a strong **E**-quotient.

REMARK 3.15

(a) Let $X \xrightarrow{q} Q$ be a strong **E**-quotient of the family $(M_i \xrightarrow{m_i} X)_{i \in I}$ and let $M_{j_0} \xrightarrow{m_{j_0}}$ be a non-trivial element of this family. By condition 3.14(a) i), $q \circ m_{j_0}$ factors through T. Call its image $T \xrightarrow{t_Q} Q$. Clearly we have that $m_{j_0} \leq q^{-1}(t_Q)$. Condition 3.14(a) ii) and the disjointness of the family $(M_i \xrightarrow{m_i} X)_{i \in I}$ imply that $m_{j_0} = q^{-1}(t_Q)$.

(b) We observe that if **E** is a class of episinks then any two strong **E**-quotients with respect to the same family of $\mathcal{M}$-subobjects $(M_i \xrightarrow{m_i} X)_{i \in I}$ must be isomorphic. The same conclusion can be drawn if we require uniqueness of the morphism h in property iii).

(c) Consider the category **Top** of topological spaces with the (episink,embedding)-factorization structure. If $X \xrightarrow{q} Q$ is a topological quotient, then it can be easily seen that q is a strong **E**-quotient with respect to the family of subspaces $(q^{-1}\{t_i\})_{t_i \in Q}$. On the other hand, let $X \xrightarrow{q} Q$ be a strong **E**-quotient with respect to a disjoint family of non-empty subspaces $(M_i \xrightarrow{m_i} X)_{i \in I}$. Clearly q is a surjective continuous function. Consider the topological quotient that is obtained by identifying the points of X that lie in the same subspace M_i, that is for each $x, y \in X$ we define the relation: $x\mathcal{R}y$ if and only if either $x = y$ or there is an element $i \in I$ such that $x, y \in M_i$. If P is the induced topological quotient, then from property iii) of strong **E**-quotients we obtain a continuous function $Q \xrightarrow{h} P$ such that $h \circ q = p$. Moreover, since q is constant on the fibers of p and P has the quotient topology, we obtain a continuous function $P \xrightarrow{k} Q$ such that $k \circ p = q$. This together with $h \circ q = p$ yields a homomorphism between P and Q. Thus, we can conclude that **Top** has strong **E**-quotients.

(d) It is very easy to see that in the category **Ab** of abelian groups with the (episink,injective)-factorization structure the strong **E**-quotients are precisely the surjective homomorphisms. Consequently **Ab** has strong **E**-quotients.

(e) Clearly, the category **Grp** of groups with the (episink,injective)-factorization structure does not have strong **E**-quotients since in this case not every subgroup in normal.

THEOREM 3.16

Suppose that $\mathbf{E}$ *is a class of episinks that is closed under the formation of pullbacks along elements of* $\mathcal{M}$, $\mathcal{X}$ *is* $\mathbf{E}$*-cowell powered with products and strong* $\mathbf{E}$*-quotients and that* $\mathcal{N}$ *is closed under the formation of direct images.*

Let $\mathcal{A}$ *be a class of objects that is isomorphism closed and contains all trivial objects. Assume that there is a weakly hereditary closure operator* C *such that* $\mathcal{A} \subseteq I_{\mathcal{N}}(T_{\mathcal{N}}(D_{\mathcal{N}}(C)))$ *and that the class of strong* $\mathbf{E}$*-quotients is closed under the formation of pullbacks along* C*-closures of* $\mathcal{M}$*-subobjects. Then,* $\mathcal{A}$ *is an* $\mathcal{N}$*-connectedness class if and only if it satisfies the following conditions:*

(a) $\mathcal{A}$ *is closed under* $\mathbf{E}$*-quotients;*

(b) for every $\mathcal{M}$*-subobject* $M \xrightarrow{m} X$ *that contains* T, $M \in \mathcal{A}$ *implies that its* C*-closure* $M^{C} \in \mathcal{A}$*;*

(c) if $(M_i \xrightarrow{m_i} X)_{i\in I}$ *is a family of* $\mathcal{M}$*-subobjects such that each* $M_i \in \mathcal{A}$ *and* $\cap M_i$ *contains* T*, then also* $\vee M_i \in \mathcal{A}$*;*

(d) if $X \xrightarrow{q} Q$ *is a strong* $\mathbf{E}$*-quotient such that* $Q \in \mathcal{A}$ *and for every morphism* $T \xrightarrow{t_Q} Q$, $q^{-1}(T) \in \mathcal{A}$*, then* $X \in \mathcal{A}$*.*

Proof:

($\Rightarrow$). (a). Assume that $\mathcal{A}$ is an $\mathcal{N}$-connectedness class, i.e., as observed in Remark 2.12, $\mathcal{A} = I_{\mathcal{N}}(T_{\mathcal{N}}(D_{\mathcal{N}}(J_{\mathcal{N}}(\mathcal{A}))))$. Let $X \xrightarrow{f} Y$ be an $\mathcal{X}$-morphism with $X \in \mathcal{A}$ and $f \in \mathbf{E}$. If X contains T, then the result follows from Proposition 3.7. If X is empty, then as a consequence of Assumption 3.2(a) we have that $f \in \mathcal{M} \cap \mathbf{E}$ and so it is an isomorphism. Thus, $Y \in \mathcal{A}$.

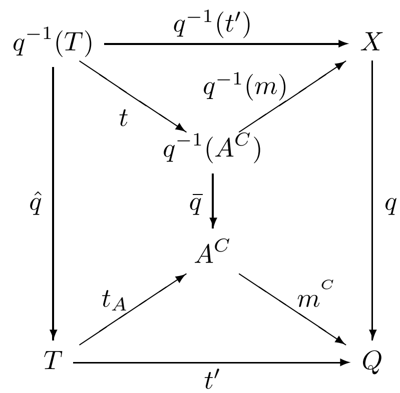
(b) follows from [C_5, Proposition 3.2(b)].

(c) follows from [C_5, Proposition 3.3(a)].

(d). Now let $X \xrightarrow{q} Q$ be a strong $\mathbf{E}$-quotient for the non-empty disjoint family of non-empty$\mathcal{M}$-subobjects $(M_i \xrightarrow{m_i} X)_{i\in I}$, such that Q is $(J_{\mathcal{N}}(\mathcal{A}), \mathcal{N})$-connected. Let (e_i, t_i) be the $(\mathbf{E}, \mathcal{M})$-factorization of $q \circ m_i$ with $T \xrightarrow{t_i} Q$. Clearly we have that $m_i \leq q^{-1}(t_i)$. By assumption we have that $q^{-1}(T)$ is $(J_{\mathcal{N}}(\mathcal{A}), \mathcal{N})$-connected. Notice that under our assumptions, $D_{\mathcal{N}}(J_{\mathcal{N}}(\mathcal{A})) = \Delta_{\mathcal{N}}(\mathcal{A})$ is $\mathbf{E}$-reflective in $\mathcal{X}$ (cf. Remark 3.12 and [HS, Theorem 37.1]). So, let us consider the $\mathbf{E}$-reflection $X \xrightarrow{r_X} rX$ into $D_{\mathcal{N}}(J_{\mathcal{N}}(\mathcal{A}))$. Since $q^{-1}(T)$ contains T, from Proposition 2.10(a) and Lemma 3.4 we have that $r_X \circ q^{-1}(t_i)$ factors through T and consequently so does $r_X \circ m_i$. Thus, by definition of strong $\mathbf{E}$-quotient, there exists a morphism $Q \xrightarrow{p} rX$ such that $p \circ q = r_X$. Notice that since $\mathbf{E}$ is a class of episinks, the fact that r_X and q belong to $\mathbf{E}$ implies that also p belongs to $\mathbf{E}$. Since Q is $(J_{\mathcal{N}}(\mathcal{A}), \mathcal{N})$-connected and $p \in \mathbf{E}$ then, from (a) rX is too. However, the fact that $rX \in D_{\mathcal{N}}(J_{\mathcal{N}}(\mathcal{A}))$ implies that id_{r_X} factors through T. This clearly implies that $rX \simeq T$. Consequently from Proposition 3.9, X is $(J_{\mathcal{N}}(\mathcal{A}), \mathcal{N})$-connected, i.e., $X \in \mathcal{A}$.

($\Leftarrow$). Now let us assume that $\mathcal{A}$ contains all trivial objects and satisfies conditions (a) through (d). To show that $\mathcal{A}$ is an $\mathcal{N}$-connectedness class we use the characterization in Theorem 3.8. Clearly, if X is non-trivial and belongs to $\mathcal{A}$, then any image of it is non-trivial and from (a) it satisfies the condition in Theorem 3.8.

Conversely, let $X \in \mathcal{X}$ be non-trivial. For any morphism $T \xrightarrow{t_i} X$ consider the family of all $\mathcal{M}$-subobjects $M_i \xrightarrow{m_i} X$ containing $T \xrightarrow{t_i} X$ such that $M_i \in \mathcal{A}$. Clearly condition (c) implies that $\vee M_i \in \mathcal{A}$. Call this object A_{t_i}. This yields a family $(A_{t_i} \xrightarrow{a_i} X)_{i \in I}$ of non-isomorphic $\mathcal{M}$-subobjects (the isomorphic ones are identified) such that $A_{t_i} \in \mathcal{A}$ for every $i \in I$. Notice that if $|I| > 1$, then $\cap A_{t_i}$ does not contain T. As a matter of fact, the existence of a morphism $T \xrightarrow{t} \cap A_{t_i}$ would yield a morphism $T \xrightarrow{t_0} X$. Condition (c) would imply that $A_{t_0} \simeq A_{t_i}$, for each $i \in I$, that is $|I| = 1$, which is a contradiction. Let $X \xrightarrow{q} Q$ be the strong $\mathbf{E}$-quotient of the family $(A_{t_i} \xrightarrow{a_i} X)_{i \in I}$. Since X is non-trivial, Q cannot be empty, so we can consider a morphism $T \xrightarrow{t_q} Q$. By the property ii) of strong $\mathbf{E}$-quotients, we have that $q^{-1}(T)$ belongs to $\mathcal{A}$. If $Q \simeq T$ then $X \simeq q^{-1}(T)$ and so X belongs to $\mathcal{A}$. So, let us assume that Q is non-trivial. Now, from the condition in Theorem 3.8, Q has a non-trivial $\mathcal{M}$-subobject $A \xrightarrow{m} Q$ with $A \in \mathcal{A}$. Consider its C-closure $A^{C} \xrightarrow{m^{C}} Q$. By condition (b), $A^{C} \in \mathcal{A}$. Let $T \xrightarrow{t_A} A^{C}$ be a morphism and consider the following commutative diagram



where t is the morphism induced by the universal property of pullbacks. Now, since $X \xrightarrow{q} Q$ is the strong $\mathbf{E}$-quotient of the family $(A_{t_i} \xrightarrow{a_i} X)_{i \in I}$, again from property ii) of strong $\mathbf{E}$-quotients, we have that either $q^{-1}(T) \simeq A_{t_j}$ for some $j \in I$ or $q^{-1}(T) \simeq T$. So, $q^{-1}(T) \in \mathcal{A}$. Notice that in the case that $q^{-1}(T) \simeq T$, since $q^{-1}(t') = t_k$ for some $k \in I$ with $T \xrightarrow{t_k} X$, by property i) of strong $\mathbf{E}$-quotients, we easily conclude that $A_{t_k} \xrightarrow{a_k} X$ must factor through $T \xrightarrow{t'} Q$. This clearly yields that $q^{-1}(T) \simeq A_{t_k}$. Thus, in any case we have that $q^{-1}(T) \simeq A_{t_j}$ for some $j \in I$. Thus, $q^{-1}(T) \in \mathcal{A}$ for every morphism $T \xrightarrow{t_A} A^{C}$. Now, since the right and the outer squares of the above diagram are pullbacks, so is the left one. Notice that our assumptions on C and $\mathcal{A}$ imply that $\bar{q}$ is a strong $\mathbf{E}$-quotient. Thus from (d), $q^{-1}(A^{C}) \in \mathcal{A}$ and consequently, by construction of A_{t_j} we have that $A_{t_j} \simeq q^{-1}(A^{C})$. Therefore $q^{-1}(T) \xrightarrow{t} q^{-1}(A^{C})$ is an isomorphism. So,

$\bar{q} \circ t = t_A \circ \hat{q}$ implies $\bar{q} = t_A \circ \hat{q} \circ t^{-1}$. Now, notice that $t_A \in \mathcal{M}$. Since $\bar{q} \in \mathbf{E}$, by assumption we have that $\hat{q} \in \mathbf{E}$ as a pullback of $\bar{q}$ along t_A. Thus, $\hat{q} \circ t^{-1} \in \mathbf{E}$. The uniqueness of $(\mathbf{E}, \mathcal{M})$-factorizations implies that t_A is an isomorphism. Clearly, this contradicts the fact that A^C is non-trivial. □

REMARK 3.17

We would like to observe that the hypotheses of Theorem 3.16 are not as strong as they may first appear. As a matter of fact, from [C_2, Proposition 2.7] we know that for any class of $\mathcal{X}$-objects $\mathcal{A}$, the closure operator $J_{\mathcal{N}}(\mathcal{A})$ is always weakly hereditary and clearly satisfies $\mathcal{A} \subseteq I_{\mathcal{N}}(T_{\mathcal{N}}(D_{\mathcal{N}}(J_{\mathcal{N}}(\mathcal{A}))))$. Moreover, if in **Ab** we consider the (episinks,monomorphism)-factorization structure, due to the fact that in this case the strong **E**-quotients are exactly the surjective homomorphisms, we have that the pullback condition in the hypotheses of Theorem 3.16 is satisfied for any closure operator C.

In the category **Top** with the (episink,embedding)-factorization structure, since as observed in Remark 3.15, the strong **E**-quotients are precisely the topological quotients, the Kuratowski closure K certainly satisfies the hypotheses of Theorem 3.16. Since, if $\mathcal{N}$ is the class of all non-empty embeddings, $I_{\mathcal{N}}(T_{\mathcal{N}}(D_{\mathcal{N}}(K))))$ consists of all connected topological spaces, the above theorem clearly applies to any subclass $\mathcal{A}$ of connected topological spaces satisfying conditions (a) through (d). Notice that it was proved in [AW, Lemma 3.11] that condition (b) of the above theorem is always satisfied for the Kuratowski closure by any class of topological spaces satisfying (a), (c) and (d). Therefore, in this case we obtain as a special case the characterization of topological connectednesses given by Arhangel'skii and Wiegandt [AW, Theorem 3.10]. We further observe that in [AW, Proposition 4.2] it was shown that connected topological spaces is the largest non-trivial connectedness in **Top**, therefore our special case characterizes all non-trivial connectednesses in **Top**.

Several examples that illustrate the above theory can be found in [C_5]. Here we conclude with a few examples that show that in those cases in which the above concepts of $\mathcal{N}$-constant, $\mathcal{N}$-fixed and $\mathcal{N}$-dependent do not agree with the classical notion of constant function, we obtain some new Galois correspondences.

EXAMPLE 3.18

Consider the category **Top** of topological spaces with $\mathcal{M}$ consisting of all embeddings and $\mathcal{N}$ all nonempty clopen subsets. If $\mathcal{A}$ is the class **Discr** of discrete topological spaces, then for every $\mathcal{M}$-subobject of $X \in$ **Top**, $M^{T_{\mathcal{N}}(\mathcal{A})} = \cap\{f^{-1}(f(M)) : X \xrightarrow{f} Y, Y \text{ discrete }\}$. Consequently, $I_{\mathcal{N}}(T_{\mathcal{N}}(\mathcal{A})) = \{X :$ for every non-empty clopen subset $M \subseteq X, M^{T_{\mathcal{N}}(\mathcal{A})} = X\} = \{X :$ for every non-empty clopen $M \subseteq X, X \xrightarrow{f} Y$ and Y discrete $, f(X) = f(M)\}$. Clearly $I_{\mathcal{N}}(T_{\mathcal{N}}(\mathcal{A}))$ contains the class **Conn** of connected topological spaces. On the other hand, if X is not connected then there is a non-empty clopen subset $M \neq X$. As a consequence we can find a morphism $X \xrightarrow{f} D_2$, with D_2 being the two point discrete topological space with underlying set $\{0,1\}$ and f defined as $f(M) = 0, f(X-M) = 1$. Hence, X does not belong to $I_{\mathcal{N}}(T_{\mathcal{N}}(\mathbf{Discr}))$ and so $I_{\mathcal{N}}(T_{\mathcal{N}}(\mathbf{Discr})) = \mathbf{Conn}$. Clearly, if $\mathcal{B}$ consists of all connected topological spaces, then $J_{\mathcal{N}}(\mathcal{B})$ is the discrete closure and $D_{\mathcal{N}}(J_{\mathcal{N}}(\mathcal{B})) = \mathbf{Top}$. Using the properties of Galois connections we obtain that $I_{\mathcal{N}}(T_{\mathcal{N}}(\mathbf{Top})) = \mathbf{Conn}$. This is clearly different from the classical correspondence between connected and totally disconnected topological spaces.

We also observe that if $\mathcal{A}$ is a class of connected topological spaces, then $T_{\mathcal{N}}(\mathcal{A})$ is the indiscrete closure and clearly $I_{\mathcal{N}}(T_{\mathcal{N}}(\mathcal{A})) = \mathbf{Top}$. On the other hand, if $\mathcal{B}$=$\mathbf{Top}$, and N is a non-empty clopen subset of X, then using the morphism $X \xrightarrow{id_X} X$, from Proposition 2.3, we obtain that $N^{J_{\mathcal{N}}(\mathcal{B})} = X$. Consequently, $X \in D_{\mathcal{N}}(J_{\mathcal{N}}(\mathcal{B}))$ if and only if for every non-empty clopen subset N of X, $N = N^{J_{\mathcal{N}}(\mathcal{B})} = X$, i.e., $D_{\mathcal{N}}(J_{\mathcal{N}}(\mathcal{B}))$ consists of all connected topological spaces.

In conclusion, this Galois connection yields the following pairs of fixed points: $(\mathbf{Top}, \mathbf{Conn})$ and $(\mathbf{Conn}, \mathbf{Top})$.

EXAMPLE 3.19

Consider the category $\mathbf{Grp}$ of groups with $\mathcal{M}$ consisting of all monomorphisms and let $\mathcal{N}$ be the class of all normal subgroups different from zero. If Y is simple, then any morphism $X \xrightarrow{f} Y$ is $\mathcal{N}$-dependent, so for instance $\mathbf{Z}_{(\mathbf{3})} \xrightarrow{\mathbf{id}} \mathbf{Z}_{(\mathbf{3})}$ is an $\mathcal{N}$-dependent morphism that is not constant. Now let us consider the class $\mathcal{B}$ of all simple groups. Consider the $\mathcal{M}$-subobject $M \xrightarrow{m} Y$, $Y \in \mathbf{Grp}$. Since for every $X \in \mathcal{B}$ the only $n \in \mathcal{N}_X$ is X itself we have that if $n_f \leq m$ then also $(id_X)_f \leq m$ and so $m^{J_{\mathcal{N}}(\mathcal{B})} \simeq m$. Thus $J_{\mathcal{N}}(\mathcal{B})$ is the discrete closure operator. Consequently $D_{\mathcal{N}}(J_{\mathcal{N}}(\mathcal{B})) = \mathbf{Grp}$. Now, let $\mathcal{A}$=$\mathbf{Grp}$. Then it is easy to see that for an $\mathcal{M}$-subobject $M \xrightarrow{m} X$ different from zero, $m^{T_{\mathcal{N}}(\mathcal{A})}$ is the normal closure of M, that is the intersection of all normal subgroups of X containing M. It follows that $I_{\mathcal{N}}(T_{\mathcal{N}}(\mathbf{Grp}))$ consists of all simple groups. On the other side, for every class of simple groups $\mathcal{A}$, $T_{\mathcal{N}}(\mathcal{A})$ is the indiscrete closure and $I_{\mathcal{N}}(T_{\mathcal{N}}(\mathcal{A})) = \mathbf{Grp}$. Now let $\mathcal{B}$= $\mathbf{Grp}$ and let $Y \in D_{\mathcal{N}}(J_{\mathcal{N}}(\mathcal{B}))$. Consider a normal subgroup N of Y different from zero. Clearly using the morphism $Y \xrightarrow{id_Y} Y$, from Proposition 2.3 we obtain that $N^{J_{\mathcal{N}}(\mathcal{B})} = Y$. However, if $Y \in D_{\mathcal{N}}(J_{\mathcal{N}}(\mathcal{B}))$, then we must have that $N^{J_{\mathcal{N}}(\mathcal{B})} = N$. So, $N = Y$, that is Y is a simple group. Now, if Y is simple, then any $\mathcal{N}$-subobject of Y is $J_{\mathcal{N}}(\mathcal{B})$-closed by default, since the only normal subgroup of Y different from zero is Y itself. In conclusion, $D_{\mathcal{N}}(J_{\mathcal{N}}(\mathcal{B}))$ consists of all simple groups.

Let $\mathcal{A}$= $\mathbf{Ab}$ and let $M \xrightarrow{m} X$ be an $\mathcal{M}$-subobject. Using the characterization in 2.3 we have that for every subgroup P normal in Y, $Y \in \mathbf{Ab}$ and $X \xrightarrow{f} Y$, $f^{-1}(P)$ is normal in X and satisfies $X/f^{-1}(P) \simeq f(X)/(P \cap f(X))$ is abelian. On the other hand, if N is a normal subgroup of X containing M such that X/N is abelian, then N must contain the subgroup X'

generated by the commutators of X. Consider the homomorphism $X \xrightarrow{q} X/X'$. Since $X' \subseteq N$ we obtain a morphism $X/X' \xrightarrow{t} X/N$ such that $t \circ q = q'$ with $X \xrightarrow{q'} X/N$. Therefore, we have that $N = q^{-1}(t^{-1}(\{0\}))$. If $t^{-1}(\{0\}) \neq \{0\}$, that is $N \neq X'$, then N occurs in the construction of $T_{\mathcal{N}}(\mathcal{A})$. In conclusion, $M^{T_{\mathcal{N}}(\mathcal{A})}$ consists of the intersection of all normal subgroups of X containing M and properly containing the commutator subgroup of X (we recall that $X' \subseteq N$ is equivalent to X/N being abelian.) Thus $I_{\mathcal{N}}(T_{\mathcal{N}}(\mathbf{Ab}))$ consists of all groups X that do not have any proper normal subgroup that properly contains the commutator subgroup. We call this the class of *quasi-perfect groups*. We observe that in the case that $\mathcal{N}$ consists of all normal subgroups then $I_{\mathcal{N}}(T_{\mathcal{N}}(\mathbf{Ab}))$ consists of all perfect subgroups (cf. [C_3, Example 3.5]).

EXAMPLE 3.20

Consider the category $\mathbf{Ab}$ of abelian groups with $\mathcal{M}$ consisting of all monomorphisms and let $\mathcal{N}$ be the class of all torsion subgroups different from zero. Clearly, if $\mathbf{Z}$ is the additive group of integers, the quotient morphism $\mathbf{Z} \xrightarrow{\mathbf{q}} \mathbf{Z}/\mathbf{2Z}$ is $\mathcal{N}$-dependent but not constant. Let $\mathcal{A}$ be the class $\mathbf{TF}$ of all torsion free abelian groups. Since for every $Y \in \mathcal{A}$ there is no $n \in \mathcal{N}_X$, we have that for every $\mathcal{M}$-subobject $M \xrightarrow{m} X$, $M^{T_{\mathcal{N}}(\mathcal{F})} = X$, that is, $T_{\mathcal{N}}(\mathcal{F})$ is the indiscrete closure operator. Consequently, $I_{\mathcal{N}}(T_{\mathcal{N}}(\mathcal{F})) = \mathbf{Ab}$. Now let $\mathcal{B}= \mathbf{Ab}$ and let $0 \neq M \leq Y$ be a torsion subgroup. Then, using the morphism $Y \xrightarrow{id_Y} Y$, from Proposition 2.3, we obtain that $M^{J_{\mathcal{N}}(\mathcal{B})} = Y$. Clearly as a consequence we obtain that $D_{\mathcal{N}}(J_{\mathcal{N}}(\mathcal{B})) = \mathbf{TF}$.

Let $\mathcal{A}= \mathbf{Ab}$. It is easy to see that for every torsion subgroup $0 \neq M \leq X$, $M^{T_{\mathcal{N}}(\mathcal{A})} = M$. Consequently $I_{\mathcal{N}}(T_{\mathcal{N}}(\mathbf{Ab})) = \{X \in \mathbf{Ab} : \forall 0 \neq M \leq X, M \text{ torsion }, X = M^{T_{\mathcal{N}}(\mathbf{Ab})} = M\} = \mathbf{TF}$. If $\mathcal{B}$=$\mathbf{TF}$, the clearly $J_{\mathcal{N}}(\mathcal{B})$ is the discrete closure operator and consequently $D_{\mathcal{N}}(J_{\mathcal{N}}(\mathcal{B})) = \mathbf{Ab}$.

Let $\mathcal{B}$=$\mathbf{T}$, the class of all torsion abelian groups and let $0 \neq M \leq X$ be a torsion subgroup. Then using as a morphism the inclusion of the torsion of X, $t(X) \xrightarrow{t} X$, from proposition 2.3, we obtain that $M^{J_{\mathcal{N}}(\mathcal{B})} = t(X)$. So, $D_{\mathcal{N}}(J_{\mathcal{N}}(\mathcal{B})) = \{X \in \mathbf{Ab} : \forall 0 \neq M \leq X, M \text{ torsion }, M = M^{J_{\mathcal{N}}(\mathcal{B})} = t(X)\}$. In other words, this class consists of all abelian groups X such that $t(X)$ does not have any non-trivial subgroup. This class properly contains $\mathbf{TF}$ since any finite group of prime order belongs to it but not to $\mathbf{TF}$. This clearly differs from the classical correspondence between torsion and torsion-free abelian groups.

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