CLOSURE OPERATOR CONSTRUCTIONS DEPENDING ON ONE PARAMETER

G. Castellini * D. Holgate †

ABSTRACT: Let $\mathcal{X}$ be an $(\mathbf{E},\mathcal{M})$-category for sinks. For each subclass $\mathcal{N}$ of $\mathcal{M}$, two new Galois connections that generalize the Clementino-Tholen connectedness and separation Galois connections are introduced. In particular, closure operator constructions that generalize the regular and coregular closure operators are given. A big diagram of several different types of Galois connections is built. The usefulness of the dependence on the parameter $\mathcal{N}$ is shown in that many previously obtained closure constructions can be deduced from the diagram by choosing $\mathcal{N}$ in different ways.

KEY WORDS: Galois connection, (regular and coregular) closure operator.

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0 INTRODUCTION

Let $\mathcal{X}$ be an $(\mathbf{E},\mathcal{M})$-category for sinks, let $CL(\mathcal{X},\mathcal{M})$ denote the conglomerate of all closure operators on $\mathcal{X}$ with respect to $\mathcal{M}$ with the usual pointwise order and let $S(\mathcal{X})$ denote the conglomerate of all subclasses of objects of $\mathcal{X}$, ordered by inclusion. For each subclass $\mathcal{N}$ of $\mathcal{M}$ and closure operator C, the function $CL(\mathcal{X},\mathcal{M}) \xrightarrow{A} S(\mathcal{X})^{\mathbf{op}}$ defined by: $A(C) = \{X \in \mathcal{X} :$ every $m \in \mathcal{N}$ with domain X is C-closed$\}$ is shown to preserve suprema and consequently it gives rise to a Galois connection $CL(\mathcal{X},\mathcal{M}) \underset{P}{\overset{A}{\underset{\longleftarrow}{\overset{\perp}{\longrightarrow}}}} S(\mathcal{X})^{\mathbf{op}}$. Similarly, the function $CL(\mathcal{X},\mathcal{M}) \xrightarrow{B} S(\mathcal{X})$ defined by: $B(C) = \{X \in \mathcal{X} :$ every $m \in \mathcal{N}$ with domain X is C-dense$\}$ preserves infima and so yields a Galois connection $S(\mathcal{X}) \underset{B}{\overset{Q}{\underset{\longleftarrow}{\overset{\perp}{\longrightarrow}}}} CL(\mathcal{X},\mathcal{M})$. In Section 2, constructive descriptions of the closure operators $P(\mathcal{A})$ and $Q(\mathcal{B})$ for $\mathcal{A},\mathcal{B} \subseteq \mathcal{X}$ are presented.

The above Galois connections yield as special cases the Clementino-Tholen connectedness and separation Galois connections ([CT]) and have been used ([CH$_1$]) to provide a link between the connectedness notion with respect to a closure operator introduced in [CT] and the one by Castellini ([C$_{1-5}$]).

The assignments A and B have an obvious duality to each other – one depending on closedness, the other on density. This duality extends to $P(\mathcal{A})$ and $Q(\mathcal{B})$ which are generalizations of the regular closure operator and its dual the coregular closure operator (cf. [DG], [DT] and [CT]).

In Section 3, we see how the present constructions do indeed generalize and simplify previous constructions, particularly in [CKS$_2$]. These newly introduced Galois connections are then pasted together

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with some previously introduced ones in order to build a big diagram of Galois connections, all depending on the parameter $\mathcal{N}$. The final aim of this paper is to show that this dependence on $\mathcal{N}$ is useful since the variation of the parameter $\mathcal{N}$ enables us to obtain many existing closure operator constructions by focusing on different parts of the diagram.

We use the terminology of [AHS] throughout the paper[1].

1 PRELIMINARIES

Throughout we consider a category $\mathcal{X}$ and a fixed class $\mathcal{M}$ of $\mathcal{X}$-monomorphisms, which contains all $\mathcal{X}$-isomorphisms and is closed under composition . It is assumed that $\mathcal{X}$ is $\mathcal{M}$-complete; i.e.,

> Pullbacks of $\mathcal{M}$-morphisms exist and belong to $\mathcal{M}$, and multiple pullbacks of (possibly large) families of $\mathcal{M}$-morphisms with common codomain exist and belong to $\mathcal{M}$.

One of the consequences of the above assumptions is that there is a uniquely determined class $\mathbf{E}$ of sinks in $\mathcal{X}$ such that $\mathcal{X}$ is an $(\mathbf{E},\mathcal{M})$-*category for sinks*. This implies the following features of $\mathcal{M}$ and $\mathbf{E}$ (cf. [AHS] for the dual case):

PROPOSITION 1.1 *(1) Every isomorphism is in both $\mathcal{M}$ and $\mathbf{E}$ (as a singleton sink).*

(2) $\mathcal{M}$ is closed under $\mathcal{M}$-relative first factors, i.e., if $n \circ m \in \mathcal{M}$, and $n \in \mathcal{M}$, then $m \in \mathcal{M}$.

(3) $\mathcal{M}$ and $\mathbf{E}$ are closed under composition, in particular for $\mathbf{E}$ this has the consequence that if $(X_i \xrightarrow{e_i} Y)_{i\in I}$ is a sink in $\mathbf{E}$ and the morphism $Y \xrightarrow{f} Z$ (seen as a singleton sink) belongs to $\mathbf{E}$, then so does the sink $(X_i \xrightarrow{f\circ e_i} Z)_{i\in I}$.

(4) Pullbacks of $\mathcal{X}$-morphisms in $\mathcal{M}$ exist and belong to $\mathcal{M}$.

(5) The $\mathcal{M}$-subobjects of every $\mathcal{X}$-object form a (possibly large) complete lattice; suprema are formed via $(\mathbf{E},\mathcal{M})$-factorizations and infima are formed via intersections. $\square$

If $X \xrightarrow{f} Y$ is an $\mathcal{X}$-morphism and $M \xrightarrow{m} X$ is an $\mathcal{M}$-subobject, then $M \xrightarrow{e_{f\circ m}} M_f \xrightarrow{m_f} Y$ will denote the $(\mathbf{E},\mathcal{M})$-factorization of $f \circ m$. $M_f \xrightarrow{m_f} Y$ will be called the direct image of m along f and $M \xrightarrow{e_{f\circ m}} M_f$ will be called the restriction of the morphism f to the $\mathcal{M}$-subobject m. If $N \xrightarrow{n} Y$ is an $\mathcal{M}$-subobject, then the pullback $f^{-1}(N) \xrightarrow{f^{-1}(n)} X$ of n along f will be called the inverse image of n along f. Whenever no confusion is likely to arise, we will denote the morphism $e_{f\circ m}$ simply e_f.

DEFINITION 1.2 *A closure operator C on $\mathcal{X}$ (with respect to $\mathcal{M}$) is a family $\{(\ )^C_X\}_{X\in\mathcal{X}}$ of functions on the $\mathcal{M}$-subobject lattices of $\mathcal{X}$ with the following properties that hold for each $X \in \mathcal{X}$:*

(a) [extension] $m \leq (m)^C_X$, for every $\mathcal{M}$-subobject $M \xrightarrow{m} X$;

(b) [order-preservation] $m \leq n \Rightarrow (m)^C_X \leq (n)^C_X$ for every pair of $\mathcal{M}$-subobjects of X;

(c) [continuity] If $f^{-1}(m)$ is the inverse image of the $\mathcal{M}$-subobject $M \xrightarrow{m} Y$ along $X \xrightarrow{f} Y$ and $f^{-1}((m)^C_Y)$ is the inverse image of the closure of m along f, then $(f^{-1}(m))^C_X \leq f^{-1}((m)^C_Y)$.

[1] Paul Taylor's Commutative Diagrams in TEX macro package was used to typeset most of the diagrams in this paper.

Condition (a) implies that for every closure operator C on $\mathcal{X}$, every $\mathcal{M}$-subobject $M \xrightarrow{m} X$ has a canonical factorization

$$\begin{tikzcd} M \arrow[r, "t"] \arrow[dr, "m"'] & (M)^{C}_{X} \arrow[d, "(m)^{C}_{X}"] \\ & X \end{tikzcd}$$

where $(m)^{C}_{X}$ is called the C-*closure* of the subobject m.

When no confusion is likely we will write m^{C} rather than $(m)^{C}_{X}$ and for notational symmetry we will denote the morphism t by m_{C}.

REMARK 1.3 *(1) Notice that in the above definition, under condition (b), the continuity condition (c) is equivalent to the following statement concerning direct images: if $M \xrightarrow{m} X$ is an $\mathcal{M}$-subobject and $X \xrightarrow{f} Y$ is a morphism, then $((m)^{C}_{Y})_f \le (m_f)^{C}_{Y}$, i.e., the direct image of the closure of m is less than or equal to the closure of the direct image of m; (cf. [DG]).*

(2) Under condition (a), both order-preservation and continuity, i.e., conditions (b) and (c) together are equivalent to the following: given $\mathcal{M}$-subobjects $M \xrightarrow{m} X$ and $N \xrightarrow{n} Y$, if f and g are morphisms such that $n \circ g = f \circ m$, then there exists a unique morphism d such that the following diagram commutes.

$$\begin{tikzcd} M \arrow[rr, "g"] \arrow[dd, "m"'] \arrow[dr, "m_{C}"] & & N \arrow[dd, "n"] \arrow[dr, "n_{C}"] & \\ & M^{C} \arrow[rr, "d"] \arrow[dl, "m^{C}"] & & N^{C} \arrow[dl, "n^{C}"] \\ X \arrow[rr, "f"'] & & Y & \end{tikzcd}$$

DEFINITION 1.4 *Given a closure operator C, we say that $m \in \mathcal{M}$ is C-closed if m_{C} is an isomorphism. An $\mathcal{X}$-morphism f is called C-dense if for every $(\mathbf{E}, \mathcal{M})$-factorization (e, m) of f we have that m^{C} is an isomorphism. We call C idempotent provided that m^{C} is C-closed for every $m \in \mathcal{M}$. C is called weakly hereditary if m_{C} is C-dense for every $m \in \mathcal{M}$.*

Notice that Definition 1.2(c) implies that inverse images of C-closed $\mathcal{M}$-subobjects are C-closed.

We denote the collection of all closure operators on $\mathcal{X}$ with respect to $\mathcal{M}$ by $CL(\mathcal{X}, \mathcal{M})$ pre-ordered as follows: $C \sqsubseteq D$ if $m^{C} \le m^{D}$ for all $m \in \mathcal{M}$ (where $\le$ is the usual order on subobjects). Notice that arbitrary suprema and infima exist in $CL(\mathcal{X}, \mathcal{M})$, they are formed pointwise in the $\mathcal{M}$-subobject fibers. The symbols $iCL(\mathcal{X}, \mathcal{M})$ and $wCL(\mathcal{X}, \mathcal{M})$ will be used to denote all idempotent and all weakly hereditary closure operators, respectively.

For more background on closure operators see, e.g., [CKS_{1-2}], [DG] and [DGT]. For a recent survey on the same topic, one could check [C_7]. Detailed proofs can be found in [H_1] and [DT].

DEFINITION 1.5 *For pre-ordered classes $(X, \leq)$ and $(Y, \leq)$, a Galois connection* $X \underset{G}{\overset{F}{\rightleftarrows}} Y$ *consists of order preserving functions F and G that satisfy $F \dashv G$, i.e., $x \leq G(F(x))$ for every $x \in X$ and $F(G(y)) \leq y$ for every $y \in Y$. (G is adjoint and has F as coadjoint or left adjoint).*

If $x \in X$ and $y \in Y$ are such that $F(x) = y$ and $G(y) = x$, then x and y are said to be corresponding fixed points of the Galois connection (X, F, G, Y).

An order preserving function $(X, \leq) \xrightarrow{H} (Y, \leq)$ is adjoint (coadjoint, respectively) if it preserves infima (suprema). In this case the coadjoint (adjoint) is given by $I(y) := \bigwedge\{x \in X : y \leq H(x)\}$ $(J(y) := \bigvee\{x \in X : H(x) \leq y\})$.

Some authors consider Galois connections to be order *reversing*. We avoid this as it prohibits composition of Galois connections. Properties and many examples of Galois connections can be found in [EKMS].

2 TWO NEW CLOSURE OPERATOR CONSTRUCTIONS

Let $S(\mathcal{X})$ denote the collection of all subclasses of objects of $\mathcal{X}$, ordered by inclusion and let $\mathcal{N}$ be a fixed subclass of $\mathcal{M}$. The aim of this Section is to introduce two new closure operator constructions that depend on the parameter $\mathcal{N}$.

PROPOSITION 2.1 *Let $CL(\mathcal{X}, \mathcal{M}) \xrightarrow{A} S(\mathcal{X})^{\mathbf{op}}$ be defined by:*

$$A(C) = \{X \in \mathcal{X} : \text{ every } m \in \mathcal{N} \text{ with domain } X \text{ is } C\text{-closed}\}$$

then, A preserves suprema and thus forms a Galois connection $CL(\mathcal{X}, \mathcal{M}) \underset{P}{\overset{A}{\rightleftarrows}} S(\mathcal{X})^{\mathbf{op}}$ *with adjoint*

$$P(\mathcal{A}) = \bigvee\{C \in CL(\mathcal{X}, \mathcal{M}) : A(C) \supseteq \mathcal{A}\}.$$

Proof: It follows since $m \in \mathcal{M}$ is $\bigvee_I C_i$-closed iff m is C_i-closed for each $i \in I$. (Note that $m^{\vee_I C_i} = \bigvee_I m^{C_i} \leq m \Leftrightarrow m^{C_i} \leq m$ for each $i \in I$.) $\square$

PROPOSITION 2.2 *Let $CL(\mathcal{X}, \mathcal{M}) \xrightarrow{B} S(\mathcal{X})$ be defined by:*

$$B(C) = \{X \in \mathcal{X} : \text{ every } m \in \mathcal{N} \text{ with domain } X \text{ is } C\text{-dense}\}$$

then, B preserves infima and thus forms a Galois connection $S(\mathcal{X}) \underset{B}{\overset{Q}{\rightleftarrows}} CL(\mathcal{X}, \mathcal{M})$ *with coadjoint*

$$Q(\mathcal{B}) = \bigwedge\{C \in CL(\mathcal{X}, \mathcal{M}) : B(C) \supseteq \mathcal{B}\}.$$

Proof: It follows by observing that $m \in \mathcal{M}$ is $\bigwedge_I C_i$-dense iff m is C_i-dense for each $i \in I$. $(id_X \leq m^{\wedge_I C_i} = \bigwedge_I m^{C_i} \Leftrightarrow id_X \leq m^{C_i}$ for each $i \in I$.) $\square$

REMARK 2.3 *It is important to observe that under the assumption that $\mathcal{M}$ contains all regular monomorphisms, if $\mathcal{N}$ consists of all diagonal morphisms then for every $C \in CL(\mathcal{X}, \mathcal{M})$ we obtain* *special cases of the above definitions:* $A(C) = \{X \in \mathcal{X} : \delta_X$ *is* C*-closed*$\}$ *and* $B(C) = \{X \in \mathcal{X} : \delta_X$ *is* C*-dense*$\}$*. These definitions give rise to two well known Galois connections that have been widely used in the literature (cf. [DG], [CKS*$_2$*] and [CT]). In this case the closure operator* $P(\mathcal{A})$ *is the regular closure, while* $Q(\mathcal{A})$ *is its dual or the "coregular" closure induced by the subcategory* $\mathcal{A}$*.*

Next we present some more practical descriptions of the closure operators $P(\mathcal{A})$ and $Q(\mathcal{B})$.

PROPOSITION 2.4 *Let* $\mathcal{A} \in S(\mathcal{X})^{\mathbf{op}}$*. For every* $X \in \mathcal{X}$ *and for every* $\mathcal{M}$*-subobject* $M \xrightarrow{m} X$*, consider all commutative squares of the form*

$$\begin{array}{ccc} M & \xrightarrow{m} & X \\ {\scriptstyle r_i}\downarrow & & \downarrow{\scriptstyle s_i} \\ A_i & \xrightarrow[n_i]{} & B_i \end{array}$$

with $A_i \in \mathcal{A}$ *and* $n_i \in \mathcal{N}$*, indexed by* I*. Form all pullbacks* m_i *of* n_i *along* s_i *and set* $C_{\mathcal{A}}(m) = \bigwedge_I m_i$*.*

For every $\mathcal{A} \in S(\mathcal{X})^{\mathbf{op}}$ *we have that the function* $C_{\mathcal{A}}$ *that to every* $\mathcal{M}$*-subobject* $M \xrightarrow{m} X$ *associates* $C_{\mathcal{A}}(m)$ *is an idempotent closure operator on* $\mathcal{X}$ *and* $C_{\mathcal{A}}(m) \simeq m^{P(\mathcal{A})}$*.*

Proof: We observe that the stated construction is a special case of the following construction which yields an idempotent closure operator (cf. [CKS$_2$, Theorem 2.3]). However, for completeness we include a short proof.

Let $\mathcal{F} \subseteq \mathcal{M}$ be closed under pullbacks and for $m \in \mathcal{M}$ define $m^{C_{\mathcal{F}}} = \bigwedge\{m' \in \mathcal{F} : m \leq m'\}$.

Clearly by construction $C_{\mathcal{F}}$ is extensive and order preserving. To show continuity, consider any morphism $X \xrightarrow{f} Y$ and $M \xrightarrow{m} Y \in \mathcal{M}$ and note:

$$\begin{array}{rcl} f^{-1}(m^{C_{\mathcal{F}}}) & = & f^{-1}(\bigwedge\{m' \in \mathcal{F} : m \leq m'\}) \\ & =^{(1)} & \bigwedge\{f^{-1}(m') : m' \in \mathcal{F}, m \leq m'\} \\ & \geq^{(2)} & \bigwedge\{f^{-1}(m') : m' \in \mathcal{F}, f^{-1}(m) \leq f^{-1}(m')\} \\ & \geq^{(3)} & \bigwedge\{\bar{m} : \bar{m} \in \mathcal{F}, f^{-1}(m) \leq \bar{m}\} \\ & = & (f^{-1}(m))^{C_{\mathcal{F}}} \end{array}$$

where (1) follows since pullbacks and intersections commute, (2) follows since $m \leq m' \Rightarrow f^{-1}(m) \leq f^{-1}(m')$ and (3) since each $f^{-1}(m) \in \mathcal{F}$.

As an infimum construction, $C_{\mathcal{F}}$ is clearly idempotent, since for $m' \in \mathcal{F}$, $m^{C_{\mathcal{F}}} \leq m' \Leftrightarrow m \leq m'$.

If we set $\mathcal{F} = \{f^{-1}(n) : n \in \mathcal{N}, dom(n) \in \mathcal{A}, cod(n) = cod(f)\}$ then our construction $C_{\mathcal{A}}(m) = m^{C_{\mathcal{F}}}$ for any $m \in \mathcal{M}$ and we have an idempotent closure operator as claimed.

It remains to show that $C_{\mathcal{A}}(m) \simeq m^{P(\mathcal{A})}$. Since for every $A \in \mathcal{A}$ and $\mathcal{N}$-subobject $A \xrightarrow{n} X$, the square $id_X \circ n = n \circ id_A$ is a pullback, we have that $\mathcal{A} \subseteq A(C_{\mathcal{A}})$ and so by definition of P we obtain that $C_{\mathcal{A}} \sqsubseteq P(\mathcal{A})$. On the other hand, from the property that $\mathcal{A} \subseteq A(P(\mathcal{A}))$, we have that each $\mathcal{N}$-subobject $A_i \xrightarrow{n_i} B_i$ with $A_i \in \mathcal{A}$ is $P(\mathcal{A})$-closed. Thus, so is each m_i as a pullback of a $P(\mathcal{A})$-closed subobject and consequently $\bigwedge_I m_i = C_{\mathcal{A}}(m)$ is $P(\mathcal{A})$-closed too. Hence, $P(\mathcal{A}) \sqsubseteq C_{\mathcal{A}}$ and the equality follows. $\square$

LEMMA 2.5 *Let* $C \in CL(\mathcal{X}, \mathcal{M})$ *and let* $(e_i)_{i \in I}$ *be an* $\mathbf{E}$*-sink. Fix* $j \in I$ *and assume that for each* $i \neq j$ *there are morphisms* r_i, f_i *such that* $e_i \circ f_i = e_j \circ r_i$*. If each* f_i*,* $i \neq j$ *is* C*-dense, then* e_j *is* C*-dense.*

Proof: Let us consider the following commutative diagram:

$$\begin{tikzcd}
X_i \arrow[rr, "f_i"] \arrow[dr, "a_i"'] \arrow[dd, "r_i"'] & & Y_i \arrow[dd, "e_i"] \\
& M_i \arrow[ur, "m_i"'] \arrow[dd, "d_i"] & \\
Y_j \arrow[rr, "e_j"] \arrow[dr, "a_j"'] & & B \\
& M_j \arrow[ur, "m_j"'] &
\end{tikzcd}$$

where $m_i \circ a_i = f_i$ and $m_j \circ a_j = e_j$ are $(\mathbf{E}, \mathcal{M})$-factorizations and d_i is the $(\mathbf{E}, \mathcal{M})$-diagonal. By taking the C-closures of each m_i and of m_j, we obtain the next commutative diagram:

$$\begin{tikzcd}
M_i \arrow[rr, "m_i"] \arrow[dr, "(m_i)_C"'] \arrow[dd, "d_i"'] & & Y_i \arrow[dd, "e_i"] \\
& (M_i)^C \arrow[ur, "(m_i)^C"'] \arrow[dd, "h_i"] & \\
M_j \arrow[rr, "m_j"] \arrow[dr, "(m_j)_C"'] & & B \\
& (M_j)^C \arrow[ur, "(m_j)^C"'] &
\end{tikzcd}$$

where h_i is the morphism induced by property 1.3(2) of closure operators. However, $(m_i)^C$ is an isomorphism, since each f_i is C-dense. So, we obtain the following commutative square:

$$\begin{tikzcd}
Y_i \arrow[r, "e_i"] \arrow[d, "k_i"'] & B \arrow[d, "id_B"] \arrow[dl, "d"] \\
(M_j)^C \arrow[r, "(m_j)^C"'] & B
\end{tikzcd}$$

where $k_i = h_i \circ ((m_i)^C)^{-1}$ for $i \neq j$, $k_j = (m_j)_C \circ a_j$ and d is the $(\mathbf{E}, \mathcal{M})$-diagonal. Clearly, $(m_j)^C$ is a monomorphism and a retraction and consequently an isomorphism. Thus, we conclude that e_j is C-dense. $\square$

PROPOSITION 2.6 *Let $\mathcal{B} \in S(\mathcal{X})$. For every $X \in \mathcal{X}$ and for every $\mathcal{M}$-subobject $M \xrightarrow{m} X$, consider all commutative squares of the form*

$$\begin{tikzcd}
A_i \arrow[r, "n_i"] \arrow[d, "r_i"'] & B_i \arrow[d, "s_i"] \\
M \arrow[r, "m"'] & X
\end{tikzcd}$$

with $A_i \in \mathcal{B}$ and $n_i \in \mathcal{N}$, indexed by I. Take the $(\mathbf{E}, \mathcal{M})$-factorization of the sink $(s_i)_{i\in I} \cup \{m\}$. Thus we obtain the following commutative diagram

$$\begin{tikzcd}
A_i \arrow[rr, "n_i"] \arrow[dd, "r_i"'] & & B_i \arrow[dl, "e_i"'] \arrow[dd, "s_i"] \\
& Q \arrow[dr, "\bar{m}"] & \\
M \arrow[ur, "k"] \arrow[rr, "m"'] & & X
\end{tikzcd}$$

where $(e_i)_{i\in I} \cup \{k\} \in \mathbf{E}$ and $\bar{m} \in \mathcal{M}$ satisfy $\bar{m} \circ e_i = s_i$ for every $i \in I$ and $\bar{m} \circ k = m$. Set $C^{\mathcal{B}}(m) = \bar{m}$.

For every $\mathcal{B} \in S(\mathcal{X})$, the function $C^{\mathcal{B}}$ is a weakly hereditary closure operator on $\mathcal{X}$ and $C^{\mathcal{B}}(m) \simeq m^{Q(\mathcal{B})}$.

Proof: Clearly, by construction we have that $m \leq C^{\mathcal{B}}(m)$.

To show order preservation and continuity we prove the equivalent condition in Remark 1.3(2). Consider the commutative diagram below detailing the construction of $C^{\mathcal{B}}(m) = \bar{m}$ for $m \in \mathcal{M}$ where $A_i \in \mathcal{B}$, $n_i \in \mathcal{N}$, and also $n \in \mathcal{M}$:

$$\begin{tikzcd}
B_i \arrow[rr, "s_i"] \arrow[dr, "e_i"] & & X \arrow[r, "f"] & Y \\
& Q \arrow[ur, "\bar{m}"] & & \\
A_i \arrow[uu, "n_i"] \arrow[rr, "r_i"'] & & M \arrow[ul, "k"'] \arrow[uu, "m"'] \arrow[r, "g"'] & N \arrow[uu, "n"']
\end{tikzcd}$$

Let $((e'_i) \cup \{k'\}, m')$ then be the $(\mathbf{E}, \mathcal{M})$-factorization of the sink $(f \circ s_i) \cup \{n\}$. Denoting the domain of m' as Q', we have the following commutative diagram:

$$\begin{tikzcd}
\bullet \arrow[r, "(e_i) \cup \{k\}"] \arrow[d, "(e'_i) \cup \{k' \circ g\}"'] & Q \arrow[d, "f \circ \bar{m}"] \\
Q' \arrow[r, "m'"'] & Y
\end{tikzcd}$$

The $(\mathbf{E}, \mathcal{M})$-diagonalization property gives a diagonal $Q \xrightarrow{d} Q'$ for the above square that in particular yields $m' \circ d = f \circ \bar{m}$. However, since in the construction of $C^{\mathcal{B}}(n)$ there are further squares than the ones appearing in the above diagram, we have that $m' \leq C^{\mathcal{B}}(n)$, or there is a d' such that $m' = C^{\mathcal{B}}(n) \circ d'$. The morphism $d' \circ d$ is the one we seek to conclude that $C^{\mathcal{B}}$ is a closure operator as claimed.

Next we show that $C^{\mathcal{B}}$ is weakly hereditary. Consider again the commutative diagram (the left-hand part of the double square above) used in the construction of $C^{\mathcal{B}}(m)$ where $A_i \in \mathcal{B}$, $n_i \in \mathcal{N}$ and $((e_i)_{i\in I} \cup \{k\}, \bar{m})$ is the $(\mathbf{E}, \mathcal{M})$-factorization of $(s_i) \cup \{m\}$. As a member of $\mathcal{N}$ with domain in $\mathcal{B}$, each

n_i is clearly $C^{\mathcal{B}}$-dense (consider the commutative square $id_{B_i} \circ n = n \circ id_{A_i}$ in the construction of $C^{\mathcal{B}}(n)$ for such an n). Thus by Lemma 2.5, k is also $C^{\mathcal{B}}$-dense and $C^{\mathcal{B}}$ is weakly hereditary.

It remains to be shown that $C^{\mathcal{B}}(m) \simeq m^{Q(\mathcal{B})}$. Since for every $A \in \mathcal{B}$ and $\mathcal{N}$-subobject $A \xrightarrow{n} X$, n is $C^{\mathcal{B}}$-dense it follows that $\mathcal{B} \subseteq B(C^{\mathcal{B}})$. Consequently, by definition of Q we obtain that $Q(\mathcal{B}) \sqsubseteq C^{\mathcal{B}}$. Conversely, from the property that $\mathcal{B} \subseteq B(Q(\mathcal{B}))$, we have that each $\mathcal{N}$-subobject $A_i \xrightarrow{n_i} B_i$ with $A_i \in \mathcal{B}$ is $Q(\mathcal{B})$-dense. Thus, from Lemma 2.5, the morphism k is also $Q(\mathcal{B})$-dense. Thus we obtain the following commutative diagram:

$$\begin{tikzcd}
M \arrow[rr, "id_M"] \arrow[dr, "k_{Q(\mathcal{B})}"] \arrow[dd, "k"'] & & M \arrow[dr, "m_{Q(\mathcal{B})}"] \arrow[dd, "m"] & \\
& M_Q^{Q(\mathcal{B})} \arrow[rr, "d"] \arrow[dl, "k^{Q(\mathcal{B})}"] & & M_X^{Q(\mathcal{B})} \arrow[dl, "m^{Q(\mathcal{B})}"] \\
Q \arrow[rr, "C^{\mathcal{B}}(m)"'] & & X &
\end{tikzcd}$$

where $k^{Q(\mathcal{B})}$ is an isomorphism. Consequently $C^{\mathcal{B}}(m) \leq m^{Q(\mathcal{B})}$ and $C^{\mathcal{B}}(m) \simeq m^{Q(\mathcal{B})}$. $\square$

It may be worth mentioning that both closure operator constructions introduced in Propositions 2.4 and 2.6 work if I is empty.

We close this Section by observing that the closure operator constructions and related Galois connections introduced in this Section have been used in a parallel paper ([CH$_1$]) to provide a link between the notion of connectedness with respect to a closure operator introduced by Castellini ([C$_{1-5}$]) and the one by Clementino and Tholen ([CT]). Moreover, the above Galois connections are also being used in a forthcoming paper to study a notion of absolutely closed object in an arbitrary category ([CH$_2$]).

3 A DIAGRAM OF GALOIS CONNECTIONS OF CLOSURE OPERATORS

In this Section we show that the closure operators described in Propositions 2.4 and 2.6 are strongly related to some classical closure operators constructions.

First we see that the present descriptions simplify and extend some previous constructions in the literature. Second, when considered in conjunction with a number of prior constructions the usefulness of the parameter $\mathcal{N}$ is borne out. Finally, a global summary of a number of results involving closure operators and Galois connections is provided.

From now on, we assume that $\mathcal{M}$ contains all regular monomorphisms.

$S(\mathcal{M})$ will denote the conglomerate of all subclasses of $\mathcal{M}$, ordered by inclusion, and $\mathcal{N}$ is a fixed subclass of $\mathcal{M}$. For $X \in \mathcal{X}$ set $\mathcal{N}^X = \{n \in \mathcal{N} : dom(n) = X\}$ and dually $\mathcal{N}_X = \{n \in \mathcal{N} : cod(n) = X\}$.

PROPOSITION 3.1 *Define $S(\mathcal{X}) \xrightarrow{H'} S(\mathcal{M})$ and $S(\mathcal{M}) \xrightarrow{K'} S(\mathcal{X})$ by:*

$H'(\mathcal{A}) = \{n \in \mathcal{N}^X : X \in \mathcal{A}\}$

$K'(\mathcal{M}') = \{X \in \mathcal{X} : n \in \mathcal{N}^X \Rightarrow n \in \mathcal{M}'\}$.

Then $S(\mathcal{X}) \underset{K'}{\overset{H'}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} S(\mathcal{M})$ *is a Galois connection.*

Proof: Clearly both H' and K' are order-preserving.

If $X \in \mathcal{A}$, then every $n \in \mathcal{N}^X$ also belongs to $H'(\mathcal{A})$, consequently $X \in (K' \circ H')(\mathcal{A})$. On the other hand, if $n \in (H' \circ K')(\mathcal{M}')$, then $n \in \mathcal{N}^X$ with $X \in K'(\mathcal{M}')$. This implies that $n \in \mathcal{M}'$. $\square$

As a consequence we obtain that $S(\mathcal{M})^{\mathbf{op}} \underset{H'^{\mathbf{op}}}{\overset{K'^{\mathbf{op}}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} S(\mathcal{X})^{\mathbf{op}}$ is also a Galois connection, where $H'^{\mathbf{op}}$ and $K'^{\mathbf{op}}$ are defined just like H' and K', respectively.

In [CKS$_2$] a number of Galois connections were introduced in a study of the regular closure operator and its dual. As already remarked in 2.3, our operators P and Q render the regular closure and its dual as a special case. The next two results demonstrate that the current closure operator constructions both generalize and greatly simplify some constructions of [CKS$_2$].

DEFINITION 3.2 [CKS$_2$] *A subclass* $\mathcal{M}'$ *of* $\mathcal{M}$ *is called* **E***-sink stable, if for every commutative square*

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "m"'] & N \arrow[d, "n"] \\ X \arrow[r, "g"'] & Y \end{tikzcd}$$

with $n \in \mathcal{M}$ *and the* 2*-sink* $(g, n) \in \mathbf{E}$ *we have that* $m \in \mathcal{M}'$ *implies* $n \in \mathcal{M}'$.

Notice that a byproduct of Lemma 2.5 is that the class of all C-dense $\mathcal{M}$-subobjects is **E**-sink stable.

$S_{es}(\mathcal{M})$ will denote the collection of all **E**-sink stable subclasses of $\mathcal{M}$, ordered by inclusion and $S_{pb}(\mathcal{M})$ will denote the collection of all subclasses of $\mathcal{M}$ that are closed under the formation of pullbacks, ordered by inclusion.

We recall from [CKS$_2$, Theorem 2.4 and Proposition 2.7] that the four assignments described below yield Galois connections $iCL(\mathcal{X}, \mathcal{M}) \underset{R^*}{\overset{R_*}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} S_{pb}(\mathcal{M})^{\mathbf{op}}$ and $S_{pb}(\mathcal{M})^{\mathbf{op}} \underset{Q^*}{\overset{Q_*}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} S(\mathcal{M})^{\mathbf{op}}$.

(i) $R_*(C) = \{m \in \mathcal{M} : m \text{ is } C\text{-closed}\}$;

(ii) $m^{R^*(\mathcal{M}')} = \bigwedge\{m' \in \mathcal{M}' : M' \xrightarrow{m'} X \text{ and } m \leq m'\}$, for $M \xrightarrow{m} X$ in $\mathcal{M}$;

(iii) $Q_*(\mathcal{M}') = \mathcal{M}'$;

(iv) $Q^*(\mathcal{M}') = \{m \in \mathcal{M} : m \text{ is a pullback of some } m' \in \mathcal{M}'\}$.

Our aim is to show that these Galois connections, together with the one of Proposition 3.1 provide a factorization of the Galois connection $CL(\mathcal{X}, \mathcal{M}) \underset{P}{\overset{A}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} S(\mathcal{X})^{\mathbf{op}}$.

THEOREM 3.3 *For every subclass $\mathcal{N} \subseteq \mathcal{M}$, we have (up to isomorphism) the following commutative diagram of Galois connections:*

$$\begin{tikzcd} CL(\mathcal{X},\mathcal{M}) \arrow[r, shift left, "A"] \arrow[r, phantom, "\perp"] \arrow[d, shift left, "\overline{R}_*"] \arrow[d, phantom, "\vdash"] & S(\mathcal{X})^{\mathbf{op}} \arrow[l, shift left, "P"] \arrow[d, shift left, "H'^{\mathbf{op}}"] \arrow[d, phantom, "\dashv"] \\ S_{pb}(\mathcal{M})^{\mathbf{op}} \arrow[u, shift left, "\overline{R}^*"] \arrow[r, shift left, "Q_*"] \arrow[r, phantom, "\perp"] & S(\mathcal{M})^{\mathbf{op}} \arrow[l, shift left, "Q^*"] \arrow[u, shift left, "K'^{\mathbf{op}}"] \end{tikzcd}$$

Proof: First we observe that the Galois connection $CL(\mathcal{X},\mathcal{M}) \underset{\bar{R}^*}{\overset{\bar{R}_*}{\rightleftarrows}} S_{pb}(\mathcal{M})^{\mathbf{op}}$ ($\perp$) is simply the extension of $iCL(\mathcal{X},\mathcal{M}) \underset{R^*}{\overset{R_*}{\rightleftarrows}} S_{pb}(\mathcal{M})^{\mathbf{op}}$ ($\perp$) to $CL(\mathcal{X},\mathcal{M})$.

The side and bottom Galois connections give rise by composition to a new Galois connection that will turn out to be $CL(\mathcal{X},\mathcal{M}) \underset{P}{\overset{A}{\rightleftarrows}} S(\mathcal{X})^{\mathbf{op}}$ ($\perp$). In fact, let $C \in CL(\mathcal{X},\mathcal{M})$, then we have that $\bar{R}_*(C) = \{m \in \mathcal{M} : m \text{ is } C\text{-closed}\}$. Consequently, $K'^{\mathbf{op}}(Q_*(\bar{R}_*(C))) = K'^{\mathbf{op}}(\bar{R}_*(C)) = K'^{\mathbf{op}}(\{m \in \mathcal{M} : m \text{ is } C\text{-closed}\}) = \{X \in \mathcal{X} : n \in \mathcal{N}^X \Rightarrow n \text{ is } C\text{-closed}\} = A(C)$.

Since the two functions in a Galois connection determine each other up to isomorphism, we conclude that for any subcategory $\mathcal{A} \in S(\mathcal{X})^{\mathbf{op}}$, $P(\mathcal{A}) \simeq (\bar{R}^* \circ Q^* \circ H'^{\mathbf{op}})(\mathcal{A})$. □

REMARK 3.4 *(a) Commutativity of the above diagram is only in the direction of the Galois connections.*

(b) Observe that for $\mathcal{A} = \mathcal{X}$, we obtain that $P(\mathcal{X}) = (\bar{R}^* \circ Q^*)(\mathcal{N})$, and if $\mathcal{N}$ is stable under pullback then $P(\mathcal{X}) = \bar{R}^*(\mathcal{N})$. In other words, the closure operator construction $\bar{R}^*$ is just a special case of the P construction. Restriction to pullback stable classes can be avoided.

Now we recall from [CKS$_2$, Theorem 2.4 and Proposition 2.7] that two further Galois connections $S_{es}(\mathcal{M}) \underset{K^*}{\overset{K_*}{\rightleftarrows}} wCL(\mathcal{X},\mathcal{M})$ ($\perp$) and $S(\mathcal{M}) \underset{L^*}{\overset{L_*}{\rightleftarrows}} S_{es}(\mathcal{M})$ ($\perp$) are yielded by the assignments:

(v) $K^*(C) = \{m \in \mathcal{M} : m \text{ is } C\text{-dense}\}$;

(vi) $m^{K_*(\mathcal{M}')} = \bigvee\{(N \xrightarrow{n} X) \in \mathcal{M} : \exists (M \xrightarrow{t} N) \in \mathcal{M}' \text{ with } n \circ t = m\}$ for $M \xrightarrow{m} X \in \mathcal{M}$;

(vii) $L^*(\mathcal{M}') = \mathcal{M}'$;

(viii) $L_*(\mathcal{M}') = \{n \in \mathcal{M} : n \circ f = g \circ m \text{ for some } m \in \mathbf{M}' \text{ and some } \mathcal{X}\text{-morphisms } f \text{ and } g$ with $(g,n) \in \mathbf{E}\}$.

These Galois connections are now seen to factor through $S(\mathcal{X}) \underset{B}{\overset{Q}{\rightleftarrows}} CL(\mathcal{X},\mathcal{M})$ ($\perp$).

THEOREM 3.5 *For every subclass $\mathcal{N} \subseteq \mathcal{M}$, we have (up to isomorphism) the following commutative diagram of Galois connections:*

$$\begin{tikzcd}
S(\mathcal{X}) \arrow[r, shift left, "Q"] \arrow[r, phantom, "\perp"] \arrow[d, shift left, "H'"] \arrow[d, phantom, "\vdash"] & CL(\mathcal{X},\mathcal{M}) \arrow[l, shift left, "B"] \arrow[d, shift left, "\overline{K}^*"] \arrow[d, phantom, "\dashv"] \\
S(\mathcal{M}) \arrow[u, shift left, "K'"] \arrow[r, shift left, "L_*"] \arrow[r, phantom, "\perp"] & S_{es}(\mathcal{M}) \arrow[l, shift left, "L^*"] \arrow[u, shift left, "\overline{K}_*"]
\end{tikzcd}$$

Proof: First we observe that the Galois connection $S_{es}(\mathcal{X}) \underset{\bar{K}^*}{\overset{\bar{K}_*}{\rightleftarrows}} CL(\mathcal{X},\mathcal{M})$ is simply the extension of $S_{es}(\mathcal{X}) \underset{K^*}{\overset{K_*}{\rightleftarrows}} wCL(\mathcal{X},\mathcal{M})$ to $CL(\mathcal{X},\mathcal{M})$.

The side and bottom Galois connections give rise by composition to a new Galois connection that will turn out to be $S(\mathcal{X}) \underset{B}{\overset{Q}{\rightleftarrows}} CL(\mathcal{X},\mathcal{M})$. Let $C \in CL(\mathcal{X},\mathcal{M})$. Then, we have that $K'(L^*(\bar{K}^*(C))) = K'(\{m \in \mathcal{M} : m \text{ is } C\text{-dense}\}) = \{X \in \mathcal{X} : n \in \mathcal{N}^X \Rightarrow n \text{ is } C\text{-dense}\} = B(C)$. Clearly, we have used the fact that $\bar{K}^*(C)$ is **E**-sink stable.

Since the two functions in a Galois connection determine each other up to isomorphism, we conclude that for any subcategory $\mathcal{B} \in S(\mathcal{X})$, $Q(\mathcal{B}) \simeq (\bar{K}_* \circ L_* \circ H')(\mathcal{B})$. □

REMARK 3.6 *For $\mathcal{B} = \mathcal{X}$, observe that $Q(\mathcal{X}) = (\bar{K}_* \circ L_*)(\mathcal{N})$. In other words, for any subclass $\mathcal{N}$ of $\mathcal{M}$, the construction Q bypasses the concept of* **E***-sink stability and accomplishes in one step what in [CKS$_2$] was accomplished in two, that is first enlarge the class $\mathcal{N}$ so to make it* **E***-sink stable and then apply the closure operator construction given by $\bar{K}_*$.*

In his study of categorical connectedness and disconnectedness, Castellini introduced the Galois connections below. We recall them here as, like the closure operator constructions of the current article, they depend on a parameter $\mathcal{N}$ and in conjunction with our current Galois connections provide a global perspective on a number of previous closure operator constructions.

(1) In [C$_2$] the Galois connection $S(\mathcal{X}) \underset{K}{\overset{H}{\rightleftarrows}} S(\mathcal{M})$ is introduced, where

$H(\mathcal{A}) = \{n \in \mathcal{N}_X : X \in \mathcal{A}\}$;

$K(\mathcal{M}') = \{X \in \mathcal{X} : n \in \mathcal{N}_X \Rightarrow n \in \mathcal{M}'\}$.

As a consequence we obtain that $S(\mathcal{M})^{\mathbf{op}} \underset{H^{\mathbf{op}}}{\overset{K^{\mathbf{op}}}{\rightleftarrows}} S(\mathcal{X})^{\mathbf{op}}$ is also a Galois connection. $H^{\mathbf{op}}$ and $K^{\mathbf{op}}$ are defined just like H and K, respectively.

(2) The Galois connection $CL(\mathcal{X},\mathcal{M}) \underset{T}{\overset{D}{\rightleftarrows}} S(\mathcal{X})^{\mathbf{op}}$ was presented in [C$_1$], with

$D(C) = \{X \in \mathcal{X} :$ every $n \in \mathcal{N}_X$ is C-closed$\}$;

$T(\mathcal{A}) = \bigvee\{C \in CL(\mathcal{X},\mathcal{M}) : D(C) \supseteq \mathcal{A}\}$.

(3) Also from [C$_2$], the Galois connection $S(\mathcal{X}) \underset{I}{\overset{J}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} CL(\mathcal{X},\mathcal{M})$ with

$$I(C) = \{X \in \mathcal{X} : \text{ every } n \in \mathcal{N}_X \text{ is } C\text{-dense}\};$$
$$J(\mathcal{B}) = \bigwedge\{C \in CL(\mathcal{X},\mathcal{M}) : I(C) \supseteq \mathcal{B}\}.$$

(4) Recalling from [C$_4$] that a morphism $X \xrightarrow{f} Y$ is called $\mathcal{N}$-*dependent* if for every $n \in \mathcal{N}_X$ and every $p \in \mathcal{N}_Y$, $n_f \leq p$ implies $f^{-1}(p) \simeq id_X$. The Galois connection $S(\mathcal{X}) \underset{\nabla}{\overset{\Delta}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} S(\mathcal{X})^{\mathbf{op}}$ is defined by

$$\nabla(\mathcal{A}) = \{X \in \mathcal{X} : \forall Y \in \mathcal{A}, X \xrightarrow{f} Y \textit{ is } \mathcal{N}\text{-dependent}\};$$
$$\Delta(\mathcal{B}) = \{Y \in \mathcal{X} : \forall X \in \mathcal{B}, X \xrightarrow{f} Y \textit{ is } \mathcal{N}\text{-dependent}\}.$$

(5) Finally we recall from [CH$_1$] a Galois connection given by morphism orthogonality. For $\mathcal{P}, \mathcal{Q} \subseteq Mor\mathcal{X}$, we write $\mathcal{P} \perp \mathcal{Q}$ if for every commutative diagram:

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "p"'] & Y \arrow[d, "q"] \\
W \arrow[r, "v"'] & Z
\end{tikzcd}$$

with $p \in \mathcal{P}$ and $q \in \mathcal{Q}$, there is a unique morphism $W \xrightarrow{d} Y$ such that $d \circ p = u$ and $q \circ d = v$. Using the classical terminology, this means that every element of $\mathcal{P}$ is "left orthogonal" to every element of $\mathcal{Q}$ (or equivalently, every element of $\mathcal{Q}$ is "right orthogonal" to every element of $\mathcal{P}$).

We now define a Galois connection $S(\mathcal{X}) \underset{\lambda}{\overset{\rho}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} S(\mathcal{X})^{\mathbf{op}}$ via

$$\rho(\mathcal{B}) = \{Y \in \mathcal{X} : \forall X \in \mathcal{B}, \mathcal{N}^X \perp \mathcal{N}^Y\};$$
$$\lambda(\mathcal{A}) = \{X \in \mathcal{X} : \forall Y \in \mathcal{A}, \mathcal{N}^X \perp \mathcal{N}^Y\}.$$

By putting together all the Galois connections introduced above, we obtain the following commutative diagram of Galois connections:

$$\begin{tikzcd}
S(\mathcal{M}) \arrow[d, "(L_*, L^*)"'] & S(\mathcal{X}) \arrow[l, "(H,K)"'] \arrow[r, "(\Delta,\nabla)"] \arrow[d, "(J,I)"'] & S(\mathcal{X})^{\mathbf{op}} & S(\mathcal{M})^{\mathbf{op}} \arrow[l, "(K^{\mathbf{op}}, H^{\mathbf{op}})"'] \\
S_{es}(\mathcal{M}) \arrow[r, "(\overline{K}_*, \overline{K}^*)"] & CL(\mathcal{X},\mathcal{M}) \arrow[r, shift left, "id_{CL(\mathcal{X},\mathcal{M})}"] & CL(\mathcal{X},\mathcal{M}) \arrow[l, shift left, "id_{CL(\mathcal{X},\mathcal{M})}"] \arrow[u, "(D,T)"'] \arrow[r, "(\overline{R}_*, \overline{R}^*)"] \arrow[d, "(A,P)"] & S_{pb}(\mathcal{M})^{\mathbf{op}} \arrow[u, "(Q_*, Q^*)"'] \arrow[d, "(Q_*, Q^*)"] \\
S(\mathcal{M}) \arrow[u, "(L_*, L^*)"] & S(\mathcal{X}) \arrow[l, "(H',K')"] \arrow[r, "(\rho,\lambda)"'] \arrow[u, "(Q,B)"] & S(\mathcal{X})^{\mathbf{op}} & S(\mathcal{M})^{\mathbf{op}} \arrow[l, "(K'^{\mathbf{op}}, H'^{\mathbf{op}})"]
\end{tikzcd}$$

In this diagram we have used single arrows to specify the direction of the Galois connections, hoping to make more clear the way they compose and the involved factorizations. In the label of each arrow, the first element of each pair is the coadjoint function in the direction of the arrow.

The commutativity of the single squares that form the above diagram are obtained as follows: from left to right the lower squares follow from Theorem 3.5, [CH_1, Theorem 2.5] and Theorem 3.3, respectively. The upper squares follow from [C_2, Theorem 2.26], [C_4, Theorem 2.8] and [C_2, Theorem 2.26], respectively.

The interesting aspect of the above diagram is that by choosing $\mathcal{N}$ in different ways, it yields many different particular closure operator constructions that have appeared in the literature. For instance:

(a) we have already observed that for $\mathcal{N}$ consisting of diagonal morphisms, A and B yield the separation and connectedness Galois connections used in [CT]. P and Q render the regular and coregular closure operators respectively;

(b) the lower middle square provides a description in terms of morphism orthogonality of their composition. The Galois connection (ρ, λ) is the classical connectedness-disconnectedness connection of Preuss, Herrlich, Arhangelskii & Wiegand and is analyzed in [CH_1];

(c) the upper middle square wraps up Castellini's notions of connectedness and disconnectedness in an arbitrary category (cf. [C_{4-6}]) and yields the most classical results in concrete situations when $\mathcal{N} = \mathcal{M}$;

(d) for $\mathcal{N} = \mathcal{M}$, T yields the splitting closure operator (cf. [BGH]);

(e) for $\mathcal{A} = \mathcal{X}$, as already observed, $P(\mathcal{X})$ yields a very classical closure operator construction induced by the class $\mathcal{N}$ in which the $\mathcal{N}$-subobjects turn out to be closed. Moreover, the class of $P(\mathcal{X})$-closed $\mathcal{M}$-subobjects is the smallest class of closed $\mathcal{M}$-subobjects with respect to a closure operator that contains $\mathcal{N}$;

(f) similarly, for $\mathcal{B} = \mathcal{X}$, $Q(\mathcal{X})$ yields a "dual" closure operator construction induced by the class $\mathcal{N}$ in which the $\mathcal{N}$-subobjects turn out to be dense (cf. [CKS_2]). Moreover, the class of $Q(\mathcal{X})$-dense $\mathcal{M}$-subobjects is the smallest class of dense $\mathcal{M}$-subobjects with respect to a closure operator that contains $\mathcal{N}$;

(g) for any full subcategory $\mathcal{A}$ of $\mathcal{X}$, if $\mathcal{N} = Mor\mathcal{A} \cap \mathcal{M}$ then $P(\mathcal{X})$ generalizes the pullback closure construction and coincides with it if $\mathcal{A}$ is **E**-reflective and **E**-morphisms are epic;

(h) as a final observation we notice that for $\mathcal{A} = \mathcal{X}$, $P(\mathcal{X}) = T(\mathcal{X})$ and $J(\mathcal{X}) = Q(\mathcal{X})$.

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Gabriele Castellini, Department of Mathematics, University of Puerto Rico, Mayagüez campus, P.O. Box 9018, Mayagüez, PR 00681-9018, U.S.A. e-mail: castell@cs.uprm.edu

David Holgate, Department of Mathematics, University of Stellenbosch, Stellenbosch 7600, South Africa. e-mail: dh2@sun.ac.za