CLOSURE OPERATORS AND POLARITIES

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ABSTRACT: Basic results are obtained concerning Galois connections between collections of closure operators (of various types) and collections consisting of subclasses of (pairs of) morphisms in $\mathscr{M}$ for an $\langle \boldsymbol{E}, \mathscr{M} \rangle$-category $\mathscr{X}$. In effect, the "lattice" of closure operators on $\mathscr{M}$ is shown to be equivalent to the fixed point lattice of the polarity induced by the orthogonality relation between composable pairs of morphisms in $\mathscr{M}$.

KEY WORDS: Galois connection, polarity, closure operator, closure operator, composable pair of morphisms, factorization structure for sinks.

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0. INTRODUCTION

To study closure operators we introduce two new concepts that are reasonably simple, yet apparently quite powerful. The first one is the relation $\perp$, which naturally extends earlier notions of "diagonalizability" or "orthogonality". The second one is the subcollection $\mathscr{M} \diamond \mathscr{M}$ of $\mathscr{M} \times \mathscr{M}$ for $\mathscr{M} \subseteq \boldsymbol{Mor}(\mathscr{X})$ that consists of the composable pairs in $\mathscr{M}$.

In Section 1 we recall the basic categorical approach to factorization systems on a class $\mathscr{M}$ of morphisms, and relate it to the classical topological closure operators. We then collect some familiar results regarding such closure operators (in the setting of $\langle \boldsymbol{E}, \mathscr{M} \rangle$-categories) that are needed later, and in the process we introduce the relation $\perp$.

In Section 2 we introduce composable pairs and restrict $\perp$ accordingly. This leads to the Galois connections mentioned in the abstract. We show that classes of composable pairs of $\mathscr{M}$-elements are related to general closure operators in essentially the same way as subclasses of $\mathscr{M}$ are related to closure operators that are necessarily both idempotent and weakly hereditary (cf. Diagram (2-00)). Indeed the lattice of all closure operators on $\mathscr{M}$ is essentially the lattice of Galois fixed points with respect to the natural polarity induced by the restriction of $\perp$ to $\mathscr{M} \diamond \mathscr{M}$. The collection of closure operators on $\mathscr{M}$ that are both idempotent and weakly hereditary appears twice as such a lattice of Galois fixed points — first with respect to a suitable restriction of $\perp$ to $\mathscr{M}$, and second with respect to the composite of the two natural Galois "insertions" of idempotent (respectively weakly hereditary) closure operators into all closure operators. This two-fold appearance helps to explain why the idempotent weakly hereditary

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closure operators have been dominant in the early work in the field (cf. [3], [9], [11] and [13]). The analogue between the results for $\mathscr{M}$ and $\mathscr{M} \diamond \mathscr{M}$ seems to make a strong case for the study of classes of composable pairs.

Applications and examples can be found in section 3.

1. PRELIMINARIES

Our main tool will be a notion of orthogonality that generalizes the one introduced by Tholen (cf. [17]), and encompasses part of the defining properties of factorization structures for sinks (cf. Definition 1.04) and for sources, as well as one of the essential features of closure operators (cf. Definition 1.02). Throughout we work in a category $\mathscr{X}$.

1.00 Definition. (cf. [13]) A pair $\langle \boldsymbol{a}, a' \rangle$ consisting of a sink $\boldsymbol{a} = \langle A_i \xrightarrow{a_i} A' \rangle_I$ and a morphism $A' \xrightarrow{a'} A''$ is called **left orthogonal to** a pair $\langle b, \boldsymbol{b}' \rangle$ consisting of a morphism $B \xrightarrow{b} B'$ and a source $\boldsymbol{b}' = \langle B' \xrightarrow{b'_j} B''_j \rangle_J$, written as $\langle \boldsymbol{a}, a' \rangle \perp \langle b, \boldsymbol{b}' \rangle$, iff for any sink $\boldsymbol{f} = \langle A_i \xrightarrow{f_i} B \rangle_I$ and any source $\boldsymbol{f}'' = \langle A'' \xrightarrow{f''_j} B''_j \rangle_J$ with the property that for each $i \in I$ and each $j \in J$ the outer square of following diagram commutes

$$\begin{tikzcd}
A_i \arrow[rrr, "f_i"] \arrow[dr, "a_i"] \arrow[dd, "a' \circ a_i"'] & & & B \arrow[dl, "b"'] \arrow[dd, "b'_j \circ b"] \\
& A' \arrow[r, "f'"] \arrow[dl, "a'"] & B' \arrow[dr, "b'_j"'] & \\
A'' \arrow[rrr, "f''_j"'] & & & B''_j
\end{tikzcd} \tag{1-00}$$

there exists a unique $\mathscr{X}$-morphism $A' \xrightarrow{f'} B'$ such that all inner trapezoids commute. In this case the pair $\langle b, \boldsymbol{b}' \rangle$ is called **right orthogonal to** $\langle \boldsymbol{a}, a' \rangle$. We write $\boldsymbol{a} \perp \boldsymbol{b}'$ rather than $\langle \boldsymbol{a}, id \rangle \perp \langle id, \boldsymbol{b}' \rangle$, i.e., we suppress the morphism-part of a pair in case it is an isomorphism.

Notice that a sink $\boldsymbol{a}$ and an object X are *separated* in the sense of Pumplün and Röhrl (cf. [14]), if $\boldsymbol{a}$ is left-orthogonal to the 2-source $X \xleftarrow{id_X} X \xrightarrow{id_X} X$, and that $\boldsymbol{a}$ is an epi-sink iff $\boldsymbol{a}$ and every $\mathscr{X}$-object are separated in this sense.

Now consider the following characterization of continuous functions between topological spaces $\langle X, t \rangle$ and $\langle Y, s \rangle$, where " $^-$ " denotes the usual topological closure.

1.01 Proposition. *A function $X \xrightarrow{g} Y$ is continuous from $\langle X, t \rangle$ to $\langle Y, s \rangle$ iff for every $M \subseteq X$ and every $N \subseteq Y$ the direct image of $\overline{M}$ along g is contained in $\overline{N}$ provided that the direct image of M along g is contained in N .* $\square$

Diagrammatically the situation can be described as follows (the vertical and diagonal arrows are inclusions): If a function f exists [3] that makes the outer square commute, then there exists a unique function $\overline{M} \xrightarrow{d} \overline{N}$ that makes the lower trapezoid commute (and by default makes the upper trapezoid commute as well).

$$\begin{tikzcd} M \arrow[rrr, "f"] \arrow[ddd] \arrow[dr] & & & N \arrow[ddd] \arrow[dl] \\ & \overline{M} \arrow[r, "d"] \arrow[ddl] & \overline{N} \arrow[ddr] & \\ & & & \\ X \arrow[rrr, "g"'] & & & Y \end{tikzcd} \tag{1-01}$$

The factorizations of the inclusions $M \longrightarrow X$ and $N \longrightarrow Y$ are clearly orthogonal to each other in the sense of Definition 1.00. Moreover, we see that the classical topological closure operators (= increasing, idempotent, order-preserving functions that preserve finite unions) on the subspace lattices of topological spaces do not just exist in isolation, but are also linked together quite strongly. In fact, the way they are linked determines the notion of continuity.

Let $\mathscr{M}$ be an arbitrary class of morphisms in $\mathscr{X}$ (corresponding to the embeddings in ***Top***). We regard $\mathscr{M}$ as a full subcategory of the arrow category $\mathscr{X}/\mathscr{X}$; its objects are all morphisms in $\mathscr{M}$, and an $\mathscr{M}$**-morphism** $\langle f, g\rangle$ from $m \in \mathscr{M}$ to $n \in \mathscr{M}$ is a pair of $\mathscr{X}$-morphisms that satisfy $g \circ m = n \circ f$. The **domain functor** $\mathscr{M} \xrightarrow{U} \mathscr{X}$ maps $\langle f, g\rangle$ to f, while the **codomain functor** V maps $\langle f, g\rangle$ to g.

1.02 DEFINITION. A **closure** (resp. **density**) **operator** $F = \langle (\,)_F, (\,)^F\rangle$ on $\mathscr{M}$ maps each $m \in \mathscr{M}$ to a pair $\langle m_F, m^F\rangle$ with $m = m^F \circ m_F$ and $m^F \in \mathscr{M}$ (resp. $m_F \in \mathscr{M}$) such that $F(m) \perp F(n)$ for all $n \in \mathscr{M}$.

If F is a closure or density operator, $m \in \mathscr{M}$ is called F**-closed** (resp. F**-dense**) if m_F (resp. m^F) is an isomorphism. $\nabla_*(F)$ and $\Delta^*(F)$ denote the classes of F-closed and F-dense members of $\mathscr{M}$, respectively.

1.03 REMARKS. (0) A succinct categorical formulation of the concept of closure operator, first proposed by Dikranjan and Giuli [9], views $(\,)^F$ as an endofunctor $\mathscr{M} \xrightarrow{(\,)^F} \mathscr{M}$ that satisfies $V * (\,)^F = V$, and views $(\,)_F$ as the domain-part of a natural transformation $id_{\mathscr{M}} \xrightarrow{\delta} (\,)^F$ that satisfies $id_V * \delta = id_V$. The uniqueness part of the orthogonality condition then says that $\langle \delta, (\,)^F\rangle$ is a pre-reflection in the sense of Börger [1], cf. also [18].

(1) If $\mathscr{M}$ is **stable under** $\mathscr{M}$**-relative first factors**, i.e., if $p \circ n \in \mathscr{M}$ and $p \in \mathscr{M}$ implies $n \in \mathscr{M}$, then every closure operator on $\mathscr{M}$ is a density operator as well. Furthermore,

[3] Such a function will necessarily be unique.

if F is a closure operator on a class $\mathscr{M}$ of monomorphisms, as we will assume later, the uniqueness part of the orthogonality condition is automatically satisfied.

In order to have analogues to the complete subobject lattices of topological spaces, throughout the remainder of the paper we assume that we are working in a category $\mathscr{X}$ that is sufficiently nice to support certain constructions. Specifically, we require $\mathscr{X}$ to be an $\langle \boldsymbol{E}, \mathscr{M} \rangle$-category for sinks. For easy reference we recall the definition.

1.04 Definition. $\mathscr{X}$ is called an **$\langle \boldsymbol{E}, \mathscr{M} \rangle$-category for sinks** if there exists a collection $\boldsymbol{E}$ of $\mathscr{X}$-sinks and a class $\mathscr{M}$ of $\mathscr{X}$-morphisms such that:

(0) each of $\boldsymbol{E}$ and $\mathscr{M}$ is stable under compositions with isomorphisms;

(1) $\mathscr{X}$ has $\langle \boldsymbol{E}, \mathscr{M} \rangle$-factorizations (of sinks); i.e., each sink $\boldsymbol{s}$ in $\mathscr{X}$ has a factorization $\boldsymbol{s} = m \circ \boldsymbol{e}$ with $\boldsymbol{e} \in \boldsymbol{E}$ and $m \in \mathscr{M}$, and

(2) every sink $\boldsymbol{e} \in \boldsymbol{E}$ is left orthogonal to every morphism $m \in \mathscr{M}$.

That $\mathscr{X}$ is an $\langle \boldsymbol{E}, \mathscr{M} \rangle$-category for sinks entails certain features of the class $\mathscr{M}$, namely:

1.05 Proposition. (0) *Every m in $\mathscr{M}$ is a monomorphism.*

(1) *$\mathscr{M}$ is stable under $\mathscr{M}$-relative first factors.*

(2) *$\mathscr{M}$ is closed under composition.*

(3) *Pullbacks of $\mathscr{M}$-elements along $\mathscr{X}$-morphisms exist and belong to $\mathscr{M}$.*

(4) *The $\mathscr{M}$-subobjects of every $\mathscr{X}$-object form a (possibly large) complete lattice; suprema are formed via $\langle \boldsymbol{E}, \mathscr{M} \rangle$-factorizations.* $\square$

$\leq$ denotes the usual pre-order on $\mathscr{M}$-subobjects. To minimize problems resulting from the fact that this pre-order in general is not antisymmetric, we assume that for every sink $\boldsymbol{s}$ in $\mathscr{X}$ a specific $\langle \boldsymbol{E}, \mathscr{M} \rangle$-factorization has been chosen. (This choice need not be canonical in any sense.) Now we can speak about *the* infimum and *the* supremum of a sink consisting of $\mathscr{M}$-elements, as well as of *the* pullback of an $\mathscr{M}$-element. We continue to use the term **lattice** for pre-ordered classes that are finitely complete and finitely cocomplete. A lattice is called **complete** if each subclass has an infimum or, equivalently, each subclass has a supremum.

1.06 Definition. A closure operator F on $\mathscr{M}$ is called

(0) **idempotent** iff m^F is F-closed for every $m \in \mathscr{M}$, i.e., iff $(\,)^F * (\,)^F \cong (\,)^F$;

(1) **weakly hereditary** iff m_F is F-dense for every $m \in \mathscr{M}$, i.e., iff $(\,)_F * (\,)_F \cong (\,)_F$.

$\boldsymbol{CL}(\mathscr{M})$ denotes the collection of all closure operators on $\mathscr{M}$, pre-ordered by $F \sqsubseteq G$ iff $m^F \leq m^G$ for all $m \in \mathscr{M}$, while $\boldsymbol{iCL}(\mathscr{M})$ and $\boldsymbol{wCL}(\mathscr{M})$ stand for the subcollections of idempotent and weakly hereditary closure operators, respectively. We write $\boldsymbol{iwCL}(\mathscr{M})$ for the intersection of these two collections.

Two other important concepts, *hereditary closure operators* and *modal closure operators*, are investigated in a subsequent paper [4].

Notice that under our assumptions arbitrary suprema and infima of closure operators exist. They are formed pointwise in the fibers. (N.B., in [13] the terms *proper* and *strict* were used instead of *idempotent* and *weakly hereditary*, respectively.)

1.07 PROPOSITION. (cf. [9] and [13]) *$\boldsymbol{wCL}(\mathscr{M})$ is closed under the formation of suprema in $\boldsymbol{CL}(\mathscr{M})$, whereas $\boldsymbol{iCL}(\mathscr{M})$ is closed under the formation of infima in $\boldsymbol{CL}(\mathscr{M})$.* □

Thus every closure operator F has an **idempotent hull** (i.e., reflection) $F^{\mathrm{i}} \in \boldsymbol{iCL}(\mathscr{M})$ as well as a **weakly hereditary core** (i.e., coreflection) $F_{\mathrm{w}} \in \boldsymbol{wCL}(\mathscr{M})$.

1.08 PROPOSITION. (cf. [10], [11] and [13]) (0) *If F is idempotent, so is F_{w}.*

(1) *If F is weakly hereditary, so is F^{i}.* □

Notice, however, that in general ${F^{\mathrm{i}}}_{\mathrm{w}}$ and ${F_{\mathrm{w}}}^{\mathrm{i}}$ need not agree.

1.09 LEMMA. *Let F be a closure operator.*

(0) *If m is F-dense and n is F-closed then $m \perp n$.*

(1) *$m \in \mathscr{M}$ is F-dense iff m is left-orthogonal to every pair $F(n)$ with $n \in \mathscr{M}$ iff $m \perp F(m)$. If F is idempotent, then $m \in \mathscr{M}$ is F-dense iff m is left-orthogonal to every F-closed element of $\mathscr{M}$ iff $m \perp m^F$.*

(2) *$n \in \mathscr{M}$ is F-closed iff n is right-orthogonal to every pair $F(m)$ with $m \in \mathscr{M}$ iff $F(n) \perp n$. If F is weakly hereditary, then $n \in \mathscr{M}$ is F-closed iff n is right-orthogonal to every F-dense element of $\mathscr{M}$ iff $n_F \perp n$.*

(3) *F is idempotent iff $m^F \cong \inf\{\, p \in \mathscr{M} \mid m = p \circ n \text{ and } p \in \nabla_*(F)\,\}$ for each $m \in \mathscr{M}$. Thus an idempotent closure operator F is essentially determined by its class $\nabla_*(F)$ of F-closed $\mathscr{M}$-elements (cf.* [13]*).*

(4) *F is weakly hereditary iff $m^F \cong \sup\{\, p \in \mathscr{M} \mid m = p \circ n \text{ and } n \in \Delta^*(F)\,\}$ for each $m \in \mathscr{M}$. Thus a weakly hereditary closure operator F is essentially determined by its class $\Delta^*(F)$ of F-dense $\mathscr{M}$-elements (cf.* [13]*).*

(5) *If F is idempotent and weakly hereditary, then $\Delta^*(F)$ and $\nabla_*(F)$ determine each other via the orthogonality relation $\perp$ (cf.* [9] *and* [13]*).* □

For more background on closure operators see, e.g., [2], [9], [11], and [13].

The above lemma (in particular part (5)) suggests that polarities induced by suitable restrictions of the orthogonality relation $\perp$ will play a significant role in the study of closure operators.

Below we recall some facts about Galois connections between pre-ordered classes, especially between power collections, and introduce some convenient notation. A more detailed account in the case of partially ordered sets can be found in [12]. There are only minor technical differences between the theories of Galois connections in the setting of partially ordered sets and in the setting of pre-ordered classes.

1.10 DEFINITION. For pre-ordered classes $\mathscr{A} = \langle A, \sqsubseteq \rangle$ and $\mathscr{B} = \langle B, \sqsubseteq \rangle$ a **Galois connection** $\mathscr{A} \xrightarrow{\pi = \langle \pi_*, \pi^* \rangle} \bullet\, \mathscr{B}$ consists of two functions $A \underset{\pi_*}{\overset{\pi^*}{\leftrightarrows}} B$ that satisfy $a \sqsubseteq \pi^*(b)$ iff $\pi_*(a) \sqsubseteq b$ for all $a \in A$ and $b \in B$. (Order-preservation then is automatic.) π^* is the **adjoint** part and π_* is the **coadjoint** part of π. We write $\mathscr{A}^\pi$ and $\mathscr{B}_\pi$ for the classes of Galois fixed points $\{ a \in A \mid a \cong \pi^* \pi_*(a) \}$ and $\{ b \in B \mid \pi_* \pi^*(b) \cong b \}$ with the induced orders, respectively. π is called a (**co**)**reflection** iff π^* (respectively π_*) is a one-to-one function, and an **equivalence** iff $\langle \pi^*, \pi_* \rangle$ is a Galois connection from $\mathscr{B}$ to $\mathscr{A}$. (Notice that π restricts to an equivalence from $\mathscr{A}^\pi$ to $\mathscr{B}_\pi$.)

The **composite** $\mathscr{A} \xrightarrow{\rho \circ \pi} \bullet\, \mathscr{C}$ of two Galois connections $\mathscr{A} \xrightarrow{\pi} \bullet\, \mathscr{B}$ and $\mathscr{B} \xrightarrow{\rho} \bullet\, \mathscr{C}$ is defined as $\langle \rho_* \circ \pi_*, \pi^* \circ \rho^* \rangle$, and the **dual** $\mathscr{B}^{\mathrm{op}} \xrightarrow{\pi^{\mathrm{op}}} \bullet\, \mathscr{A}^{\mathrm{op}}$ of a Galois connection $\mathscr{A} \xrightarrow{\pi} \bullet\, \mathscr{B}$ is given by $\pi^{\mathrm{op}} = \langle \pi^*, \pi_* \rangle$.

If $\mathscr{A} \xrightarrow{\varphi} \bullet\, \mathscr{C}$ factors as $\mathscr{A} \xrightarrow{\dot\varphi} \bullet\, \mathscr{B} \xrightarrow{\ddot\varphi} \bullet\, \mathscr{C}$ and if $\mathscr{B}$ is equivalent to both fixed point classes $\mathscr{A}^\varphi$ and $\mathscr{C}_\varphi$, then we call $\ddot\varphi \circ \dot\varphi$ an **essentially canonical factorization** of φ with **center** $\mathscr{B}$. The dot notation will be employed throughout to indicate essentially canonical factorizations.

1.11 PROPOSITION. (cf. [12]) *Any relation $R \subseteq A \times B$ between classes A and B induces*

(0) *a Galois connection $\boldsymbol{P}(A) \xrightarrow{\varphi} \bullet\, \boldsymbol{P}(B)^{\mathrm{op}}$, called a* **polarity**, *the coadjoint and adjoint parts of which are defined by*

$$\varphi_*(U) := \{ b \in B \mid \forall_{a \in U} \ \langle a, b \rangle \in R \} \qquad \text{for } U \subseteq A$$
$$\varphi^*(V) := \{ a \in A \mid \forall_{b \in V} \ \langle a, b \rangle \in R \} \qquad \text{for } V \subseteq B$$

(1) *a Galois connection $\boldsymbol{P}(A) \xrightarrow{\psi} \bullet\, \boldsymbol{P}(B)$, called an* **axiality**, *the coadjoint and adjoint parts of which are defined by*

$$\psi_*(U) := \{ b \in B \mid \exists_{a \in A} \ \langle a, b \rangle \in R \textit{ and } a \in U \} \qquad \textit{for } U \subseteq A$$
$$\psi^*(V) := \{ a \in A \mid \forall_{b \in B} \ \langle a, b \rangle \in R \textit{ implies } b \in V \} \qquad \textit{for } V \subseteq B$$

($\psi_*(U)$ *is commonly known as the* **direct image** *of U under R.*) □

2. HOW TO CONSTRUCT CLOSURE OPERATORS

Let $\mathscr{M} \diamond \mathscr{M}$ be the class of composable pairs of morphisms in $\mathscr{M}$, i.e., the pullback of the domain functor U along the codomain functor V. In this section we construct the following commutative diagram of Galois connections between lattices

$$\begin{tikzcd}[column sep=small]
\boldsymbol{P}(\mathscr{M} \diamond \mathscr{M}) \arrow[rr, -{Circle}, "\dot\omega"] & & \boldsymbol{CL}(\mathscr{M}) \arrow[rr, -{Circle}, "\ddot\omega"] \arrow[dr, -{Circle}, "\dot\nabla"] & & \boldsymbol{P}(\mathscr{M} \diamond \mathscr{M})^{\mathrm{op}} \arrow[dd, -{Circle}, "\curlywedge"] \\
& \boldsymbol{wCL}(\mathscr{M}) \arrow[ur, -{Circle}, "\ddot\Delta"] \arrow[dr, -{Circle}, "\dot\epsilon"'] & & \boldsymbol{iCL}(\mathscr{M}) \arrow[dr, -{Circle}, "\ddot\nabla"'] & \\
\boldsymbol{P}(\mathscr{M}) \arrow[uu, -{Circle}, "\curlyvee"] \arrow[ur, -{Circle}, "\dot\Delta"'] \arrow[rr, -{Circle}, "\dot\upsilon"'] & & \boldsymbol{iwCL}(\mathscr{M}) \arrow[rr, -{Circle}, "\ddot\upsilon"'] \arrow[ur, -{Circle}, "\ddot\epsilon"'] & & \boldsymbol{P}(\mathscr{M})^{\mathrm{op}}
\end{tikzcd} \tag{2-00}$$

In the course of the construction we will see that

- The restriction of the relation $\perp$ to $\mathscr{M}$ induces a polarity $\boldsymbol{P}(\mathscr{M}) \xrightarrow{\upsilon} \boldsymbol{P}(\mathscr{M})^{\mathrm{op}}$ that has (by virtue of Lemma 1.09(5)) an essentially canonical factorization $\upsilon = \ddot\upsilon \circ \dot\upsilon$ with a center that consists of all idempotent and weakly hereditary closure operators on $\mathscr{M}$, i.e., $\boldsymbol{iwCL}(\mathscr{M})$ is equivalent to the Galois fixed points of υ.
- The polarity $\boldsymbol{P}(\mathscr{M} \diamond \mathscr{M}) \xrightarrow{\omega} \boldsymbol{P}(\mathscr{M} \diamond \mathscr{M})^{\mathrm{op}}$ induced by the restriction of the relation $\perp$ to $\mathscr{M} \diamond \mathscr{M}$ admits an essentially canonical factorization $\ddot\omega \circ \dot\omega$ with a center consisting of all closure operators on $\mathscr{M}$, i.e., $\boldsymbol{CL}(\mathscr{M})$ is equivalent to the lattices of Galois fixed points of ω. Moreover, this factorization "lifts" the essentially canonical factorization $\ddot\upsilon \circ \dot\upsilon$.
- The function $\boldsymbol{CL}(\mathscr{M}) \xrightarrow{\Delta^*} \boldsymbol{P}(\mathscr{M})$ that maps each closure operator to its class of dense $\mathscr{M}$-elements is in fact the adjoint part of a Galois connection $\boldsymbol{P}(\mathscr{M}) \xrightarrow{\Delta} \boldsymbol{CL}(\mathscr{M})$ that factors through $\dot\omega$ by means of an axiality $\curlyvee$, and has an essentially canonical factorization $\Delta = \ddot\Delta \circ \dot\Delta$ with a center that consists of all weakly hereditary closure operators on $\mathscr{M}$. (Here $\dot\Delta^*$ maps a closure operator to its weakly hereditary core.) In particular, $\boldsymbol{wCL}(\mathscr{M})$ is equivalent to the lattice of Galois fixed points of Δ.
- Analogous to the above, the function $\boldsymbol{CL}(\mathscr{M}) \xrightarrow{\nabla_*} \boldsymbol{P}(\mathscr{M})^{\mathrm{op}}$ that maps each closure operator to its class of closed $\mathscr{M}$-elements turns out to be coadjoint, and the resulting Galois connection $\boldsymbol{CL}(\mathscr{M}) \xrightarrow{\nabla} \boldsymbol{P}(\mathscr{M})^{\mathrm{op}}$ factors through the analogue $\curlywedge$ of $\curlyvee$ by means of $\ddot\omega$, and has an essentially canonical factorization $\nabla = \ddot\nabla \circ \dot\nabla$ that has a center consisting of all idempotent closure operators on $\mathscr{M}$. (Here $\ddot\nabla_*$ maps a closure operator to its idempotent hull.) Now $\boldsymbol{iCL}(\mathscr{M})$ is equivalent to the lattice of Galois fixed points of ∇.
- The composite $\dot\nabla \circ \ddot\Delta$ is a Galois connection $\boldsymbol{wCL}(\mathscr{M}) \xrightarrow{\epsilon} \boldsymbol{iCL}(\mathscr{M})$ with the property that the Galois fixed point lattices $\boldsymbol{wCL}(\mathscr{M})^{\epsilon}$ and $\boldsymbol{iCL}(\mathscr{M})_{\epsilon}$ actually coincide; in fact, they are equal to $\boldsymbol{iwCL}(\mathscr{M})$. Because of Proposition 1.08 $\dot\nabla$ admits a "weakly

hereditary" restriction $\dot{\epsilon}$, and $\ddot{\Delta}$ admits an "idempotent" restriction $\ddot{\epsilon}$ that constitute an essentially canonical factorization of ϵ. Hence $\boldsymbol{iwCL}(\mathscr{M})$ appears as a Galois fixed point lattice for the second time.

A crucial ingredient for our approach will be the interplay between inverse images and direct images. The codomain functor V is well-known to support such constructs. Recall that for an $\mathscr{X}$-object X the V-fibre V/X consists of all $\mathscr{M}$-subobjects of X, i.e., elements of $\mathscr{M}$ with codomain X, equipped with the usual pre-order $m \leq p$ iff $p = n \circ m$ for some (necessarily unique) n.

2.00 Proposition. *Every $\mathscr{X}$-morphism $X \xrightarrow{f} Y$ induces a V-***inverse image function** *$V/Y \xrightarrow{f^{\leftarrow}} V/X$ (that maps each $\mathscr{M}$-subobject of Y to its pullback along f) and a V-***direct image function** *$V/X \xrightarrow{f_{\rightarrow}} V/Y$ (that maps each $\mathscr{M}$-subobject m of X to the $\mathscr{M}$-component of the chosen $\langle \boldsymbol{E}, \mathscr{M}\rangle$-factorization of $f \circ m$). Moreover, $\langle f_{\rightarrow}, f^{\leftarrow}\rangle$ is a Galois connection; in particular, $f_{\rightarrow}$ (up to isomorphism) preserves suprema of $\mathscr{M}$-subobjects.* $\square$

To compare different factorizations of a morphism in $\mathscr{M}$ we introduce a pre-order $\ll$ on $\mathscr{M} \diamond \mathscr{M}$. Notice that $\mathscr{M} \diamond \mathscr{M}$ can be viewed as a category whose morphisms from $\langle n, p\rangle$ to $\langle q, r\rangle$ are triples $\langle a, b, c\rangle$ of $\mathscr{X}$-morphisms that satisfy $q \circ a = b \circ n$ and $r \circ b = c \circ p$.

2.01 Definition. The **composition functor** $\mathscr{M} \diamond \mathscr{M} \xrightarrow{W} \mathscr{M}$ maps $\langle n, p\rangle \xrightarrow{\langle a,b,c\rangle} \langle q, r\rangle$ to the $\mathscr{M}$-morphism $p \circ n \xrightarrow{\langle a,c\rangle} r \circ q$. For each $m \in \mathscr{M}$ its W**-fiber,** i.e., the comma category W/m, is pre-ordered by $\langle n, p\rangle \ll \langle q, r\rangle$ iff there exists a (necessarily unique) $\mathscr{X}$-morphism b such that $\langle n, p\rangle \xrightarrow{\langle id,b,id\rangle} \langle q, r\rangle$ is an $\mathscr{M} \diamond \mathscr{M}$-morphism.

The W-fibers form (possibly large) complete lattices with respect to $\ll$. Intersections and $\langle \boldsymbol{E}, \mathscr{M}\rangle$-factorizations of the (collections of) second components yield infima and suprema, respectively. Inside of W-fibers $\langle n, p\rangle \perp \langle q, r\rangle$ implies $\langle n, p\rangle \ll \langle q, r\rangle$, and closure operators F and G satisfy $F \sqsubseteq G$ iff $F(m) \ll G(m)$ for all $m \in \mathscr{M}$.

2.02 Definition. For $m \xrightarrow{\langle f,g\rangle} n$ in $\boldsymbol{Mor}(\mathscr{M})$ consider $\langle q, r\rangle \in W/m$ and $\langle s, t\rangle \in W/n$.

(0) $\langle q, r\rangle$ is called the W**-inverse image** of $\langle s, t\rangle$ along $\langle f, g\rangle$ (denoted by $\langle f, g\rangle^{\leftarrow}(\langle s, t\rangle)$) iff there exists an $\mathscr{X}$-morphism d such that $\bullet \xleftarrow{r} \bullet \xrightarrow{d} \bullet$ is the pullback of $\bullet \xrightarrow{g} \bullet \xleftarrow{t} \bullet$ and q satisfies $d \circ q = s \circ f$. Given $D \subseteq \mathscr{M} \diamond \mathscr{M}$ we write D_{ii} for the smallest subclass of $\mathscr{M}$ that contains D and is closed under isomorphisms and W-inverse images.

(1) $\langle s, t\rangle$ is called the W**-direct image** of $\langle q, r\rangle$ along $\langle f, g\rangle$ (denoted by $\langle f, g\rangle_{\rightarrow}(\langle q, r\rangle)$) iff there exists an $\mathscr{X}$-morphism d such that $\langle\langle d, s\rangle, t\rangle$ is the $\langle \boldsymbol{E}, \mathscr{M}\rangle$-factorization of the sink $\bullet \xrightarrow{g \circ r} \bullet \xleftarrow{n} \bullet$. Given $C \subseteq \mathscr{M} \diamond \mathscr{M}$ we write C^{di} for the smallest subclass of $\mathscr{M}$ that contains C and is closed under isomorphisms and W-direct images.

2.03 Proposition. (0) *Every $\mathscr{M}$-morphism $m \xrightarrow{\langle f,g\rangle} n$ induces a Galois connection $\langle\langle f, g\rangle_{\rightarrow}, \langle f, g\rangle^{\leftarrow}\rangle$.*

(1) *Both of the operations* $(\,)_{\mathrm{ii}}$ *and* $(\,)^{\mathrm{di}}$ *on* $\boldsymbol{P}(\mathscr{M} \diamond \mathscr{M})$ *are idempotent.*

Proof. (0) One easily verifies that $\langle f,g\rangle_{\rightarrow}(\langle q,r\rangle) \ll \langle s,t\rangle \iff \langle q,r\rangle \ll \langle f,g\rangle^{\leftarrow}(\langle s,t\rangle)$.

(1) $(\,)_{\mathrm{ii}}$ is idempotent since for any two $\mathscr{M}$-morphisms $m \xrightarrow{\langle f,g\rangle} n$ and $n \xrightarrow{\langle h,k\rangle} p$ we have $(\langle f,g\rangle^{\leftarrow})(\langle h,k\rangle^{\leftarrow}) \cong \langle h\circ f, k\circ g\rangle^{\leftarrow}$. But (0) implies $(\langle h,k\rangle_{\rightarrow})(\langle f,g\rangle_{\rightarrow}) \cong \langle h\circ f, k\circ g\rangle_{\rightarrow}$, so $(\,)^{\mathrm{di}}$ is idempotent as well. □

In order to factor the Galois connection ω , we first characterize its fixed points.

2.04 LEMMA. *If* $C \subseteq \mathscr{M} \diamond \mathscr{M}$ *and* $m \in \mathscr{M}$ *we have* $\sup_{\ll}\big(C^{\mathrm{di}} \cap W/m\big) \in \omega_*(C)$.

Proof. Write $\langle a,b\rangle$ for the supremum and consider $\langle q,r\rangle \in C$. For every $\mathscr{M}$-morphism $r\circ q \xrightarrow{\langle f,g\rangle} m$ the W-direct image of $\langle q,r\rangle$ along $\langle f,g\rangle$ is used in the construction of the supremum, hence $\langle q,r\rangle \perp \langle a,b\rangle$. □

2.05 THEOREM. $C \subseteq \mathscr{M}\diamond\mathscr{M}$ *is a fixed point of* $\omega^*\circ\omega_*$ *iff* C *satisfies the following conditions*

(C0) C *is closed under the formation of* W*-direct images.*

(C1) $\sup_{\ll}\big(C \cap W/m\big) \in C$ *for every* $m \in \mathscr{M}$.

(C2) $C \cap W/m$ *is a lower segment for every* $m \in \mathscr{M}$.

Proof. $(\Rightarrow)$. Let C be a fixed point of $\omega^*\circ\omega_*$.

(C0) If $\langle s,t\rangle$ is a direct image of $\langle q,r\rangle \in C$ along $r\circ q \xrightarrow{\langle f,g\rangle} t\circ s$, there exists a unique c with $\langle c,s\rangle \in \boldsymbol{E}$ as well as $c\circ q = s\circ f$ and $t\circ c = g\circ r$. For $\langle n,p\rangle \in \omega_*(C)$ consider an $\mathscr{M}$-morphism $t\circ s \xrightarrow{\langle h,k\rangle} p\circ n$. Since $\langle q,r\rangle \perp \langle n,p\rangle$, there exists a unique e with $e\circ q = n\circ h\circ f$ and $p\circ e = k\circ g\circ r$. But now $\langle c,s\rangle \perp p$ implies the existence of a unique d such that $d\circ c = e$ as well as $d\circ s = n\circ h$ and $p\circ d = k\circ t$. Hence $\langle s,t\rangle \perp \langle n,p\rangle$, and consequently $\langle s,t\rangle \in \omega^*\omega_*(C) = C$.

(C1) Write $\langle a,b\rangle$ for the supremum. By (C0) and Lemma 2.04 $\langle a,b\rangle$ belongs to $\omega_*(C)$. Dually, $\langle c,d\rangle := \inf_{\ll}\big(\omega_*(C) \cap W/m\big) \in \omega^*\omega_*(C) = C$, and hence $\langle c,d\rangle \ll \langle a,b\rangle$. But clearly $\langle a,b\rangle \ll \langle c,d\rangle$, and so $\langle a,b\rangle \cong \langle c,d\rangle$, which implies $\langle a,b\rangle \in C$.

(C2) $\langle n,p\rangle \ll \langle q,r\rangle \in C \cap W/m$ and $\langle q,r\rangle \perp \langle s,t\rangle \in \omega_*(C)$ implies $\langle n,p\rangle \perp \langle q,r\rangle$, since t is mono.

$(\Leftarrow)$. Suppose that $C \subseteq \mathscr{M}\diamond\mathscr{M}$ satisfies (C0) – (C2). For $\langle n,p\rangle \in \omega^*\omega_*(C)$ set $m = p\circ n$. By (C0) and Lemma 2.04 $\langle a,b\rangle := \sup_{\ll}\big(C \cap W/m\big)$ belongs to $\omega_*(C)$, which implies $\langle n,p\rangle \ll \langle a,b\rangle$. By (C1) $\langle a,b\rangle$ belongs to C as well, so (C2) yields $\langle n,p\rangle \in C$. □

Notice that conditions (C1) and (C2) together are equivalent to

- $C \cap W/m$ is a principal ideal for every $m \in \mathscr{M}$.

2.06 Proposition. $\boldsymbol{P}(\mathscr{M}\diamond\mathscr{M}) \xrightarrow{\dot\omega_*} \boldsymbol{CL}(\mathscr{M})$ *and* $\boldsymbol{P}(\mathscr{M}\diamond\mathscr{M})^{\mathrm{op}} \xrightarrow{\ddot\omega^*} \boldsymbol{CL}(\mathscr{M})$ *defined by*

$$\dot\omega_*(C)(m) := \sup_{\ll}\bigl(C^{\mathrm{di}} \cap W/m\bigr) \qquad \text{and} \qquad \ddot\omega^*(D)(m) := \inf_{\ll}\bigl(D_{\mathrm{ii}} \cap W/m\bigr) \tag{2-01}$$

for $C, D \subseteq \mathscr{M}\diamond\mathscr{M}$ *and* $m \in \mathscr{M}$ *, both preserve order.*

Proof. To show that $\dot\omega_*(C)$ is a closure operator, consider an $\mathscr{M}$-morphism $m \xrightarrow{\langle f,g\rangle} n$. Since $\langle f,g\rangle_\rightarrow$ as a coadjoint preserves suprema, the W-direct image of $\dot\omega_*(C)(m)$ along $\langle f,g\rangle$ is isomorphic to $\sup_{\ll}\{\,\langle f,g\rangle_\rightarrow(\langle q,r\rangle) \mid \langle q,r\rangle \in C^{\mathrm{di}} \cap W/m\,\}$. Now Proposition 2.03(1) implies $\langle f,g\rangle_\rightarrow(\dot\omega_*(C)(m)) \ll \sup_{\ll}\bigl(C^{\mathrm{di}} \cap W/n\bigr) = \dot\omega_*(C)(n)$.

That $\dot\omega_*$ preserves order follows since $(\,)^{\mathrm{di}}$ does. The proof for $\ddot\omega^*$ is similar. $\square$

For an idempotent and weakly hereditary closure operator the classes $\Delta^*(F)$ of F-dense $\mathscr{M}$-elements and $\nabla_*(F)$ of F-closed $\mathscr{M}$-elements are corresponding fixed points for the polarity υ induced by the restriction of $\perp$ to $\mathscr{M}$ (cf. Lemma 1.09(5)). For a general closure operator we need to consider classes of composable pairs to obtain a similar result.

2.07 Definition. For a closure operator F define the classes $\dot\omega^*(F)$ of **relatively** F**-dense pairs** and $\ddot\omega_*(F)$ of **relatively** F**-closed pairs**, respectively, as

$$\dot\omega^*(F) = \omega^*\{\,F(m) \mid m \in \mathscr{M}\,\} \qquad \text{and} \qquad \ddot\omega_*(F) = \omega_*\{\,F(m) \mid m \in \mathscr{M}\,\} \tag{2-02}$$

2.08 Theorem. (0) $\boldsymbol{CL}(\mathscr{M}) \xrightarrow{\dot\omega^*} \boldsymbol{P}(\mathscr{M}\diamond\mathscr{M})$ *and* $\boldsymbol{CL}(\mathscr{M}) \xrightarrow{\ddot\omega_*} \boldsymbol{P}(\mathscr{M}\diamond\mathscr{M})^{\mathrm{op}}$ *both preserve order.*

(1) *For any closure operator* F *the classes of relatively* F*-dense pairs and of relatively* F*-closed pairs are corresponding fixed points under the polarity* ω*, i.e., we have*

$$\omega_*\dot\omega^*(F) = \ddot\omega_*(F) \qquad \text{and} \qquad \omega^*\ddot\omega_*(F) = \dot\omega^*(F)$$

(2) *For any two classes* $C, D \subseteq \mathscr{M}\diamond\mathscr{M}$ *with* $\omega_*(C) = D$ *and* $\omega^*(D) = C$ *the closure operators* $\dot\omega_*(C)$ *and* $\ddot\omega^*(D)$ *are isomorphic. Moreover, any closure operator* F *isomorphic to these two satisfies* $\{\,F(m) \mid m \in \mathscr{M}\,\} \subseteq C \cap D$*, and* $\dot\omega^*(F) = C$ *as well as* $\ddot\omega_*(F) = D$*.*

(3) $\dot\omega_* \circ \dot\omega^* \cong id_{\boldsymbol{CL}(\mathscr{M})} \cong \ddot\omega^* \circ \ddot\omega_*$*.*

(4) $\ddot\omega \circ \dot\omega$ *is an essentially canonical factorization of* ω*.*

Proof. (0) If $F \sqsubseteq G$ then $F(m) \ll G(m)$ for every $m \in \mathscr{M}$. Hence $\langle n,p\rangle \perp F(m)$ implies $\langle n,p\rangle \perp G(m)$. This shows that $\dot\omega^*(F) \subseteq \dot\omega^*(G)$. The result for $\ddot\omega_*$ follows dually.

(1) By the definition of a closure operator we have $\{\,F(m) \mid m \in \mathscr{M}\,\} \subseteq \dot\omega^*(F) \cap \ddot\omega_*(F)$. Therefore $\dot\omega^*(F) \supseteq \omega^*\ddot\omega_*(F)$ and $\omega_*\dot\omega^*(F) \subseteq \ddot\omega_*(F)$. On the other hand for $\langle n,p\rangle \in$

$\dot{\omega}^*(F)$ and $\langle q,r\rangle \in \ddot{\omega}_*(F)$ we have $\langle n,p\rangle \ll F(p\circ n) \perp F(r\circ q) \ll \langle q,r\rangle$ and hence $\langle n,p\rangle \perp \langle q,r\rangle$, which yields the other inclusions.

(2) That $\dot{\omega}_*(C)$ and $\ddot{\omega}^*(D)$ are isomorphic follows from Lemma 2.04 and Theorem 2.05, and their duals: For any $m \in \mathscr{M}$ both $\sup_{\ll}\big(C \cap W/m\big)$ and $\inf_{\ll}\big(D \cap W/m\big)$ belong to $C \cap D$ and hence are isomorphic. Moreover, any closure operator F isomorphic to $\dot{\omega}_*(C)$ and $\ddot{\omega}^*(D)$ satisfies $\{\,F(m) \mid m \in \mathscr{M}\,\} \subseteq C \cap D$, which in turn yields $\dot{\omega}^*(F) \supseteq C$ and $\ddot{\omega}_*(F) \supseteq D$. But C and D as well as $\dot{\omega}^*(F)$ and $\ddot{\omega}_*(F)$ are corresponding fixed points under the polarity ω, which implies that $\dot{\omega}^*(F) = C$ and $\ddot{\omega}_*(F) = D$.

(3) For $F \in \boldsymbol{CL}(\mathscr{M})$ by (1) the classes $\dot{\omega}^*(F)$ and $\ddot{\omega}_*(F)$ satisfy the hypothesis of (2), hence $\dot{\omega}_*\dot{\omega}^*(F)$ and $\ddot{\omega}^*\ddot{\omega}_*(F)$ are isomorphic. Moreover, $\{\,\dot{\omega}_*\dot{\omega}^*(F)(m) \mid m \in \mathscr{M}\,\} \subseteq \dot{\omega}^*(F) \cap \ddot{\omega}_*(F)$. Thus for each $m \in \mathscr{M}$ we have $F(m) \perp \dot{\omega}_*\dot{\omega}^*(F)(m) \perp F(m)$, so $F(m) \ll \dot{\omega}_*\dot{\omega}^*(F)(m) \ll F(m)$, i.e., $F \cong \dot{\omega}_*\dot{\omega}^*(F)$.

(4) For $C \subseteq \mathscr{M} \diamond \mathscr{M}$ Lemma 2.04 and Theorem 2.05 imply $\dot{\omega}_*\omega^*\omega_*(C) \sqsubseteq \dot{\omega}_*(C)$. The reverse order holds by Proposition 2.06, so both closure operators are isomorphic. Hence by (2) $\ddot{\omega}_*\dot{\omega}_*(C) = \omega_*(C)$ and $\dot{\omega}^*\dot{\omega}_*(C) = \omega^*\omega_*(C)$. Similarly one obtains $\dot{\omega}^* \circ \ddot{\omega}^* = \omega^*$ and $\ddot{\omega}_* \circ \ddot{\omega}^* = \omega_* \circ \omega^*$. Therefore $\omega = \ddot{\omega} \circ \dot{\omega}$ and $\dot{\omega}^* \circ \dot{\omega}_*$ is increasing, while $\ddot{\omega}_* \circ \ddot{\omega}^*$ is decreasing. This together with (3) establishes the desired adjunctions. $\square$

2.09 Definition. (cf. 1.11(1)) Let $\boldsymbol{P}(\mathscr{M}) \xrightarrow{\curlyvee\ \bullet} \boldsymbol{P}(\mathscr{M} \diamond \mathscr{M})$ be the axiality induced by the dual of the graph of the first projection $\mathscr{M} \diamond \mathscr{M} \longrightarrow \mathscr{M}$, and let $\boldsymbol{P}(\mathscr{M} \diamond \mathscr{M})^{\mathrm{op}} \xrightarrow{\curlywedge\ \bullet} \boldsymbol{P}(\mathscr{M})^{\mathrm{op}}$ be the dual of the axiality induced similarly by the second projection $\mathscr{M} \diamond \mathscr{M} \longrightarrow \mathscr{M}$.

Notice that $\omega \circ \curlyvee$ is the polarity induced by the restriction of $\perp$ to $\mathscr{M} \times (\mathscr{M} \diamond \mathscr{M})$, while $\curlywedge \circ \omega$ is the polarity induced by the restriction of $\perp$ to $(\mathscr{M} \diamond \mathscr{M}) \times \mathscr{M}$.

2.10 Theorem. (0) *$\dot{\omega}_*\curlyvee_*(A)$ is weakly hereditary for every $A \in \boldsymbol{P}(\mathscr{M})$.*

(1) *$\ddot{\omega}^*\curlywedge^*(B)$ is idempotent for every $B \in \boldsymbol{P}(\mathscr{M})^{\mathrm{op}}$.*

(2) *$\curlyvee^*\dot{\omega}^*(F) = \Delta^*(F)$ and $\curlywedge_*\ddot{\omega}_*(F) = \nabla_*(F)$ for every $F \in \boldsymbol{CL}(\mathscr{M})$. In particular, Δ^* is adjoint, while ∇_* is coadjoint.*

Proof. (0) Fix $A \subseteq \mathscr{M}$ and set $F = \dot{\omega}_*\curlyvee_*(A)$. We claim that $\langle m_{FF}, m^F \circ m_F{}^F\rangle$ is relatively F-closed, i.e., belongs to $\ddot{\omega}_*(F) = \omega_*\curlyvee_*(A)$, for any $m \in \mathscr{M}$. Both $F(m)$ and $F(m_F)$ have this property. Consider an $\mathscr{M}$-morphism $n \xrightarrow{\langle f,g\rangle} m$ and $\langle q,r\rangle \in \curlyvee_*(A)$ with $r \circ q = n$. Since $\langle q,r\rangle \perp F(m)$ there exists a unique d with $d \circ q = m_F \circ f$ and $g \circ r = m^F \circ d$. But $\langle q, id\rangle \in \curlyvee_*(A)$ implies $\langle q, id\rangle \perp F(m_F)$, so there exists a unique d' with $d' \circ q = f$ and $m_F{}^F \circ d' = d$. Thus $\langle q,r\rangle \perp \langle m_{FF}, m^F \circ m_F{}^F\rangle$, which establishes the claim. Since $F(m)$ is relatively F-dense as well, we have $F(m) \perp \langle m_{FF}, m^F \circ m_F{}^F\rangle$, which forces $m_F{}^F$ to be an isomorphism. Hence m_F is F-dense.

(1) Similar.

(2) By Lemma 1.09(1) we have

$$\curlyvee^{*}\dot{\omega}^{*}(F) = \{\, n \in \mathscr{M} \mid \forall_{\langle q,r\rangle \in \mathscr{M}\diamond\mathscr{M}}\ q = n \Rightarrow \forall_{m\in\mathscr{M}}\ \langle q,r\rangle \perp F(m)\,\}$$
$$= \{\, n \in \mathscr{M} \mid \forall_{m\in\mathscr{M}}\ n \perp F(m)\,\} = \Delta^{*}(F)\ .$$

The same kind of reasoning establishes the second equality. □

We are now in a position to define the remaining Galois connections in Diagram (2-00).

2.11 Definition. (0) Define $\dot{\Delta}$ as the codomain restriction of $\dot{\omega}\circ\curlyvee$ to $\boldsymbol{wCL}(\mathscr{M})$, and $\ddot{\nabla}$ as the domain restriction of $\curlywedge\circ\ddot{\omega}$ to $\boldsymbol{iCL}(\mathscr{M})$.

(1) Set $\epsilon := \dot{\nabla}\circ\ddot{\Delta}$.

(2) Set $\dot{\upsilon} := \dot{\epsilon}\circ\dot{\Delta}$ and $\ddot{\upsilon} := \ddot{\Delta}\circ\ddot{\epsilon}$.

2.12 Proposition. (0) $\dot{\Delta}_{*}\circ\dot{\Delta}^{*} \cong id_{\boldsymbol{wCL}(\mathscr{M})}$ *and* $\ddot{\nabla}^{*}\circ\ddot{\nabla}_{*} \cong id_{\boldsymbol{iCL}(\mathscr{M})}$.

(1) $\ddot{\epsilon}\circ\dot{\epsilon}$ *is an essentially canonical factorization of* ϵ.

(2) $\ddot{\upsilon}\circ\dot{\upsilon}$ *is an essentially canonical factorization of the polarity* υ *induced by restricting* $\perp$ *to* $\mathscr{M}$.

Proof. (0) For a weakly hereditary closure operator F we have

$$\Delta^{*}\left(\dot{\Delta}_{*}\dot{\Delta}^{*}(F)\right) = \dot{\Delta}^{*}\dot{\Delta}_{*}\dot{\Delta}^{*}(F) = \dot{\Delta}^{*}(F) = \Delta^{*}(F)\ .$$

By Lemma 1.09(4) this implies that $F \cong \dot{\Delta}_{*}\dot{\Delta}^{*}(F)$. The argument for $\ddot{\nabla}$ is similar.

(1) and (2) follow from Proposition 1.08 from Lemma 1.09(5), respectively. □

This completes the construction of the commutative Diagram (2-00).

3. EXAMPLES

We start with some examples that illustrate Diagram (2-00).

3.00 Example. Let $\mathscr{M}$ be the class of embeddings in $\mathscr{X} = \boldsymbol{Top}$. If S is the Sierpinski space on the set $2 = \{0,1\}$ with $\{0\}$ open, let r_i be the embedding of the subspace $\{i\}$ into S, for $i \in 2$.

To describe the weakly hereditary closure operators $\dot{\Delta}_{*}\{r_i\} = \dot{\omega}_{*}\curlyvee_{*}\{r_i\}$ we introduce a relation R on the underlying set of a topological space X: $\langle x,y\rangle \in R$ iff $g(0) = x$ and $g(1) = y$ for some continuous $S \xrightarrow{g} X$, i.e., iff every open set about y contains x, or in other words, iff y belongs to the usual topological closure $\overline{\{x\}}$ of $\{x\}$.

For a subspace $\langle M,m\rangle$ of X, an embedding $S \xrightarrow{k} Y$, and $k\circ r_0 \xrightarrow{\langle f,h\rangle} m$, the W-direct image of $\langle r_0,k\rangle$ along $\langle f,h\rangle$ coincides with the W-direct image of $\langle r_0, id_S\rangle$ along $\langle f, h\circ k\rangle$.

Hence the $\dot{\Delta}_*\{r_0\}$-closure $\langle M^0, m^0\rangle$ of $\langle M, m\rangle$ is determined by the union of all W-direct images of $\langle r_0, id_S\rangle$ along some $\mathscr{M}$-morphism $r_0 \xrightarrow{\langle f,g\rangle} m$, i.e.,

$$M^0 = \bigcup\Big\{\, \overline{\{x\}} \mid x \in M \,\Big\} = \{\, y \in X \mid \exists_{x\in X} \ \langle x, y\rangle \in R \text{ and } x \in M \,\}$$

is precisely the direct image of M under R (cf. Proposition 1.11(1)). Similarly, $\dot{\Delta}_*\{r_1\}$ produces

$$M^1 = \Big\{\, x \in X \mid \overline{\{x\}} \cap M \neq \emptyset \,\Big\} = \{\, x \in X \mid \exists_{y\in X} \ \langle x, y\rangle \in R \text{ and } y \in M \,\}$$

which is the direct image of M under the opposite relation R^{op}. Since R is transitive, both closure operators are easily seen to be idempotent, and hence are determined by their respective classes of closed $\mathscr{M}$-elements.

Next we consider the idempotent closure operators $\ddot{\nabla}^*\{r_i\} = \ddot{\omega}^*\curlywedge^*\{r_i\}$. The $\ddot{\nabla}^*\{r_0\}$-closure of a subspace $\langle M, m\rangle$ is given by the intersection of all open subspaces (in the ordinary sense) containing M, while its $\ddot{\nabla}^*\{r_1\}$-closure is given by the intersection of all closed subspaces (in the ordinary sense) containing M, i.e., it is the usual topological closure. Clearly, both of these closure operators are weakly hereditary, and hence are determined by their respective classes of dense $\mathscr{M}$-elements. □

3.01 Example. With the hypotheses of the preceding example consider the topological space T on $3 = \{0, 1, 2\}$ with $\{0\}$ and $\{2\}$ open, and the inclusion pairs $\{0\} \xrightarrow{u} \{0, 2\} \xrightarrow{v} T$ and $\{0\} \xrightarrow{r_0} S \xrightarrow{s} T$. A similar analysis as in the preceding example yields that the $\dot{\omega}_*\{\langle u, v\rangle\}$-closure of a subspace $\langle M, m\rangle$ of X is the inclusion into X of the subspace

$$\{\, z \in X \mid \exists_{y\in X} \ \exists_{x\in M} \ \langle x, y\rangle, \langle z, y\rangle \in R \,\} = \{\, z \in X \mid \exists_{y\in M^0} \ \langle z, y\rangle \in R \,\} = (M^0)^1$$

which is the direct image of M under the product relation $R \circ R^{\mathrm{op}}$. Hence $\dot{\omega}_*\{\langle u, v\rangle\} = \dot{\Delta}_*\{r_1\} \circ \dot{\Delta}_*\{r_0\}$. This closure operator is neither idempotent nor weakly hereditary, i.e., it cannot be recovered from its collection of closed $\mathscr{M}$-elements, nor from its collection of dense $\mathscr{M}$-elements. $\dot{\omega}_*\{\langle r_0, s\rangle\}$ is just $\dot{\Delta}_*\{r_0\}$.

On the other hand, the $\ddot{\omega}^*\{\langle r_0, s\rangle\}$-closure of $\langle M, m\rangle$ in X is the intersection of all closed subspaces N (in the ordinary sense) such that M is contained in the interior of N. This closure operator is weakly hereditary, but not idempotent. So the collection of dense $\mathscr{M}$-elements does characterize it. $\ddot{\omega}^*\{\langle u, v\rangle\}$ coincides with $\ddot{\nabla}^*\{r_0\}$. □

3.02 Example. For $\mathscr{X} = \boldsymbol{Pos}$ and $\mathscr{M} = \boldsymbol{Mono}(\boldsymbol{Pos})$ let $\mathbf{1} \xrightarrow{f} \mathbf{2}$ and $\mathbf{1} \xrightarrow{t} \mathbf{2}$ be the embeddings of the singleton poset into the two-element chain $\{0, 1\} = 2$ with $0 < 1$ that take values 0 and 1, respectively. Then $\dot{\Delta}_*\{t\}$ as well as $\ddot{\nabla}^*\{f\}$ map a subset M of a poset X to its **down-closure**, the smallest down-closed set that contains M, while both $\dot{\Delta}_*\{f\}$ and $\ddot{\nabla}^*\{t\}$ map M to its **up-closure**, the smallest up-closed set that contains M. Clearly, both of these closure operators are weakly hereditary and idempotent. □

3.03 Example. In the same setting as above consider the inclusion u of $\{1\}$ into $\mathbf{3}$, the three-element chain $\{0,1,2\} = 3$ with $0 < 1 < 2$. Then $\dot{\Delta}_*\{u\}$ maps a subset M of a poset X to the union of the down-closure of M with the up-closure of M. This closure operator is weakly hereditary but not idempotent. On the other hand, $\ddot{\nabla}_*\{u\}$ maps M to the intersection of the down-closure of M with the up-closure of M. This is the "interval hull" of M, i.e., the smallest subset that for any $a, b \in M$ contains each $c \in X$ with $a \leq c \leq b$. Clearly, $\ddot{\nabla}_*\{u\}$ is both idempotent and weakly hereditary. □

In the next three examples we take a closer look at the Salbany closure operator $\kappa^*(\mathscr{C})$ induced by a subclass $\mathscr{C}$ of $\boldsymbol{Ob}(\mathscr{X})$ (cf. [15]). We have to require that $\mathscr{X}$ has equalizers and that $\mathscr{M}$ contains all regular monomorphisms, or equivalently, that $\boldsymbol{E}$ consists of epi-sinks. The closure of $m \in \mathscr{M}$ then is the intersection of all equalizers of those parallel morphisms $\langle r, s\rangle$ from the codomain of m to some object in $\mathscr{C}$ that satisfy $r \circ m = s \circ m$. Salbany closure operators are always idempotent, and hence are determined by their collections of closed $\mathscr{M}$-elements.

3.04 Example. If X is a topological space, the *b-closure* of a subspace $\langle M, m\rangle$ consists of all those points $x \in X$ with the property that $M \cap \overline{\{x\}} \cap N$ is not empty, for each neighborhood N of x. This yields a closure operator (in the sense of Definition 1.02) on the class $\mathscr{M}$ of topological embeddings. In fact, this is the Salbany closure operator $\kappa^*(\boldsymbol{Top}_0)$ induced by the class of T_0-spaces. This closure operator was shown to be weakly hereditary in [5], Example 3.01, hence it can be recovered from the class of b-dense embeddings (cf. Proposition 2.12). □

3.05 Example. If $\mathscr{C}$ is a class of Hausdorff spaces that contains a space with at least two points, then the Salbany closure operator $\kappa^*(\mathscr{C})$ is not weakly hereditary:

Consider the discrete space $\boldsymbol{N}$ of non-negative integers, and form a new space X by adding two new points a, b to $\boldsymbol{N}$ as follows.

For $c \in \{a, b\}$ the basic neighborhoods are the upper segments of $\boldsymbol{N}$ together with c. The embedding of $\boldsymbol{N}$ into X viewed as a sequence then converges to a and to b, and the subspace $\{a, b\}$ is discrete. Sequences in Hausdorff spaces have unique limits, so any two continuous functions from X into a Hausdorff space that agree on a must agree on b as well. Since $\mathscr{C}$ contains a space with at least two points, it follows that no elements of $\boldsymbol{N}$ belong to the $\kappa^*(\mathscr{C})$-closure of $\{a\}$ in X, which therefore must be $\{a, b\}$. But the $\kappa^*(\mathscr{C})$-closure of $\{a\}$ in $\{a, b\}$ is $\{a\}$. Hence the inclusion of $\{a\}$ into $\{a, b\}$ is not $\kappa^*(\mathscr{C})$-dense.

Consequently $\kappa^*(\mathscr{C})$ cannot be recovered from the class of $\kappa^*(\mathscr{C})$- dense embeddings, even though in the case that $\mathscr{C} = \boldsymbol{Top}_2$ the restriction of $\kappa^*(\mathscr{C})$ to $\mathscr{C}$ is the ordinary topological closure operator. □

Clementino in [7] characterizes a class $\mathscr{C} \subseteq \boldsymbol{Ob}(\boldsymbol{Top})$ with $\kappa^*(\mathscr{C})$ weakly hereditary as a disconnectedness, i.e., for some class $\mathscr{D}$ of spaces $Y \in \mathscr{C}$ iff every continuous function from

$X \in \mathscr{D}$ to Y is constant. This allows her to establish $\kappa^*(\boldsymbol{Top}_1)$ to be weakly hereditary. On the other hand, the preceeding example shows that $\boldsymbol{Top}_2$ is not a disconnectedness.

3.06 EXAMPLE. If X is a topological space, the Θ-*closure* of a subspace $\langle M, m\rangle$ consists of all those points $x \in X$ with the property that $M \cap \overline{N}$ is not empty, for each neighborhood N of x. This closure operator is known to be neither idempotent nor weakly hereditary, cf. Example 1.5.(b) of [8]. A space where this can easily be seen is the Simplified Arens Square, Counterexample 81 in [16]. If m is the embedding of $(0, \frac{1}{2}) \times (0, \frac{1}{2})$ into the square, then m^{Θ} is not a Θ-closed embedding. Furthermore, the embedding n of $(0, \frac{1}{2}] \times (0, \frac{1}{2}]$ into the square has the property that n_{Θ} is not Θ-dense. Therefore neither Θ-closed embeddings nor Θ-dense embeddings suffice to characterize the Θ-closure. □

$\kappa^*(\boldsymbol{Top}_{2.5})$ is known to coincide with the idempotent hull of the Θ-closure on $\boldsymbol{Top}_{2.5}$, the class of Urysohn spaces. By Example 3.05 and Proposition 1.08 neither of these operators can be weakly hereditary. However, we do not know whether they agree on all of $\boldsymbol{Top}$.

Salbany closure operators are also useful in algebraic contexts.

3.07 EXAMPLE. Let $\langle \mathscr{T}, \mathscr{F}\rangle$ be a torsion theory in the category R-$\boldsymbol{Mod}$ of left R-modules over a fixed ring R with unity, i.e., $\mathscr{T}$ and $\mathscr{F}$ are corresponding fixed points of the polarity induced by the relation on $\boldsymbol{Ob}(\mathscr{X})$ that contains all pairs $\langle X, Y\rangle$ with the property that $\boldsymbol{hom}\langle X, Y\rangle$ is a singleton set. Let $\mathscr{M}$ consist of all monomorphisms. Then $\kappa^*(\mathscr{F})$ maps a monomorphism $M \xrightarrow{m} X$ to the smallest submodule $N \xrightarrow{n} X$ that contains M and satisfies $X/N \in \mathscr{F}$. Such Salbany closure operators are always weakly hereditary, i.e., characterized by the $\mathscr{F}$-dense submodules as well as by the $\mathscr{F}$-closed submodules. A submodule M is $\mathscr{F}$-dense (resp. $\mathscr{F}$-closed) in X iff $X/M \in \mathscr{T}$ (resp. $X/M \in \mathscr{F}$).

Important special cases for $R = \boldsymbol{Z}$, i.e., R-$\boldsymbol{Mod} = \boldsymbol{Ab}$, are the following: if $\mathscr{F}$ consists of all torsion-free (resp. reduced) abelian groups, then $\mathscr{T}$ is the class of all torsion (resp. divisible) abelian groups. □

However, closure operators in algebraic contexts need not be weakly hereditary.

3.08 EXAMPLE. Let $\mathscr{X}$ be $\boldsymbol{Grp}$, the category of groups, and let $\mathscr{M}$ be the class of all monomorphisms. Mapping a subgroup M of a group X to the smallest normal subgroup containing M yields an idempotent closure operator that is not weakly hereditary (because the normal subgroup relation is not transitive). This operator is easily seen to be equal to $\ddot{\nabla}^*(\mathscr{A})$, where $\mathscr{A}$ consists of all embeddings of normal subgroups. Although the $\ddot{\nabla}^*(\mathscr{A})$-separated objects are precisely the abelian groups (cf. [6]), the Salbany closure operator $\kappa^*(\boldsymbol{Ab})$ differs from $\ddot{\nabla}^*(\mathscr{A})$: it maps M to the smallest normal subgroup N that contains M and for which X/N is abelian. Notice that $\kappa^*(\boldsymbol{Ab})$ is not weakly hereditary either. □

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