DISCONNECTEDNESS CLASSES

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ABSTRACT: Let $\mathcal{X}$ be an $(\mathbf{E},\mathcal{M})$-category for sinks. A notion of disconnectedness with respect to a closure operator C on $\mathcal{X}$ and to a class of $\mathcal{X}$-monomorphisms $\mathcal{N}$ is introduced. This gives rise to the notion of $\mathcal{N}$-disconnectedness class, a characterization of which is presented in a category with a terminal object. Some examples are provided.

KEY WORDS: Closure operator, disconnectedness, Galois connection, constant morphism.

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0 INTRODUCTION

The development of a general theory about topological connectedness and disconnectedness was started by Preuß (cf. [25]) and by Herrlich ([19]). Further literature on this topic can be found in [2-3], [13], [21], [23], [26-28] and [30-31].

Let $\mathcal{X}$ be an arbitrary category with an $(\mathbf{E},\mathcal{M})$-factorization structure for sinks and let $\mathcal{N}\subseteq\mathcal{M}$. An $\mathcal{X}$-morphism $X\xrightarrow{f}Y$ is called $\mathcal{N}$-dependent if for any $\mathcal{N}$-subobject n of X and any $\mathcal{N}$-subobject p of Y, $n_f\le p$ implies $f^{-1}(p)\simeq id_X$ (where n_f is the direct image of n along f and $f^{-1}(p)$ is the pullback of p along f [see §1]). Let $S(\mathcal{X})$ denote the collection of all subclasses of objects of $\mathcal{X}$, ordered by inclusion. For every $\mathcal{N}\subseteq\mathcal{M}$, the relation: $X\mathcal{R}_{\mathcal{N}}Y$ if and only if every $\mathcal{X}$-morphism $X\xrightarrow{f}Y$ is $\mathcal{N}$-dependent yields a Galois connection $S(\mathcal{X})\underset{\nabla'_{\mathcal{N}}}{\overset{\Delta'_{\mathcal{N}}}{\rightleftarrows}}S(\mathcal{X})^{\mathbf{op}}$. It was proved in [8] that this Galois connection factors through $CL(\mathcal{X},\mathcal{M})$, i.e., the collection of all closure operators on $\mathcal{X}$ with respect to $\mathcal{M}$, via two previously introduced Galois connections $S(\mathcal{X})\underset{I_{\mathcal{N}}}{\overset{J_{\mathcal{N}}}{\rightleftarrows}}CL(\mathcal{X},\mathcal{M})$ and $CL(\mathcal{X},\mathcal{M})\underset{T_{\mathcal{N}}}{\overset{D_{\mathcal{N}}}{\rightleftarrows}}S(\mathcal{X})^{\mathbf{op}}$.

The above factorization is used to introduce the concept of $\mathcal{N}$-disconnectedness class and the one of $\mathcal{N}$-connectedness hull of a subcategory $\mathcal{A}$ of $\mathcal{X}$. A characterization of this last notion is presented in section 2, under the assumption of $\mathcal{N}$ being closed under the formation of direct images.

However, the real purpose of the paper appears in section 3, where the assumption of the existence of a terminal object in $\mathcal{X}$ is added. This allowed us to obtain a characterization of $\mathcal{N}$-disconnectedness classes that in the category of topological spaces yields as a special case the one given by Arhangel'skii and Wiegandt, [2, Theorem 2.12]. Moreover, in the category of abelian groups, this yields the classical characterization of the torsion free part of a torsion theory. It is also worth to mention that in the process of obtaining the above characterization, we also identify those **E**-reflective subcategories in which the fibers of any reflection morphism have their reflections isomorphic to the terminal object (cf. Lemma 3.12).

The paper ends with some examples that illustrate the Galois conection $S(\mathcal{X}) \underset{\nabla'_{\mathcal{N}}}{\overset{\Delta'_{\mathcal{N}}}{\rightleftarrows}} S(\mathcal{X})^{\mathbf{op}}$ in familiar categories.

We use the terminology of [1] throughout the paper[1].

1 PRELIMINARIES

Throughout we consider a category $\mathcal{X}$ and a fixed class $\mathcal{M}$ of $\mathcal{X}$-monomorphisms, which contains all $\mathcal{X}$-isomorphisms. It is assumed that $\mathcal{X}$ is $\mathcal{M}$-complete; i.e.,

(1) $\mathcal{M}$ is closed under composition

(2) Pullbacks of $\mathcal{M}$-morphisms exist and belong to $\mathcal{M}$, and multiple pullbacks of (possibly large) families of $\mathcal{M}$-morphisms with common codomain exist and belong to $\mathcal{M}$.

One of the consequences of the above assumptions is that there is a uniquely determined class **E** of sinks in $\mathcal{X}$ such that $\mathcal{X}$ is an $(\mathbf{E}, \mathcal{M})$-*category for sinks*. This implies the following features of $\mathcal{M}$ and **E** (cf. [1] for the dual case):

PROPOSITION 1.1

(1) Every isomorphism is in both $\mathcal{M}$ and **E** *(as a singleton sink).*

(2) $\mathcal{M}$ is closed under $\mathcal{M}$-relative first factors, i.e., if $n \circ m \in \mathcal{M}$, and $n \in \mathcal{M}$, then $m \in \mathcal{M}$.

(3) $\mathcal{M}$ is closed under composition.

(4) Pullbacks of $\mathcal{X}$-morphisms in $\mathcal{M}$ exist and belong to $\mathcal{M}$.

(5) The $\mathcal{M}$-subobjects of every $\mathcal{X}$-object form a (possibly large) complete lattice; suprema are formed via $(\mathbf{E}, \mathcal{M})$-factorizations and infima are formed via intersections. □

If $X \xrightarrow{f} Y$ is an $\mathcal{X}$-morphism and $M \xrightarrow{m} X$ is an $\mathcal{M}$-subobject, then $M \xrightarrow{e_{f \circ m}} M_f \xrightarrow{m_f} Y$ will denote the $(\mathbf{E}, \mathcal{M})$-factorization of $f \circ m$. $M_f \xrightarrow{m_f} Y$ will be called the direct image of m along f and $M \xrightarrow{e_{f \circ m}} M_f$ will be called the restriction of the morphism f to the $\mathcal{M}$-subobject m. If $N \xrightarrow{n} Y$ is an $\mathcal{M}$-subobject, then the pullback $f^{-1}(N) \xrightarrow{f^{-1}(n)} X$ of n along f will be called the inverse image of n along f. Whenever no confusion is likely to arise, to simplify the notation we will denote the morphism $e_{f \circ m}$ simply e_f.

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[1] Paul Taylor's Commutative Diagrams in TEX macro package was used to typeset most of the diagrams in this paper.

DEFINITION 1.2

A *closure operator* C on $\mathcal{X}$ (with respect to $\mathcal{M}$) is a family $\{(\ )^{C}_{X}\}_{X\in\mathcal{X}}$ of functions on the $\mathcal{M}$-subobject lattices of $\mathcal{X}$ with the following properties that hold for each $X \in \mathcal{X}$:

(a) [*expansiveness*] $m \le (m)^{C}_{X}$, for every $\mathcal{M}$-subobject $M \xrightarrow{m} X$;

(b) [*order-preservation*] $m \le n \Rightarrow (m)^{C}_{X} \le (n)^{C}_{X}$ for every pair of $\mathcal{M}$-subobjects of X;

(c) [*morphism-consistency*] If p is the pullback of the $\mathcal{M}$-subobject $M \xrightarrow{m} Y$ along some $\mathcal{X}$-morphism $X \xrightarrow{f} Y$ and q is the pullback of $(m)^{C}_{Y}$ along f, then $(p)^{C}_{X} \le q$, i.e., the closure of the inverse image of m is less than or equal to the inverse image of the closure of m.

Condition (a) implies that for every closure operator C on $\mathcal{X}$, every $\mathcal{M}$-subobject $M \xrightarrow{m} X$ has a canonical factorization

$$\begin{array}{ccc} M & \xrightarrow{t} & (M)^{C}_{X} \\ & {\scriptstyle m}\searrow & \big\downarrow {\scriptstyle (m)^{C}_{X}} \\ & & X \end{array}$$

where $((M)^{C}_{X}, (m)^{C}_{X})$ is called the C-*closure* of the subobject (M, m).

When no confusion is likely we will write m^{C} rather than $(m)^{C}_{X}$ and for notational symmetry we will denote the morphism t by m_{C}.

REMARK 1.3

Notice that in the above definition, under condition (b), the morphism-consistency condition (c) is equivalent to the following statement concerning direct images: if $M \xrightarrow{m} X$ is an $\mathcal{M}$-subobject and $X \xrightarrow{f} Y$ is a morphism, then $((m)^{C}_{Y})_f \le (m_f)^{C}_{Y}$, i.e., the direct image of the closure of m is less than or equal to the closure of the direct image of m; (cf. [15]).

DEFINITION 1.4

Given a closure operator C, we say that $m \in \mathcal{M}$ is C-*closed* if m_{C} is an isomorphism. An $\mathcal{X}$-morphism f is called C-*dense* if for every $(\mathbf{E}, \mathcal{M})$-factorization (e, m) of f we have that m^{C} is an isomorphism. We call C *idempotent* provided that m^{C} is C-closed for every $m \in \mathcal{M}$. C is called *weakly hereditary* if m_{C} is C-dense for every $m \in \mathcal{M}$.

Notice that Definition 1.2(c) implies that pullbacks of C-closed $\mathcal{M}$-subobjects are C-closed.

A special case of an idempotent closure operator arises in the following way. Given any class $\mathcal{A}$ of $\mathcal{X}$-objects and $M \xrightarrow{m} X$ in $\mathcal{M}$, define $m^{\mathcal{A}}$ to be the intersection of all equalizers of pairs of $\mathcal{X}$-morphisms r, s from X to some $\mathcal{A}$-object A that satisfy $r \circ m = s \circ m$, and let $m_{\mathcal{A}} \in \mathcal{M}$ be the unique $\mathcal{X}$-morphism by which m factors through $m^{\mathcal{A}}$. It is easy to see that this gives rise to an idempotent closure operator that we will denote by $S_{\mathcal{A}}$. This generalizes the Salbany construction of closure operators induced by classes of topological spaces, cf. [29].

We denote the collection of all closure operators on $\mathcal{M}$ by $\mathbf{CL}(\mathcal{X}, \mathcal{M})$ pre-ordered as follows: $C \sqsubseteq D$ if $m^{C} \leq m^{D}$ for all $m \in \mathcal{M}$ (where $\leq$ is the usual order on subobjects). Notice that arbitrary suprema and infima exist in $\mathbf{CL}(\mathcal{X}, \mathcal{M})$, they are formed pointwise in the $\mathcal{M}$-subobject fibers.

For more background on closure operators see, e.g., [11-12] and [15-16]. For a recent survey on the same topic, one could check [9]. Detailed proofs can be found in [20] and [17].

DEFINITION 1.5

For pre-ordered classes $\mathcal{X} = (\mathbf{X}, \sqsubseteq)$ and $\mathcal{Y} = (\mathbf{Y}, \sqsubseteq)$, a *Galois connection* $\mathcal{X} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{Y}$ consists of order preserving functions F and G that satisfy $F \dashv G$, i.e., $x \sqsubseteq GF(x)$ for every $x \in \mathbf{X}$ and $FG(y) \sqsubseteq y$ for every $y \in \mathbf{Y}$. (G is adjoint and has F as coadjoint).

If $x \in \mathcal{X}$ and $y \in \mathcal{Y}$ are such that $F(x) = y$ and $G(y) = x$, then x and y are said to be corresponding fixed points of the Galois connection $(\mathcal{X}, F, G, \mathcal{Y})$.

Properties and many examples of Galois connections can be found in [18].

2 GENERAL RESULTS ABOUT C-DISCONNECTEDNESS

The aim of this section is to introduce in the category $\mathcal{X}$ a general notion of disconnectedness that depends on a given closure operator and on a chosen class of $\mathcal{M}$-subobjects. To this purpose we need to recall some results that appeared in previous papers.

Throughout the paper we will assume that $\mathcal{X}$ is an $(\mathbf{E}, \mathcal{M})$-category for sinks.

Unless otherwise specified, C will always denote a closure operator on $\mathcal{X}$ with respect to the given class $\mathcal{M}$ of $\mathcal{X}$-monomorphisms and $\mathcal{N}$ will be a subclass of $\mathcal{M}$. For $X \in \mathcal{X}$, $\mathcal{N}_X$ will denote all $\mathcal{N}$-subobjects with codomain X.

We begin by recalling the following two propositions from [4].

PROPOSITION 2.1

Let $CL(\mathcal{X}, \mathcal{M}) \xrightarrow{D_{\mathcal{N}}} S(\mathcal{X})^{\mathbf{op}}$ *and* $S(\mathcal{X})^{\mathbf{op}} \xrightarrow{T_{\mathcal{N}}} CL(\mathcal{X}, \mathcal{M})$ *be defined by:*

$D_{\mathcal{N}}(C) = \{X \in \mathcal{X} : \text{ every } n \in \mathcal{N}_X \text{ is } C\text{-closed}\}$

$T_{\mathcal{N}}(\mathcal{A}) = Sup\{C \in CL(\mathcal{X}, \mathcal{M}) : D_{\mathcal{N}}(C) \supseteq \mathcal{A}\}$.

Then, $CL(\mathcal{X}, \mathcal{M}) \underset{T_{\mathcal{N}}}{\overset{D_{\mathcal{N}}}{\rightleftarrows}} S(\mathcal{X})^{\mathbf{op}}$ *is a Galois connection.* □

PROPOSITION 2.2

Let $CL(\mathcal{X},\mathcal{M}) \xrightarrow{I_{\mathcal{N}}} S(\mathcal{X})$ *and* $S(\mathcal{X}) \xrightarrow{J_{\mathcal{N}}} CL(\mathcal{X},\mathcal{M})$ *be defined by:*

$I_{\mathcal{N}}(C) = \{X \in \mathcal{X} : \text{ every } n \in \mathcal{N}_X \text{ is } C\text{-dense}\}$

$J_{\mathcal{N}}(\mathcal{B}) = Inf\{C \in CL(\mathcal{X},\mathcal{M}) : I_{\mathcal{N}}(C) \supseteq \mathcal{B}\}.$

Then, $S(\mathcal{X}) \underset{I_{\mathcal{N}}}{\overset{J_{\mathcal{N}}}{\rightleftarrows}} CL(\mathcal{X},\mathcal{M})$ *is a Galois connection.* □

In [4] we also presented some characterizations of the functions $T_{\mathcal{N}}$ and $J_{\mathcal{N}}$. For reference purposes we collect them under the following

PROPOSITION 2.3

For every $\mathcal{A} \in S(\mathcal{X})^{\mathbf{op}}$ *and* $\mathcal{M}$*-subobject* $M \xrightarrow{m} X$*, with* $X \in \mathcal{X}$*, we have that*

$$m^{T_{\mathcal{N}}(\mathcal{A})} = \cap\{f^{-1}(n) : Y \in \mathcal{A}, X \xrightarrow{f} Y, N \xrightarrow{n} Y \in \mathcal{N}_Y \text{and } m \leq f^{-1}(n)\}.$$

Moreover, for every $\mathcal{B} \in S(\mathcal{X})$ *and* $\mathcal{M}$*-subobject* $M \xrightarrow{m} Y$*, with* $Y \in \mathcal{X}$*, we have that*

$$m^{J_{\mathcal{N}}(\mathcal{B})} = sup\left(\{m\} \cup \{(id_X)_f : X \in \mathcal{B}, X \xrightarrow{f} Y \text{ and } \exists n \in \mathcal{N}_X \text{ with } n_f \leq m\}\right).$$

□

DEFINITION 2.4

A morphism $X \xrightarrow{f} Y$ is $\mathcal{N}$-dependent if for every $n \in \mathcal{N}_X$ and every $p \in \mathcal{N}_Y$, $n_f \leq p$ implies $f^{-1}(p) \simeq id_X$.

Clearly, the above definition yields a Galois connection $S(\mathcal{X}) \underset{\nabla'_{\mathcal{N}}}{\overset{\Delta'_{\mathcal{N}}}{\rightleftarrows}} S(\mathcal{X})^{\mathbf{op}}$ where for $\mathcal{A} \in S(\mathcal{X})$, $\Delta'_{\mathcal{N}}(\mathcal{A}) = \{Y \in \mathcal{X} : \forall X \in \mathcal{A}, X \xrightarrow{f} Y$ is $\mathcal{N}$-dependent$\}$ and for $\mathcal{B} \in S(\mathcal{X})^{\mathbf{op}}$, $\nabla'_{\mathcal{N}}(\mathcal{B}) = \{X \in \mathcal{X} : \forall Y \in \mathcal{B}, X \xrightarrow{f} Y$ is $\mathcal{N}$-dependent$\}$.

In [8] we proved the following:

THEOREM 2.5

Let $\mathcal{N}$ *be a subclass of* $\mathcal{M}$*. Then the Galois connection* $S(\mathcal{X}) \underset{\nabla'_{\mathcal{N}}}{\overset{\Delta'_{\mathcal{N}}}{\rightleftarrows}} S(\mathcal{X})^{\mathbf{op}}$ *factors through* $CL(\mathcal{X},\mathcal{M})$ *via the Galois connections* $S(\mathcal{X}) \underset{I_{\mathcal{N}}}{\overset{J_{\mathcal{N}}}{\rightleftarrows}} CL(\mathcal{X},\mathcal{M})$ *and* $CL(\mathcal{X},\mathcal{M}) \underset{T_{\mathcal{N}}}{\overset{D_{\mathcal{N}}}{\rightleftarrows}} S(\mathcal{X})^{\mathbf{op}}$*.* □

Now we are ready to give the following:

DEFINITION 2.6

An $\mathcal{X}$-object X is called $(C,\mathcal{N})$-disconnected if $X \in D_{\mathcal{N}}(J_{\mathcal{N}}(I_{\mathcal{N}}(C)))$.

As a consequence of Theorem 2.5, we obtain the following alternative description of the notion of $(C,\mathcal{N})$-disconnectedness.

PROPOSITION 2.7

An $\mathcal{X}$-object X is $(C,\mathcal{N})$-disconnected if every morphism $A \xrightarrow{f} X$ with $A \in I_{\mathcal{N}}(C)$ is $\mathcal{N}$-dependent; i.e., for every $n \in \mathcal{N}_A$ and every $p \in \mathcal{N}_X$, $n_f \leq p$ implies $f^{-1}(p) \simeq id_A$. □

DEFINITION 2.8

(a) Let $\mathcal{A} \in S(\mathcal{X})^{\mathbf{op}}$. $\mathcal{A}$ is said to be a *disconnectedness class* if there is a subclass of morphisms $\mathcal{N} \subseteq \mathcal{M}$ and a closure operator C such that $\mathcal{A} = D_{\mathcal{N}}(J_{\mathcal{N}}(I_{\mathcal{N}}(C)))$.

(b) Let $\mathcal{A} \in S(\mathcal{X})^{\mathbf{op}}$ and $\mathcal{N} \subseteq \mathcal{M}$. $\mathcal{A}$ is said to be an *$\mathcal{N}$-disconnectedness class* if there is a closure operator C such that $\mathcal{A} = D_{\mathcal{N}}(J_{\mathcal{N}}(I_{\mathcal{N}}(C)))$.

REMARK 2.9

Notice that if $\mathcal{A} = D_{\mathcal{N}}(J_{\mathcal{N}}(I_{\mathcal{N}}(C)))$, then from the properties of Galois connections we have that $\mathcal{A} = D_{\mathcal{N}}(J_{\mathcal{N}}(I_{\mathcal{N}}(T_{\mathcal{N}}(\mathcal{A}))))$. Consequently, part (b) of Definition 2.8 can be also restated as follows: $\mathcal{A}$ is an $\mathcal{N}$-disconnectedness class if and only if $\mathcal{A} = D_{\mathcal{N}}(J_{\mathcal{N}}(I_{\mathcal{N}}(T_{\mathcal{N}}(\mathcal{A}))))$.

PROPOSITION 2.10

Let $\mathcal{N}$ be closed under the formation of direct images and let X and Y be two $\mathcal{X}$-objects.

(a) $X \in I_{\mathcal{N}}(T_{\mathcal{N}}(\mathcal{B}))$ if and only if for every $n \in \mathcal{N}_X$ and $X \xrightarrow{f} B$, $B \in \mathcal{B}$ we have that $f^{-1}(n_f) \simeq id_X$.

(b) $Y \in D_{\mathcal{N}}(J_{\mathcal{N}}(\mathcal{A}))$ if and only if for every $n \in \mathcal{N}_X$, $X \xrightarrow{f} Y$ with $X \in \mathcal{A}$ we have that $f^{-1}(n_f) \simeq id_X$.

Proof:

(a). If $X \in I_{\mathcal{N}}(T_{\mathcal{N}}(\mathcal{B}))$ then for every $n \in \mathcal{N}_X$, $n^{T_{\mathcal{N}}(\mathcal{B})} \simeq id_X$, that is from Proposition 2.3, $\cap\{f^{-1}(p) : B \in \mathcal{B}, X \xrightarrow{f} B, p \in \mathcal{N}_B \text{ and } n \leq f^{-1}(p)\} \simeq id_X$. This implies that for every $X \xrightarrow{f} B$, $B \in \mathcal{B}$ and $p \in \mathcal{N}_B$ such that $n \leq f^{-1}(p)$ we have that $f^{-1}(p) \simeq id_X$. Since $n \leq f^{-1}(n_f)$ and $n_f \in \mathcal{N}$ by hypothesis, for $p = n_f$ we obtain that $f^{-1}(n_f) \simeq id_X$.

Conversely, suppose that for every $X \xrightarrow{f} B$, $B \in \mathcal{B}$ and $n \in \mathcal{N}_X$, $f^{-1}(n_f) \simeq id_X$. Now let $p \in \mathcal{N}_B$ be such that $n \leq f^{-1}(p)$. Then $n_f \leq (f^{-1}(p))_f \leq p$ implies $id_X \simeq f^{-1}(n_f) \leq f^{-1}(p)$. Hence $f^{-1}(p) \simeq id_X$ and consequently $n^{T_{\mathcal{N}}(\mathcal{B})} \simeq id_X$. Thus, $X \in I_{\mathcal{N}}(T_{\mathcal{N}}(\mathcal{B}))$.

(b). Consider $X \in \mathcal{A}$, $Y \in D_{\mathcal{N}}(J_{\mathcal{N}}(\mathcal{A}))$, $n \in \mathcal{N}_X$ and $X \xrightarrow{f} Y$. By our hypothesis on $\mathcal{N}$, $n_f \in \mathcal{N}_Y$ and so $(n_f)^{J_{\mathcal{N}}(\mathcal{A})} \simeq n_f$. Notice that X occurs in the construction of $(n_f)^{J_{\mathcal{N}}(\mathcal{A})}$ (cf. Proposition 2.3) and so $(id_X)_f \leq n_f$, which implies $f^{-1}(n_f) \simeq id_X$.

Viceversa, let $p \in \mathcal{N}_Y$ and $Y \in \mathcal{X}$. If $X \in \mathcal{A}$ and $n \in \mathcal{N}_X$ is such that $n_f \leq p$, then $id_X \simeq f^{-1}(n_f) \leq f^{-1}(p)$. So, $(id_X)_f \leq (f^{-1}(p))_f \leq p$. Therefore, $p^{J_{\mathcal{N}}(\mathcal{A})} \simeq p$ and consequently $Y \in D_{\mathcal{N}}(J_{\mathcal{N}}(\mathcal{A}))$. □

DEFINITION 2.11

For every $\mathcal{A} \in S(\mathcal{X})^{\mathbf{op}}$ and $\mathcal{N} \subseteq \mathcal{M}$, $D_{\mathcal{N}}(J_{\mathcal{N}}(I_{\mathcal{N}}(T_{\mathcal{N}}(\mathcal{A}))))$ is called the $\mathcal{N}$-*disconnectedness hull* of $\mathcal{A}$.

We conclude this section with the following:

THEOREM 2.12

Let $\mathcal{N}$ be closed under the formation of direct images and let $\mathcal{A}$ be a subcategory of $\mathcal{X}$. Then, an $\mathcal{X}$-object Y belongs to the $\mathcal{N}$-disconnectedness hull of $\mathcal{A}$ if and only if for every morphism $X \xrightarrow{f} Y$, if the total source $(X \xrightarrow{f_i} A_i)_{i\in I}$ satisfies $f_i^{-1}(n_{f_i}) \simeq id_X$ for every $n \in \mathcal{N}_X$, then so does f.

Proof:

($\Leftarrow$). Consider the morphism $X \xrightarrow{f} Y$. If the total source $(X \xrightarrow{f_i} A_i)_{i\in I}$ satisfies $f_i^{-1}(n_{f_i}) \simeq id_X$ for every $n \in \mathcal{N}_X$, then from Proposition 2.10(a), $X \in I_{\mathcal{N}}(T_{\mathcal{N}}(\mathcal{A}))$. Thus, any morphism $X \xrightarrow{f} Y$ with $X \in I_{\mathcal{N}}(T_{\mathcal{N}}(\mathcal{A}))$ satisfies $f^{-1}(n_f) \simeq id_X$ for every $n \in \mathcal{N}_X$. Hence from Proposition 2.10(b) we have that $Y \in D_{\mathcal{N}}(J_{\mathcal{N}}(I_{\mathcal{N}}(T_{\mathcal{N}}(\mathcal{A}))))$.

($\Rightarrow$). If $Y \in D_{\mathcal{N}}(J_{\mathcal{N}}(I_{\mathcal{N}}(T_{\mathcal{N}}(\mathcal{A}))))$, then from Proposition 2.10(b), every morphism $X \xrightarrow{f} Y$ with $X \in I_{\mathcal{N}}(T_{\mathcal{N}}(\mathcal{A}))$ satisfies $f^{-1}(n_f) \simeq id_X$ for every $n \in \mathcal{N}_X$. Clearly $X \in I_{\mathcal{N}}(T_{\mathcal{N}}(\mathcal{A}))$ is equivalent to the total source $(X \xrightarrow{f_i} A_i)_{i\in I}$ satisfying the above condition. □

3 DISCONNECTEDNESS IN CATEGORIES WITH A TERMINAL OBJECT

The aim of this section, which is also the main purpose of the paper, is to provide a characterization of disconnectedness classes in categories with a terminal object.

So, from now on we assume that the category $\mathcal{X}$ has a terminal object T.

DEFINITION 3.1

An $\mathcal{X}$-object X is called *empty* (*non-empty*) if it does not have (it has) T as a subobject. An $\mathcal{X}$-object that is either empty or isomorphic to T is called *trivial*, otherwise it is called *non-trivial*.

DEFINITION 3.2

We say that *terminal objects detect monomorphisms* if whenever a morphism $X \xrightarrow{f} Y$ satisfies the condition that $f^{-1}(T)$ is trivial, for every morphism $T \xrightarrow{t_Y} Y$, then f is a monomorphism.

Now we make the following

ASSUMPTIONS 3.3

(a) If X is a trivial object, then any morphism with domain X belongs to $\mathcal{M}$;

(b) any morphism with domain a terminal object T belongs to $\mathcal{N}$;

(c) whenever $T \xrightarrow{t} X$ and $M \xrightarrow{m} X$ are monomorphisms such that $m \le t$, then M non-empty implies $m \simeq t$;

(d) $\mathcal{N}$ is a class of non-empty $\mathcal{M}$-subobjects;

(e) terminal objects detect monomorphisms.

REMARK 3.4

(1) Using Assumption 3.3(c), it is easy to see that if $M \xrightarrow{m} X$ is a monomorphism, then $m^{-1}(T)$ is trivial for every morphism $T \xrightarrow{t} X$.

(2) It is important to observe that if X is empty, then as a consequence of Assumption 3.3(d) it cannot have any $\mathcal{N}$-subobject.

The following result has a crucial importance for the rest of the paper. However, since a more general version of it was proved in [8], we omit its proof.

LEMMA 3.5 (cf. [8, Lemma 3.4])

Let $X \xrightarrow{f} Y$ be a morphism with X non-empty. Then, the following are equivalent:

(a) f is $\mathcal{N}$-dependent;

(b) f factors through T. □

PROPOSITION 3.6

Every $\mathcal{N}$-disconnectedness class contains all trivial objects.

Proof:

We recall from Proposition 2.7, that $\mathcal{N}$-disconnectedness classes can be described via $\mathcal{N}$-dependent morphisms.

If $X \simeq T$, then clearly any morphism $Y \xrightarrow{f} X$ with $Y \in I_{\mathcal{N}}(C)$ factors through T, so we can apply Lemma 3.5. If X is empty then from Remark 3.4 X does not have any $\mathcal{N}$-subobject. Consequently any morphism $Y \xrightarrow{f} X$ is $\mathcal{N}$-dependent. □

REMARK 3.7

If $\mathcal{A}$ is an $\mathcal{N}$-disconnectedness class then, from Remark 2.9, $\mathcal{A} = D_{\mathcal{N}}(J_{\mathcal{N}}(I_{\mathcal{N}}(T_{\mathcal{N}}(\mathcal{A}))))$. Thus, from [8, Lemma 3.4], [5, Proposition 2.10] and [8, Proposition 3.11] we obtain that if $\mathcal{X}$ is **E**-cowell powered with products, then $\mathcal{A}$ is an **E**-reflective subcategory (cf. [1, Theorem 16.8]). The next step is to identify those **E**-reflective subcategories that can be seen as $\mathcal{N}$-disconnectednesses. This is taken care by the following results.

We recall the next two definitions from [8].

DEFINITION 3.8

A non-empty family $(M_i \xrightarrow{m_i} X)_{i\in I}$ of $\mathcal{M}$-subobjects of X is said to be disjoint if $\cap M_i$ is empty or $|I| = 1$.

DEFINITION 3.9

(a) We say that a non-empty disjoint family $(M_i \xrightarrow{m_i} X)_{i\in I}$ of non-empty $\mathcal{M}$-subobjects of X has a *strong* **E**-*quotient* if there is an **E**-morphism $X \xrightarrow{q} Q$ such that:

i) $q \circ m_i$ factors through T for every $i \in I$;

ii) for every morphism $T \xrightarrow{t_Q} Q$ we have that either $q^{-1}(t_Q) \simeq id_T$ or there is an element $i_0 \in I$ such that $m_{i_0} = q^{-1}(t_Q)$;

iii) for any **E**-morphism $X \xrightarrow{g} Y$ such that $g \circ m_i$ factors through T for every $i \in I$, there exists a morphism $Q \xrightarrow{h} Y$ such that $h \circ q = g$.

(b) An $\mathcal{X}$-morphism $X \xrightarrow{q} Q$ is called a *strong* **E**-*quotient* if there is a non-empty disjoint family $(M_i \xrightarrow{m_i} X)_{i\in I}$ of non-empty $\mathcal{M}$-subobjects of X, that has q as a strong **E**-quotient.

(c) We say that $\mathcal{X}$ has strong **E**-quotients if for any $X \in \mathcal{X}$, any non-empty disjoint family of non-empty $\mathcal{M}$-subobjects $(M_i \xrightarrow{m_i} X)_{i\in I}$ has a strong **E**-quotient.

Some remarks about the concept of strong **E**-quotient can be found in [8]. Here we just observe that if **E** is a class of episinks, then any two strong **E**-quotients with respect to the same family of $\mathcal{M}$-subobjects must be isomorphic. Therefore in this case, up to isomorphism, one can speak of "the" strong **E**-quotient of that family.

The proofs of the following two lemmas follow from some straightforward categorical arguments, so we omit them.

LEMMA 3.10

If $M_i \xrightarrow{m_i} M$ is a family of monomorphisms and $M \xrightarrow{m} X$ is a monomorphism, then $\cap(m \circ m_i) \simeq m \circ (\cap m_i)$. □

LEMMA 3.11

If pullbacks of $\mathbf{E}$*-morphisms along* $\mathcal{M}$*-subobjects belong to* $\mathbf{E}$*, then if* $X \xrightarrow{e} Y$ *belongs to* $\mathbf{E}$ *and* $M \xrightarrow{m} Y$ *belongs to* $\mathcal{M}$*, we have that* $(e^{-1}(m))_e \simeq m$*.* □

LEMMA 3.12

Assume that $\mathcal{X}$ *has strong* $\mathbf{E}$*-quotients and let* $\mathcal{A}$ *be an* $\mathbf{E}$*-reflective subcategory of* $\mathcal{X}$ *that contains all trivial objects. For an* $\mathcal{X}$*-object* X *let* $X \xrightarrow{r_X} rX$ *be its* $\mathbf{E}$*-reflection in* $\mathcal{A}$*. Let us consider the following statements:*

(a) For every morphism $T \xrightarrow{t} rX$*,* $r(r_X^{-1}(T)) \simeq T$*;*

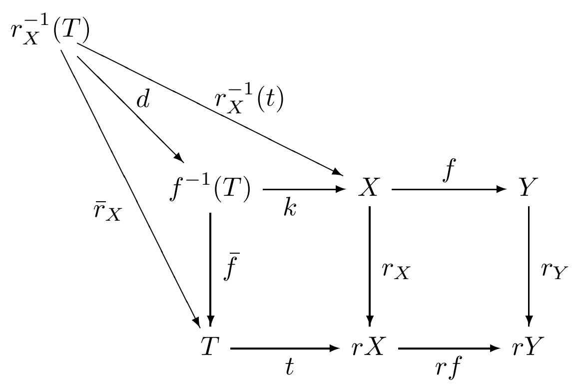
(b) for every $\mathcal{X}$*-morphism* $X \xrightarrow{f} Y$ *with* $Y \in \mathcal{A}$ *that satisfies* $f^{-1}(T) \in \mathcal{A}$ *for every morphism* $T \xrightarrow{t_Y} Y$*, we have that* $X \in \mathcal{A}$*.*

Then, if $\mathcal{A}$ *is closed under monomorphisms, (a)* $\Rightarrow$ *(b).*

Conversely, if we assume that there is a closure operator C *such that terminal objects are* C*-closed in* $\mathcal{A}$*-objects and that the restriction of any strong* $\mathbf{E}$*-quotient* q *to a* C*-closed subobject that is a pullback along* q*, is a strong* $\mathbf{E}$*-quotient. Moreover, if* $\mathbf{E}$ *consists of episinks and pullbacks of* $\mathbf{E}$*-morphisms along* $\mathcal{M}$*-subobjects belong to* $\mathbf{E}$*, then (b)* $\Rightarrow$ *(a).*

Proof:

(a) $\Rightarrow$ (b). Let $X \xrightarrow{f} Y$ be an **X**-morphism with $Y \in \mathcal{A}$. Assume that for every morphism $T \xrightarrow{t_Y} Y$, $f^{-1}(T) \in \mathcal{A}$. Consider the following commutative diagram:



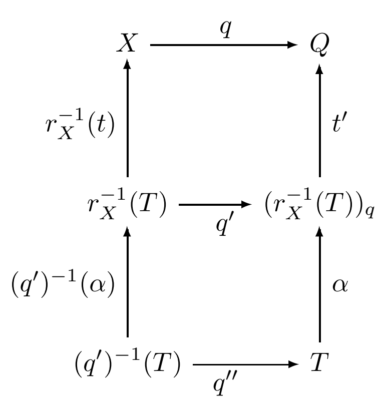
where k stands for $f^{-1}(r_Y^{-1} \circ rf \circ t)$ and d is the morphism induced by the universal property of pullbacks. By assumption, $f^{-1}(T) \in \mathcal{A}$, and since $d \in \mathcal{M}$ (cf. Proposition 1.1(2)), the closure of $\mathcal{A}$ under $\mathcal{M}$-subobjects implies that $r_X^{-1}(T) \in \mathcal{A}$. Thus we have that $r(r_X^{-1}(T)) \simeq r_X^{-1}(T)$. By assumption, $T \simeq r(r_X^{-1}(T))$ and so $r_X^{-1}(T) \simeq T$. Since by assumption, terminal objects detect monomorphisms, we conclude that r_X is a monomorphism. Finally, the closure of $\mathcal{A}$ under monomorphisms implies that $X \in \mathcal{A}$.

(b) $\Rightarrow$ (a). Let $X \in \mathcal{X}$. If rX is empty, then condition (a) is true by default. So, let rX be non-empty and let $T \xrightarrow{t} rX$ be a morphism. Consider its pullback along r_X, $r_X^{-1}(T) \xrightarrow{r_X^{-1}(t)} X$. We need to show that $r(r_X^{-1}(T)) \simeq T$.

For every morphism $T_i \xrightarrow{d_i} r(r_X^{-1}(T))$, $i \in I$, where T_i is a terminal object, consider the pullback $e^{-1}(T_i) \xrightarrow{e^{-1}(d_i)} r_X^{-1}(T)$ where, to simplify the notation we have set $e = r_{r_X^{-1}(T)}$. Consider the disjoint family of $\mathcal{M}$-subobjects of X that consists of all $\mathcal{M}$-subobjects of the form $r_X^{-1}(T_j) \xrightarrow{r_X^{-1}(t_j)} X$ which are pullbacks of the non-isomorphic $\mathcal{M}$-subobjects $T_j \xrightarrow{t_j} rX$, $j \in J$, $t_j \neq t$, together with the family $e^{-1}(T_i) \xrightarrow{r_X^{-1}(t) \circ e^{-1}(d_i)} X$, for all non-isomorphic $\mathcal{M}$-subobjects $T_i \xrightarrow{d_i} r(r_X^{-1}(T))$.

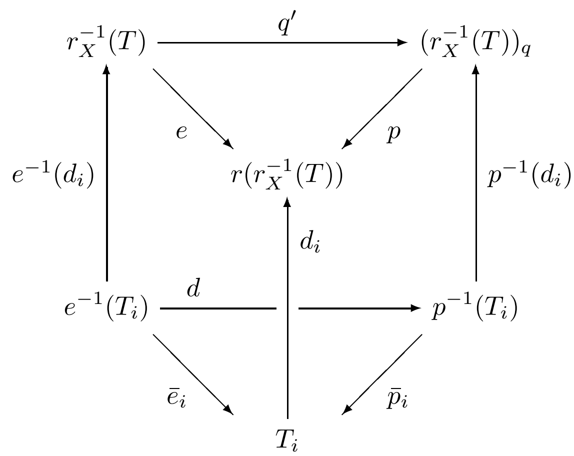
Notice that $\mathcal{M}$-subobjects of the form $r_X^{-1}(T_j) \xrightarrow{r_X^{-1}(t_j)} X$ are disjoint by construction and for the disjointness of the family $(r_X^{-1}(t) \circ e^{-1}(d_i))_{i \in I}$ one can use Lemma 3.10. Moreover, from Assumption 3.3(a) and the fact that pullbacks of **E**-morphisms along $\mathcal{M}$-subobjects belong to **E**, we can conclude that $r_X^{-1}(T)$ is non-empty and so is $r(r_X^{-1}(T))$. Consequently, the above family is non-empty.

Thus we can build the strong **E**-quotient of the above family, say $X \xrightarrow{q} Q$. Clearly, $r_X \circ r_X^{-1}(t_j)$ factors through the terminal object for every $j \in J$ and so does $r_X \circ r_X^{-1}(t) \circ e^{-1}(d_i)$, for every $i \in I$. Since $r_X \in \mathbf{E}$, from the universal property of strong **E**-quotients we obtain a morphism $Q \xrightarrow{h} rX$ such that $h \circ q = r_X$. Now, let us consider the direct image of $r_X^{-1}(T)$ along q, that is the $(\mathbf{E}, \mathcal{M})$-factorization $q \circ r_X^{-1}(t) = t' \circ q'$ with $r_X^{-1}(T) \xrightarrow{q'} (r_X^{-1}(T))_q \in \mathbf{E}$ and $(r_X^{-1}(T))_q \xrightarrow{t'} Q \in \mathcal{M}$. By our assumptions, $r_X^{-1}(T)$ is C-closed (as a pullback of a C-closed $\mathcal{M}$-subobject) and since $r_X^{-1}(T) = (h \circ q)^{-1}(T) = q^{-1}(h^{-1}(T))$, we have that q' is a strong **E**-quotient as a restriction of the strong **E**-quotient q to a C-closed $\mathcal{M}$-subobject that is a pullback along q. Next we identify the family of $\mathcal{M}$-subobjects with respect to which q' is a strong **E**-quotient. For every morphism $T \xrightarrow{\alpha} (r_X^{-1}(T))_q$ consider the commutative diagram:



The universal property of pullbacks implies that $r_X^{-1}(t) \circ (q')^{-1}(\alpha) \leq q^{-1}(t' \circ \alpha)$. Since all the $r_X^{-1}(t_j)$ are disjoint and $r_X^{-1}(t) \circ (q')^{-1}(\alpha)$ is a subobject of $r_X^{-1}(t)$, condition ii) of Definition 3.9(a) implies that either $q^{-1}(t' \circ \alpha) \simeq id_T$ or $q^{-1}(t' \circ \alpha) = r_X^{-1}(t) \circ e^{-1}(d_i)$ for some d_i. In the first case, from Assumption 3.3(c), we have that $r_X^{-1}(t) \circ (q')^{-1}(\alpha) \simeq id_T$ and consequently, so does $(q')^{-1}(\alpha)$. In the latter case we have that $q^{-1}(t' \circ \alpha) \cap r_X^{-1}(t) = (r_X^{-1}(t) \circ e^{-1}(d_i)) \cap r_X^{-1}(t) = r_X^{-1}(t) \circ e^{-1}(d_i)$. From Lemma 2.20 of [6] we have that $q^{-1}(t' \circ \alpha) \cap r_X^{-1}(t) \simeq r_X^{-1}(t) \circ (q')^{-1}(\alpha)$. Thus, $r_X^{-1}(t) \circ e^{-1}(d_i) = r_X^{-1}(t) \circ (q')^{-1}(\alpha)$ and since $r_X^{-1}(t)$ is a monomorphism, we conclude that $(q')^{-1}(\alpha) = e^{-1}(d_i)$. Moreover, for every $i \in I$, consider the $(\mathbf{E}, \mathcal{M})$-factorization (e'_i, m'_i) of $q' \circ e^{-1}(d_i)$. Then, $(e'_i, t' \circ m'_i)$ is the $(\mathbf{E}, \mathcal{M})$-factorization of $q \circ r_X^{-1}(t) \circ e^{-1}(d_i)$ and so it factors through the terminal object. Hence, from Assumption 3.3(c), so does $q' \circ e^{-1}(d_i)$. Since $r_X^{-1}(T) \xrightarrow{q'} (r_X^{-1}(T))_q$ is a strong **E**-quotient, we conclude that q' is the strong **E**-quotient of the family $\{e^{-1}(d_i)\}_{i \in I}$. Consequently, from the universal property of strong **E**-quotients there is a morphism $(r_X^{-1}(T))_q \xrightarrow{p} r(r_X^{-1}(T))$ such that $p \circ q' = e$.

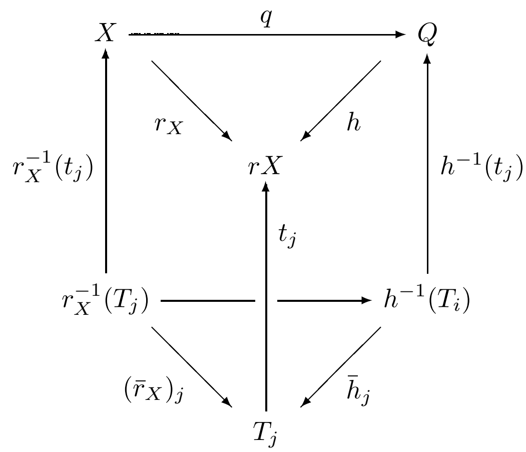
Next we show that the morphism p satisfies the requirements of property (b). For every morphism $T_i \xrightarrow{d_i} r(r_X^{-1}(T))$ consider the following commutative diagram:



where d is the unique morphism induced by the universal property of pullbacks.

From a classical property of pullback squares (cf. [1, Proposition 11.10(2)]), since the left and right squares are pullbacks, so is the outer one. Thus, $e^{-1}(T_i) \simeq (q')^{-1}(p^{-1}(T_i))$. Now, since $q' \circ e^{-1}(d_i)$ factors through the terminal object, we have that $T_i \simeq (e^{-1}(T_i))_{q'} \simeq ((q')^{-1}(p^{-1}(T_i)))_{q'} \simeq p^{-1}(T_i)$. This last isomorphism follows from Lemma 3.11. Thus, $p^{-1}(T_i) \simeq T_i \in \mathcal{A}$ and so p satisfies property (b). Consequently $(r_X^{-1}(T))_q \in \mathcal{A}$.

Consider the morphism $T_j \xrightarrow{t_j} rX$. From the following commutative diagram:



we have that $q^{-1}(h^{-1}(T_j)) = (h \circ q)^{-1}(T_j) = r_X^{-1}(T_j)$. By applying q and from Lemma 3.11 we obtain that $h^{-1}(T_j) \simeq (r_X^{-1}(T_j))_q$. Hence, for $t_j \neq t$, $(r_X^{-1}(T_j))_q \simeq T_j \in \mathcal{A}$ and for $t_j = t$, $(r_X^{-1}(T_j))_q \simeq (r_X^{-1}(T))_q \in \mathcal{A}$. Thus, from (b) $Q \in \mathcal{A}$.

As a consequence, there exists a morphism $rX \xrightarrow{k} Q$ such that $k \circ r_X = q$. This together with $h \circ q = r_X$ yields $k \circ h \circ q = q = id_Q \circ q$. Since $q \in \mathbf{E}$ and $\mathbf{E}$ consists of episinks, we obtain that $k \circ h = id_Q$. Thus h is an epimorphism (as second factor of the epimorphism $r_X = h \circ q$) and a section and so an isomorphism. So, by definition of q, $q^{-1}((r_X^{-1}(t) \circ e^{-1}(d_i))_q) \simeq r_X^{-1}(t) \circ e^{-1}(d_i)$, for every $i \in I$. However, since h is an isomorphism, we have that $q^{-1}((r_X^{-1}(t) \circ e^{-1}(d_i))_q) \simeq r_X^{-1}((r_X^{-1}(t) \circ e^{-1}(d_i))_{r_X}) \simeq r_X^{-1}(t) = r_X^{-1}(t) \circ id_{r_X^{-1}(T)}$. Hence $e^{-1}(d_i) \simeq id_{r_X^{-1}(T)}$ for every $i \in I$, since $r_X^{-1}(t)$ is a monomorphism. Finally, this implies that $r(r_X^{-1}(T)) \simeq T$. □

THEOREM 3.13

Suppose that $\mathcal{X}$ has strong $\mathbf{E}$-quotients and let $\mathcal{A}$ be an $\mathbf{E}$-reflective subcategory of $\mathcal{X}$ that contains all trivial objects. Assume that there is a closure operator C such that terminal objects are C-closed in $\mathcal{A}$-objects and that the restriction of any strong $\mathbf{E}$-quotient q to a C-closed subobject that is a pullback along q is a strong $\mathbf{E}$-quotient. Moreover, $\mathbf{E}$ consists of episinks and pullbacks of $\mathbf{E}$-morphisms along $\mathcal{M}$-subobjects belong to $\mathbf{E}$. Then the following are equivalent:

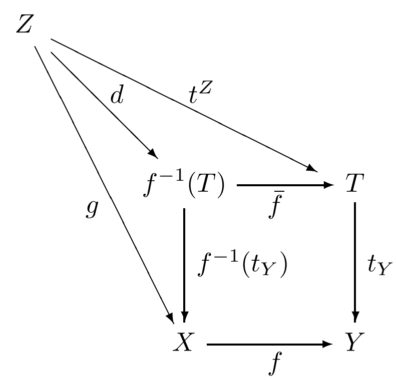
(a) $\mathcal{A}$ is an $\mathcal{N}$-disconnectedness class;

(b) for every $\mathcal{X}$-morphism $X \xrightarrow{f} Y$ with $Y \in \mathcal{A}$ that satisfies the condition that $f^{-1}(T) \in \mathcal{A}$, for every morphism $T \xrightarrow{t_Y} Y$, we have that $X \in \mathcal{A}$.

Proof:

(a) $\Rightarrow$ (b). Let $X \xrightarrow{f} Y$ be an $\mathcal{X}$-morphism with $Y \in \mathcal{A}$ such that for every morphism $T \xrightarrow{t_Y} Y$, $f^{-1}(T) \in \mathcal{A}$. The fact that $\mathcal{A}$ is an $\mathcal{N}$-disconnectedness class implies that $\mathcal{A} = D_{\mathcal{N}}(J_{\mathcal{N}}(I_{\mathcal{N}}(T_{\mathcal{N}}(\mathcal{A})))) = \Delta'_{\mathcal{N}}(I_{\mathcal{N}}(T_{\mathcal{N}}(\mathcal{A}))) = \Delta'_{\mathcal{N}}(\mathcal{B})$ with $\mathcal{B} = I_{\mathcal{N}}(T_{\mathcal{N}}(\mathcal{A}))$, (cf. Remark 2.9 and

Theorem 2.5). We need to show that $X \in \mathcal{A} = \Delta'_{\mathcal{N}}(\mathcal{B})$. Consider a morphism $Z \xrightarrow{g} X$ with $Z \in \mathcal{B}$. If Z is empty, then g is $\mathcal{N}$-dependent by default and so $X \in \Delta'_{\mathcal{N}}(\mathcal{B}) = \mathcal{A}$. Now, let Z be non-empty. Clearly, since $Y \in \mathcal{A} = \Delta'_{\mathcal{N}}(\mathcal{B})$ and $Z \in \mathcal{B}$, from Lemma 3.5 $f \circ g$ factors through T, i.e., $f \circ g = t_Y \circ t^Z$. Let us consider the following commutative diagram:



where d is the morphism induced by the universal property of pullbacks. Since $f^{-1}(T) \in \mathcal{A}$, then d factors through T and consequently so does g. Thus, from Lemma 3.5, $X \in \Delta'_{\mathcal{N}}(\mathcal{B}) = \mathcal{A}$.

(b) $\Rightarrow$ (a). Clearly $\mathcal{A} \subseteq D_{\mathcal{N}}(J_{\mathcal{N}}(I_{\mathcal{N}}(T_{\mathcal{N}}(\mathcal{A}))))$, so we just need to show that the other inclusion holds. Let $X \in D_{\mathcal{N}}(J_{\mathcal{N}}(I_{\mathcal{N}}(T_{\mathcal{N}}(\mathcal{A}))))$ and let $X \xrightarrow{r_X} rX$ be its **E**-reflection in $\mathcal{A}$. For any morphism $T \xrightarrow{t} rX$ consider the pullback $r_X^{-1}(T) \xrightarrow{r_X^{-1}(t)} X$. Notice that from Assumption 3.3(a) and from our assumptions on **E** we have that $r_X^{-1}(T)$ is non-empty. Consider the **E**-reflection $r(r_X^{-1}(T))$ of $r_X^{-1}(T)$ into $\mathcal{A}$. From Lemma 3.12 we have that $r(r_X^{-1}(T)) \simeq T$. Consequently, any morphism $r_X^{-1}(T) \xrightarrow{h} A$ with $A \in \mathcal{A}$ factors through T. So, $r_X^{-1}(T) \in I_{\mathcal{N}}(T_{\mathcal{N}}(\mathcal{A}))$. Consequently, $r_X^{-1}(t)$ factors through T. Thus, $r_X^{-1}(t) \leq t_X$ for a morphism $T \xrightarrow{t_X} X$. Now, since $r_X^{-1}(T)$ is non-empty, from condition 3.3(c) we obtain that $r_X^{-1}(T) \simeq T \in \mathcal{A}$. Hence the morphism r_X satisfies the condition in (b) and so we conclude that $X \in \mathcal{A}$. □

REMARK 3.14

We would like to observe that the hypotheses of Theorem 3.13 are not as strong as they may first appear. For instance, if in **Ab** we consider the (episinks,monomorphism)-factorization structure, since in this case, as it is easily seen, the strong **E**-quotients are exactly the surjective homomorphisms, we have that the restriction condition on **E**-quotients in the hypotheses of Theorem 3.13 is satisfied by any closure operator on **Ab**. Moreover, for any non-empty subclass $\mathcal{A} \subseteq$ **Ab**, the terminal object $\{0\}$ is $T_{\mathcal{N}}(\mathcal{A})$-closed. Consequently, since condition (b) is equivalent to the closure under group extensions, the above theorem yields the classical characterization of the torsion free part of any torsion theory in **Ab**.

In the category **Top** with the (episink,embedding)-factorization structure, since as observed in [8, Remark 3.15(c)], the strong **E**-quotients are precisely the topological quotients, the Kuratowski closure K certainly satisfies the hypotheses of Theorem 3.13 for any class of topological spaces $\mathcal{A} \subseteq \mathbf{Top_1}$. Therefore, in this case we obtain as a special case the characterization of topological disconnectedness given by Arhangel'skii and Wiegandt [2, Theorem 2.12].

Please, notice that although the above theorem seems to only characterize those disconnectedness classes contained in the category $\mathbf{Top_1}$, Arhangel'skii and Wiegandt [2, Proposition 2.10] proved that the only non-trivial disconnectedness not contained in $\mathbf{Top_1}$ is $\mathbf{Top_0}$.

4 EXAMPLES

In what follows, for the category **Top** of topological spaces we will choose as $\mathcal{M}$ the class of all extremal monomorphisms (embeddings). We recall that if **E** is the class of episinks in **Top**, then **Top** is an $(\mathbf{E}, \mathcal{M})$-category. For the category **Grp** of groups and **Ab** of abelian groups we will use the (episink,monomorphism)-factorization structure. Full details about the following examples can be found in [10] and [5-8].

EXAMPLE 4.1 (cf. [10])

Let $\mathcal{X}$ be the category **Top** and let $\mathcal{N}$ be the class of all nonempty embeddings. Notice that since $\mathcal{N}$ contains all singleton monomorphisms (i.e., morphisms with singleton domain), to say that a morphism $X \xrightarrow{f} Y$ is $\mathcal{N}$-dependent simply means that $f(X)$ is a singleton.

(a). If C is the closure operator induced by the topology, then the class $I_{\mathcal{N}}(C)$ agrees with the class **Ind** of all indiscrete topological spaces. For every $\mathcal{M}$-subobject $M \xrightarrow{m} X$, $M^{J_{\mathcal{N}}(\mathbf{Ind})}$ is the union of M with all indiscrete subobjects of X which intersect M and $(D_{\mathcal{N}} \circ J_{\mathcal{N}})(\mathbf{Ind}) = \mathbf{Top_0}$. Thus, $\mathbf{Top_0}$ is the class of $(C, \mathcal{N})$-disconnected topological spaces. As a matter of fact **Ind** and $\mathbf{Top_0}$ are corresponding fixed points of the Galois connection $(\Delta'_{\mathcal{N}}, \nabla'_{\mathcal{N}})$ of Theorem 2.5 (cf. [2]).

(b). Consider the class **Absconn** of all absolutely connected topological spaces. We recall that a topological space X is absolutely connected if it cannot be decomposed into any disjoint family $\mathcal{L}$ of nonempty closed subsets with $|\mathcal{L}| > 1$ (cf. [25]). It is well known that **Absconn** and $\mathbf{Top_1}$ are corresponding fixed points of the Galois connection $(\Delta'_{\mathcal{N}}, \nabla'_{\mathcal{N}})$ of Theorem 2.5 (cf. [2]). Consequently, $\mathbf{Absconn} = I_{\mathcal{N}}(T_{\mathcal{N}}(\mathbf{Top_1}))$ and $\mathbf{Top_1} = D_{\mathcal{N}}(J_{\mathcal{N}}(\mathbf{Absconn}))$. It was proved in [10, Example 4.3] that $T_{\mathcal{N}}(\mathbf{Top_1})$ agrees with the regular closure operator induced by $\mathbf{Top_1}$. Therefore $\mathbf{Top_1}$ is the class of $(T_{\mathcal{N}}(\mathbf{Top_1}), \mathcal{N})$-disconnected topological spaces.

(c). Consider the closure operator C that to each $\mathcal{M}$-subobject $M \xrightarrow{m} X$, associates the union of M with all connected subsets of X which intersect M. $I_{\mathcal{N}}(C)$ is the class **Conn** of all connected topological spaces and consequently $(D_{\mathcal{N}} \circ J_{\mathcal{N}})(I_{\mathcal{N}}(C))$ is the class **TDisc** of all totally disconnected topological spaces. Clearly, connected and totally disconnected topological spaces are corresponding fixed points of the Galois connection $(\Delta'_{\mathcal{N}}, \nabla'_{\mathcal{N}})$ of Theorem 2.5.

EXAMPLE 4.2

Let $\mathcal{X}$ be the category **Grp** and let $\mathcal{N} = \mathcal{M}$ be the class of all monomorphisms in **Grp**. Clearly, to say that a **Grp**-morphism $X \xrightarrow{f} Y$ is $\mathcal{N}$-dependent simply means that the image of X under f is a singleton.

Let $\mathcal{A}$ be the subcategory **Ab** of abelian groups. We have that $T_{\mathcal{N}}(\mathbf{Ab}) \simeq S_{\mathbf{Ab}}$, where $S_{\mathbf{Ab}}$ is the regular closure operator induced by **Ab**, (cf. [10, Example 4.4]). $I_{\mathcal{N}}(S_{\mathbf{Ab}})$ agrees with the class of perfect groups, i.e., $X \in I_{\mathcal{N}}(S_{\mathbf{Ab}})$ iff $X = X'$, where X' denotes the subgroup generated by the commutators. Finally, $(D_{\mathcal{N}} \circ J_{\mathcal{N}})(I_{\mathcal{N}}(S_{\mathbf{Ab}}))$ is the class of all groups which do not have any non-trivial perfect subgroup.

EXAMPLE 4.3

Let $\mathcal{X}$ be the category **Ab** and let $\mathcal{N} = \mathcal{M}$ be the class of al monomorphisms in **Ab**. Let $(\mathcal{T}, \mathcal{F})$ be a torsion theory. Clearly, $\mathcal{T}$ and $\mathcal{F}$ are corresponding fixed points of the Galois connection $(\Delta'_{\mathcal{N}}, \nabla'_{\mathcal{N}})$ of Theorem 2.5. Let $X \in \mathbf{Ab}$ and let $X \xrightarrow{r_X} rX$ be its $\mathcal{F}$-reflection. For every subobject $M \xrightarrow{m} X$ consider the closure operator C defined by $M^{C} = M + Ker(r_X)$. In particular, if $(\mathcal{T}, \mathcal{F}) =$ (Torsion,Torsion-free), then we obtain the closure operator C_1 defined by $M^{C_1} = M + Tor(X)$, where $Tor(X)$ denotes the torsion subgroup of X. If $(\mathcal{T}, \mathcal{F}) =$ (Divisible,Reduced), then we obtain the closure operator C_2 defined by $M^{C_2} = M + Div(X)$, where $Div(X)$ denotes the largest divisible subgroup of X. Clearly, $I_{\mathcal{N}}(C_1)$ consists of all torsion abelian groups and $I_{\mathcal{N}}(C_2)$ consists of all divisible abelian groups. Consequently torsion free abelian groups form the $\mathcal{N}$-disconnectedness class of C_1 and reduced abelian groups form the $\mathcal{N}$-disconnectedness class of C_2.

Here we conclude with two examples that show that in those cases in which the above concept of $\mathcal{N}$-dependent does not agree with the classical notion of constant function, we obtain some new Galois correspondences and some unusual disconnectedness classes.

EXAMPLE 4.4

Consider the category **Top** of topological spaces with $\mathcal{M}$ consisting of all embeddings and $\mathcal{N}$ all nonempty clopen subsets. If $\mathcal{A}$ is the class **Discr** of discrete topological spaces, then for every $\mathcal{M}$-subobject of $X \in \mathbf{Top}$, $M^{T_{\mathcal{N}}(\mathcal{A})} = \cap\{f^{-1}(f(M)) : X \xrightarrow{f} Y, Y \text{ discrete }\}$. Consequently, $I_{\mathcal{N}}(T_{\mathcal{N}}(\mathcal{A})) = \{X : \text{ for every non-empty clopen subset } M \subseteq X, M^{T_{\mathcal{N}}(\mathcal{A})} = X\} = \{X : \text{ for every non-empty clopen } M \subseteq X, X \xrightarrow{f} Y \text{ and } Y \text{ discrete }, f(X) = f(M)\}$. It was shown in [8, Example 3.18] that $I_{\mathcal{N}}(T_{\mathcal{N}}(\mathcal{A}))$ is the class **Conn** of connected topological spaces. Clearly, $J_{\mathcal{N}}(\mathbf{Conn})$ is the discrete closure and $D_{\mathcal{N}}(J_{\mathcal{N}}(\mathbf{Conn})) = \mathbf{Top}$. Using the properties of Galois connections we obtain that $I_{\mathcal{N}}(T_{\mathcal{N}}(\mathbf{Top})) = \mathbf{Conn}$. This is clearly different from the classical correspondence between connected and totally disconnected topological spaces.

Moreover, we also observe that if $\mathcal{A}$ is a class of connected topological spaces, then $T_{\mathcal{N}}(\mathcal{A})$ is the indiscrete closure and clearly $I_{\mathcal{N}}(T_{\mathcal{N}}(\mathcal{A})) = \mathbf{Top}$. On the other hand, if $\mathcal{B}$=$\mathbf{Top}$, and N is a non-empty clopen subset of X, then using the morphism $X \xrightarrow{id_X} X$, from Proposition 2.3, we obtain that $N^{J_{\mathcal{N}}(\mathcal{B})} = X$. Consequently, $X \in D_{\mathcal{N}}(J_{\mathcal{N}}(\mathcal{B}))$ if and only if for every non-empty clopen subset N of X, $N = N^{J_{\mathcal{N}}(\mathcal{B})} = X$, i.e., $D_{\mathcal{N}}(J_{\mathcal{N}}(\mathcal{B}))$ consists of all connected topological spaces.

In conclusion, we obtain the pairs of fixed points: $(\mathbf{Conn}, \mathbf{Top})$ and $(\mathbf{Top}, \mathbf{Conn})$. In this last case **Conn** turns out to be a disconnectedness class.

EXAMPLE 4.5

Consider the category **Grp** of groups.

(a). Let $\mathcal{N}$ consist of all inclusions of normal subgroups. Notice that in this case, $X \xrightarrow{f} Y$ is $\mathcal{N}$-dependent if and only if f is constant in the classical sense.

Let **Sim** denote the subcategory of simple groups, i.e., all those groups that have no nontrivial normal subgroups. Consider the closure operator that to each subgroup of $M \leq X$ associates the intersection of all non-zero normal subgroups K of X such that $M \leq K$. Clearly we have that $I_{\mathcal{N}}(C) = \mathbf{Sim}$ and $D_{\mathcal{N}}(J_{\mathcal{N}}(\mathbf{Sim})) = \mathbf{Simfree}$, i.e., the subcategory of all groups that have no simple subgroup different from zero. Thus, **Simfree** is the $(C, \mathcal{N})$-disconnectedness class (cf. [6, Example 2.24]).

(b). Let $\mathcal{N}$ be the class of all normal subgroups different from zero. Notice that in this case, $\mathcal{N}$-dependent does not mean constant since any non-constant homomorphism with domain a simple group is $\mathcal{N}$-dependent.

If C is the indiscrete closure operator, then $I_{\mathcal{N}}(C) = \mathbf{Grp}$. Now let $\mathcal{B}$= **Grp** and let $Y \in D_{\mathcal{N}}(J_{\mathcal{N}}(\mathcal{B}))$. Consider a normal subgroup N of Y different from zero. Clearly using the morphism $Y \xrightarrow{id_Y} Y$, from Proposition 2.3 we obtain that $N^{J_{\mathcal{N}}(\mathcal{B})} = Y$. However, if $Y \in D_{\mathcal{N}}(J_{\mathcal{N}}(\mathcal{B}))$, then we must have that $N^{J_{\mathcal{N}}(\mathcal{B})} = N$. So, $N = Y$, that is Y is a simple group. Now, if Y is simple, then any $\mathcal{N}$-subobject of Y is $J_{\mathcal{N}}(\mathcal{B})$-closed by default, since the only normal subgroup of Y different from zero is Y itself. In conclusion, $D_{\mathcal{N}}(J_{\mathcal{N}}(\mathcal{B})) = \mathbf{Sim}$. Thus **Sim** is the $(C, \mathcal{N})$-disconnectedness class.

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