CLOSURE OPERATORS AND FUZZY CONNECTEDNESS

I. W. Alderton[1] G. Castellini[2]

ABSTRACT: A general notion of connectedness with respect to a closure operator on an arbitrary category $\mathcal{X}$ is used to produce some connectedness notions in the category of fuzzy topological spaces. All these notions turn out to be connectednesses in the sense of Preuß. Some already existing notions of connectedness in the category of fuzzy topological spaces are obtained as special cases of ours.

KEY WORDS: Connectedness, fuzzy topological space, closure operator, discrete object.

AMS CLASSIFICATION: 54A40, 18A20, 54B30, 18A30, 18A99.

0 INTRODUCTION

In **FTS** several connectedness concepts have been introduced (e.g., [12], [17], [19], [20], [24] and [25]). Using results obtained in [3-5], a general notion of connectedness with respect to a closure operator C on an $(\mathbf{E}, \mathcal{M})$-category $\mathcal{X}$ and to a subclass $\mathcal{N}$ of $\mathcal{M}$ was introduced in [6]. It was shown that most of the properties of topological connectedness can be generalized to this setting. Moreover, under certain mild assumptions on $\mathcal{X}$ and $\mathcal{N}$, this notion of connectedness can be described by means of constant morphisms.

In this paper we apply the above notion of connectedness to produce further connectedness notions in **FTS**. All these new notions are of Preuß type, i.e., they can be described by means of constant morphisms (cf. [21-23]). As a consequence of the general theory developed in [6], most of the properties of topological connectedness are satisfied with respect to the closure operator that induces a given connectedness. Some already existing connectedness notions in **FTS** are obtained as particular cases of our approach. In particular, **D**-connectedness defined in [19] arises from a rather natural closure operator.

We use the categorical terminology of [1] throughout the paper.

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1 PRELIMINARIES

As usual I denotes the closed unit interval $[0,1]$. If X is any set and $\alpha \in I$, then α will also be used to denote the constant function from X into I with value α. The notion of a fuzzy topological space used in this paper is the one due to Lowen [14]:

A pair (X,τ), where X is a set and τ is contained in I^X, is called a *fuzzy topological space* (and τ is called a *fuzzy topology* on X) if:

(i) $\alpha \in \tau$ for each $\alpha \in I$

(ii) $\{W_i : i \in I\} \subseteq \tau \Rightarrow \vee\{W_i : i \in I\} \in \tau$

(iii) $W_1, W_2 \in \tau \Rightarrow W_1 \wedge W_2 \in \tau$.

If $X \xrightarrow{f} Y$ is a function, and $X \xrightarrow{U} I$ and $Y \xrightarrow{V} I$ are fuzzy sets, then the fuzzy set $f^{-1}(V) : X \longrightarrow I$ is defined by $f^{-1}(V) = V \circ f$, and $f(U) : Y \longrightarrow I$ is defined as follows:

$$f(U)(y) = \begin{cases} sup\{U(x) : x \in f^{-1}(y)\}, & \text{if } f^{-1}(y) \neq \emptyset; \\ 0, & \text{if } f^{-1}(y) = \emptyset. \end{cases}$$

If (X,τ) and (Y,σ) are fuzzy topological spaces, then a function $(X,\tau) \xrightarrow{f} (Y,\sigma)$ is said to be fuzzy continuous provided that $f^{-1}(V) \in \tau$ whenever $V \in \sigma$. The notation **FTS** will denote the category of fuzzy topological spaces and fuzzy continuous functions.

The category **FTS** has initial structures [15]: if $\{(X_i,\tau_i)\}_I$ is a family of fuzzy topological spaces, and for each $i \in I$ we have a function $X \xrightarrow{f_i} X_i$, then the fuzzy topology τ on X which is initial with respect to $(X \xrightarrow{f_i} (X_i,\tau_i))_I$ has as subbasis $\{f_i^{-1}(U_i) : i \in I, U_i \in \tau_i\}$. (The notion of subbasis for a fuzzy topology is analogous to the corresponding notion in ordinary topology.)

If $\{(X_i,\tau_i)\}_I$ is a family of fuzzy topological spaces, then their product is the fuzzy topological space $\Pi(X_i,\tau_i) = (\Pi X_i, \tau)$, where ΠX_i is the ordinary cartesian product of the sets X_i, and τ is the initial fuzzy topology with respect to the family of projections $(\Pi X_i \xrightarrow{\pi_i} (X_i,\tau_i))_I$. Note that $\Pi(X_i,\tau_i)$ is actually the categorical product in **FTS**.

For more information about fuzzy sets and the category **FTS**, the reader could consult [18].

Given a subset M of $(X,\tau) \in$ **FTS**, the initial fuzzy topology on M with respect to the inclusion $M \xrightarrow{m} X$ will be called the relative fuzzy topology on M.

From now on, $\mathcal{M}$ will denote the class of all morphisms $(M,\sigma) \xrightarrow{m} (X,\tau)$ in **FTS** where $M \xrightarrow{m} X$ is an injective function and σ is the relative fuzzy topology induced by m. Notice that since there is a bijective correspondence between subsets of X and subobjects of (X,τ) with the relative fuzzy topology, we will often make no distiction between the subset M and the subobject (M,σ), where σ is the relative fuzzy topology.

We recall that a sink $((X_i,\tau_i) \xrightarrow{f_i} (Y_i,\sigma_i))_I$ is an epi-sink if $\cup\{f_i(X_i)\} = Y$. Let **E** denote the class of all epi-sinks in **FTS**. It is easy to verify that **FTS** is an $(\mathbf{E},\mathcal{M})$-category for sinks.

DEFINITION 1.1

A closure operator C on **FTS** is a family $\{(\ )^{C}_{(X,\tau)}\}_{(X,\tau)\in\mathbf{FTS}}$ of functions on the $\mathcal{M}$-subobject lattices with the following properties which hold for every $(X,\tau)\in\mathbf{FTS}$:

(a) $M\subseteq M^{C}_{(X,\tau)}$ for every subobject M of (X,τ)

(b) $M\subseteq N\Rightarrow M^{C}_{(X,\tau)}\subseteq N^{C}_{(X,\tau)}$ for every pair of subobjects M,N of (X,τ)

(c) For every subobject N of (Y,σ) and **FTS**-morphism $(X,\tau)\xrightarrow{f}(Y,\sigma)$, $(f^{-1}(N))^{C}_{(X,\tau)}\subseteq f^{-1}(N^{C}_{(Y,\sigma)})$.

We say that the subobject M of (X,τ) is *C-closed* if $M\simeq M^{C}_{(X,\tau)}$. $M^{C}_{(X,\tau)}$ is called the *C-closure* of M. We call C *idempotent* provided that for every $(X,\tau)\in\mathbf{FTS}$, $M^{C}_{(X,\tau)}$ is C-closed for every subset M of (X,τ).

The subscripts or superscripts in $(\ )^{C}_{(X,\tau)}$ will be omitted when no confusion is possible.

For more background on closure operators see, e.g., [2], [7], [8], [9], [10] and [13]. For a detailed survey on the same topic, one could check [11].

Given a function $X\xrightarrow{U}I$, $Supp(U)$ denotes the subset of X consisting of all $x\in X$ such that $U(x)\neq 0$ and coU will denote the complement of U, that is the function $1-U$. If $X\xrightarrow{U}I$ and $X\xrightarrow{V}I$ are two functions, we recall that $U\leq V$ means that for every $x\in X$, $U(x)\leq V(x)$. Notice that any subset M of $(X,\tau)\in\mathbf{FTS}$ can be seen as a fuzzy set via its characteristic function, i.e., the function $X\xrightarrow{1_M}I$ that to each $x\in M$ associates 1 and 0 otherwise. $fcl(1_M)$ will denote the fuzzy closure of 1_M, that is, $fcl(1_M)=\wedge\{X\xrightarrow{U}I : 1_M\leq U \text{ and } coU\in\tau\}$. We say that 1_M is closed if $1_M=fcl(1_M)$. $\vee\{X\xrightarrow{U}I : U\leq V \text{ and } U\in\tau\}$.

2 CONNECTEDNESS IN FTS

We recall that in [6], Definition 2.5, a notion of connectedness with respect to a closure operator C on an $(\mathbf{E},\mathcal{M})$-category for sinks $\mathcal{X}$, was introduced. This notion was also dependent on a subclass $\mathcal{N}$ of $\mathcal{M}$. It was given the name of $(C,\mathcal{N})$-connectedness. All this was made possible using techniques and results developed in [3-5].

Set $D_{\mathcal{N}}(C)=\{X\in\mathcal{X}:\ \text{every } n\in\mathcal{N}_X \text{ is } C\text{-closed}\}$. We say that an object X is $(C,\mathcal{N})$-discrete if $X\in D_{\mathcal{N}}(C)$. We also recall from [3] that an $\mathcal{X}$-morphism $X\xrightarrow{f}Y$ is called $\mathcal{N}$-constant if f factors through the image under f of any $\mathcal{N}$-subobject of X.

It was proved in [6, Lemma 3.1] that under certain assumptions on the category $\mathcal{X}$ and on the subclass of morphisms $\mathcal{N}$, the notion of $(C,\mathcal{N})$-connectedness could be described in terms of the notion of $\mathcal{N}$-constant morphism. We recall that the needed assumptions were:

ASSUMPTIONS 2.1

(a) $\mathcal{X}$ has a terminal object T;

(b) $\mathcal{N}$ is closed under the formation of direct images and any morphism with domain T belongs to $\mathcal{N}$;

(c) T is an $\mathcal{M}$-subobject of every element of $\mathcal{N}$.

For the rest of the paper we will assume that in **FTS**, $\mathcal{N}$ consists of all $M \xrightarrow{m} X$ with $m \in \mathcal{M}$ and $M \neq \emptyset$.

Notice that in this case the notion of $\mathcal{N}$-constant morphism in **FTS** simply agrees with the classical notion of constant function.

We observe that **FTS** has a terminal object (T, τ), where T is a singleton set and $\tau = \{T \xrightarrow{\alpha} I\}_{\alpha \in I}$. Clearly, Assumptions 2.1 are satisfied. Consequently, from the above we obtain the following:

DEFINITION 2.2

Given a closure operator C on **FTS**, we say that $X \in$ **FTS** is C-connected if for every $Y \in D_{\mathcal{N}}(C)$, every morphism $X \xrightarrow{f} Y$ is constant.

Notice that in the above definition the reference to $\mathcal{N}$ was omitted since only one specific class $\mathcal{N}$ will be considered throughout the paper.

PROPOSITION 2.3

*Let $\alpha \in [0,1)$. The function c_α that to each subset M of $(X,\tau) \in$ **FTS**, associates the subset $c_\alpha(M) = (fcl(1_M))^{-1}(\alpha, 1]$ is a closure operator on **FTS**.*

Proof:

We need to show that the conditions of Definition 1.1 are satisfied.

(a). If $x \in M$, then every $X \xrightarrow{U} I$ that occurs in the construction of $c_\alpha(M)$ satisfies $U(x) = 1$. Consequently so does the infimum of all of them. Therefore $x \in c_\alpha(M)$.

(b). Let $M \subseteq N \in (X, \tau)$. Notice that since $1_M \leq 1_N$, we have that every $X \xrightarrow{U} I$ that occurs in the construction of $c_\alpha(N)$ also occurs in the construction of $c_\alpha(M)$. So, by taking the infimum we obtain that $fcl(1_M) \leq fcl(1_N)$. Consequently, we have that $c_\alpha(M) \subseteq c_\alpha(N)$.

(c). Let $(X,\tau) \xrightarrow{f} (Y,\sigma)$ be fuzzy continuous and let $N \subseteq Y$. We have that $1_N \leq \wedge\{Y \xrightarrow{U} I : 1_N \leq U \text{ and } coU \in \sigma\}$. Now, $1_{f^{-1}(N)} = f^{-1}(1_N) \leq f^{-1}(\wedge\{Y \xrightarrow{U} I : 1_N \leq U \text{ and } coU \in \sigma\}$. Since f is fuzzy continuous, we have that $fcl(f^{-1}(1_N)) = \wedge\{X \xrightarrow{V} I : 1_{f^{-1}(N)} \leq V \text{ and } coV \in \tau\} \leq f^{-1}(fcl(1_N)) = f^{-1}(\wedge\{Y \xrightarrow{U} I : 1_N \leq U \text{ and } coU \in \sigma\})$. So, let $x \in c_\alpha(f^{-1}(N))$. Then, $\wedge\{X \xrightarrow{U} I : 1_{f^{-1}(N)} \leq U \text{ and } coU \in \tau\}(x) > \alpha$ and from the above inequality, also $f^{-1}(\wedge\{Y \xrightarrow{U} I : 1_N \leq U \text{ and } coU \in \sigma\}(x) > \alpha$. Therefore, $\wedge\{Y \xrightarrow{U} I : 1_N \leq U \text{ and } coU \in \sigma\}(f(x)) > \alpha$, that is $f(x) \in c_\alpha(N)$. So, we conclude that $x \in f^{-1}(c_\alpha(N))$. □

REMARK 2.4

(a) We observe that the closure operator c_α is not idempotent for any $\alpha \in [0,1)$. As a matter of fact, given $\alpha \in [0,1)$, choose $\beta > 0$ such that $\beta < 1-\alpha$. Let $X = \{a,b,c\}$, and let τ be the fuzzy topology on X with the following collection as subbasis: $\{\gamma : \gamma \in I\} \cup \{u,v\}$, where $u(a)=0, u(b)=\beta, u(c)=1$ and $v(a)=0, v(b)=0, v(c)=\beta$. Clearly, $1-u$ and $1-v$ are fuzzy closed sets in (X,τ). Now, $fcl(1_{\{a\}}) = \wedge\{w : 1-w \in \tau \text{ and } w(a)=1\} = 1-u$. So, $(fcl(1_{\{a\}}))^{-1}(\alpha,1] = \{a,b\}$, since $1-u(b) = 1-\beta > \alpha$. Also, $fcl(1_{\{a,b\}}) = 1-v$, so $(fcl(1_{\{a,b\}}))^{-1}(\alpha,1] = \{a,b,c\}$.

(b) Notice that if in the above proposition we choose $\alpha = 0$ then we obtain as a special case that the function cl_1 that to each subset M of $(X,\tau) \in$ **FTS**, associates the subset $cl_1(M) = Supp(\wedge\{X \xrightarrow{U} I : 1_M \leq U \text{ and } coU \in \tau\})$ is a non-idempotent closure operator on **FTS**.

We recall from [26] the following:

DEFINITION 2.5

For $\alpha \in [0,1)$, we define a concrete functor $\mathbf{FTS} \xrightarrow{\mathbf{F}_\alpha} \mathbf{Top}$ as follows: if $(X,\tau) \in$ **FTS**, then $F_\alpha(X,\tau) = (X, F_\alpha\tau)$, where $F_\alpha\tau = \{M \subseteq X : 1_M \wedge \alpha \in \tau\}$.

LEMMA 2.6

*Suppose that $\alpha \in [0,1)$. Let $(X,\tau) \in$ **FTS** and let $M \subseteq X$. Then we have that $M = c_\alpha(M)$ if and only if $1_{X-M} \wedge (1-\alpha) \in \tau$.*

Proof:

($\Rightarrow$). If $M = (fcl(1_M))^{-1}(\alpha,1]$, then $1_M \vee \alpha = fcl(1_M) \vee \alpha = fcl(1_M \vee \alpha)$ and so $1-(1_M \vee \alpha) = 1_{X-M} \wedge (1-\alpha) \in \tau$.

($\Leftarrow$). $1_{X-M} \wedge (1-\alpha) \in \tau$ implies that $1_M \vee \alpha$ is closed in (X,τ). Hence, $1_M \vee \alpha = fcl(1_M \vee \alpha) = fcl(1_M) \vee \alpha$. Now, $M = 1_M^{-1}(\alpha,1] = (1_M \vee \alpha)^{-1}(\alpha,1] = (fcl(1_M) \vee \alpha)^{-1}(\alpha,1] = (fcl(1_M))^{-1}(\alpha,1] = c_\alpha(M)$. □

PROPOSITION 2.7

*Let $\alpha \in [0,1)$ and let $(X,\tau) \in$ **FTS**. The following are equivalent:*

(a) (X,τ) is c_α-discrete;

(b) $(X, F_{1-\alpha}\tau)$ is discrete.

Proof:

(a)⇒(b). If $\emptyset \neq M \subseteq X$, then $c_\alpha(X - M) = X - M$. Hence, from the previous lemma we have that $1_M \wedge (1 - \alpha) \in \tau$ and so $M \in F_{1-\alpha}\tau$.

(b)⇐(a). If $\emptyset \neq M \subseteq X$, then $X - M \in F_{1-\alpha}\tau$, so $1_{X-M} \wedge (1 - \alpha) \in \tau$. Again from the previous lemma we obtain that $M = c_\alpha(M)$. □

PROPOSITION 2.8

*Let $\alpha \in [0, 1)$ and let $(X, \tau) \in$ **FTS**. The following are equivalent:*

(a) (X, τ) is c_α-connected;

(b) $(X, F_{1-\alpha}\tau)$ is connected.

Proof:

(a)⇒(b). Suppose that $(X, F_{1-\alpha}\tau) \xrightarrow{f} (D, \delta)$ is continuous, where $D = \{0, 1\}$ and δ is the discrete topology on D. Let δ' be the fuzzy topology on D that has as subbasis $\{\gamma : \gamma \in I\} \cup \{1_{\{0\}} \wedge (1 - \alpha), 1_{\{1\}} \wedge (1 - \alpha)\}$. Now, (D, δ') is c_α-discrete because $F_{1-\alpha}\delta' = \delta$. Then, $(X, \tau) \xrightarrow{f} (D, \delta')$ is fuzzy continuous for $f^{-1}(1_{\{0\}} \wedge (1 - \alpha)) = 1_{f^{-1}(\{0\})} \wedge (1 - \alpha) \in \tau$, since $f^{-1}(\{0\}) \in F_{1-\alpha}\tau$. Similarly, $f^{-1}(1_{\{1\}}) \wedge (1 - \alpha)) \in \tau$. Hence f is constant.

(b)⇒(a). Suppose that $(X, \tau) \xrightarrow{f} (Y, \sigma)$ is fuzzy continuous and (Y, σ) is c_α-discrete. Then $(X, F_{1-\alpha}\tau) \xrightarrow{f} (Y, F_{1-\alpha}\sigma)$ is continuous and from Proposition 2.7 $(Y, F_{1-\alpha}\sigma)$ is discrete. Consequently, f is constant. □

Consequently we obtain the following characterization of c_α-connectedness:

THEOREM 2.9

*Let $\alpha \in [0, 1)$ and let $(X, \tau) \in$ **FTS**. The following are equivalent:*

(a) (X, τ) is c_α-connected;

(b) There does not exist a nonempty subset M of X such that $1_M \wedge (1 - \alpha) \in \tau$ and $1_{X-M} \wedge (1 - \alpha) \in \tau$. □

REMARK 2.10

(a) Lowen and Srivastava ([19, Definition 2.1]) gave the following definition: for $(X, \tau) \in$ **FTS** and $\alpha \in (0, 1]$, (X, τ) is 2_α-connected if there does not exist a non-empty proper subset $A \subseteq X$ such that $\{\alpha \wedge 1_A, \alpha \wedge 1_{X-A}\} \subseteq \tau$. It easily follows from Proposition 2.8 and from [19, Proposition 2.1] that our notion of c_α-connectedness agrees with that of $2_{1-\alpha}$-connectedness.

(b) As already observed in Remark 2.4(b), $c_0 = cl_1$. Consequently, from the above theorem we obtain the following: (X, τ) is cl_1-connected if and only if there does not exist a nonempty subset M of X such that $1_M \in \tau$ and $1_{X-M} \in \tau$.

PROPOSITION 2.11

*Let $\alpha \in I$. The function k_α that to each subset M of $(X, \tau) \in$ **FTS**, associates the subset $k_\alpha(M) = \cap\{\omega^{-1}[1-\alpha, 1] : \omega$ is closed in (X, τ) and $M \subseteq \omega^{-1}[1-\alpha, 1]\}$ is a closure operator on* **FTS**.

Proof:

We need to show that the conditions of Definition 1.1 are satisfied.

(a). This is straightforward.

(b). Let $M \subseteq N$ be subsets of (X, τ). Notice that every $X \xrightarrow{\omega} I$ that occurs in the construction of $k_\alpha(N)$ also occurs in the construction of $k_\alpha(M)$. So, by taking the intersection we obtain that $k_\alpha(M) \subseteq k_\alpha(N)$.

(c). Let $(X, \tau) \xrightarrow{f} (Y, \sigma)$ be fuzzy continuous and let $N \subseteq Y$. Suppose that $x \notin f^{-1}(k_\alpha(N))$. Then $f(x) \notin k_\alpha(N)$ and so there is a function $Y \xrightarrow{\omega} I$, closed in (Y, σ), such that $N \subseteq \omega^{-1}[1-\alpha, 1]$ and $f(x) \notin \omega^{-1}[1-\alpha, 1]$, i.e., $\omega(f(x)) < 1-\alpha$. Now set $\nu = \omega \circ f$. Notice that ν is closed in (X, τ) since $\nu = f^{-1}(\omega)$. Now, the fact that $\nu(x) = \omega(f(x)) < 1-\alpha$ implies that $x \notin \nu^{-1}[1-\alpha, 1]$. However, $f^{-1}(N) \subseteq \nu^{-1}[1-\alpha, 1]$, in fact if $x \in f^{-1}(N)$ then $f(x) \in N \subseteq \omega^{-1}[1-\alpha, 1]$ and so $\nu(x) = \omega(f(x)) \geq 1-\alpha$. Thus, $x \notin \cap\{\nu^{-1}[1-\alpha, 1] : \nu$ is closed in (X, τ) and $f^{-1}(N) \subseteq \nu^{-1}[1-\alpha, 1]\} = k_\alpha(f^{-1}(N))$. □

REMARK 2.12

Notice that if in the above proposition we choose $\alpha = 0$ then we obtain as a special case the function cl_2 that to each subset M of $(X, \tau) \in$ **FTS**, associates the subset $cl_2(M) = \{x : fcl(1_M)(x) = 1\}$.

We recall from [16] the following:

DEFINITION 2.13

For $\alpha \in [0, 1)$, we define a concrete functor **FTS** $\xrightarrow{\iota_\alpha}$ **Top** as follows: if $(X, \tau) \in$ **FTS**, then $\iota_\alpha(X, \tau) = (X, \iota_\alpha\tau)$, where $\iota_\alpha\tau$ is initial with respect to $(X \xrightarrow{u} (I, \{I, \emptyset, (\alpha, 1]\}))_{u \in \tau}$, i.e., $\iota_\alpha\tau = \{u^{-1}(\alpha, 1] : u \in \tau\}$.

REMARK 2.14

Let $(X, \tau) \in$ **FTS** and $\alpha \in I$. Then for each $M \subseteq X$ there exists a closed $\omega \in (X, \tau)$ such that $\omega^{-1}[1-\alpha, 1] = k_\alpha(M)$. Consequently, k_α is an idempotent closure operator.

The proof of the following lemma is quite easy, so we omit it.

LEMMA 2.15

For all fuzzy sets $X \xrightarrow{\omega} I$ *and* $\alpha \in [0,1)$*, we have:*

(a) $X - \omega^{-1}[1-\alpha, 1] = (1-\omega)^{-1}(\alpha, 1]$;

(b) $(1-\omega)^{-1}[1-\alpha, 1] = \omega^{-1}[0, \alpha]$;

(c) $X - \omega^{-1}(\alpha, 1] = (1-\omega)^{-1}[1-\alpha, 1]$. □

PROPOSITION 2.16

Let $\alpha \in [0,1)$ *and let* $(X, \tau) \in \mathbf{FTS}$*. The following are equivalent:*

(a) (X, τ) *is* k_α*-discrete;*

(b) $(X, \iota_\alpha \tau)$ *is discrete.*

Proof:

(a) $\Rightarrow$ (b). If $\emptyset \neq M \subseteq X$, then $M = k_\alpha(M)$, so $X - M = X - \omega^{-1}[1-\alpha, 1]$ for some closed ω in (X, τ) (cf. Remark 2.14). Hence, from the previous lemma we have that $X - M = (1-\omega)^{-1}(\alpha, 1] \in \iota_\alpha \tau$. The result follows since M is arbitrary.

(b) $\Rightarrow$ (a). Choose $\emptyset \neq M \subseteq X$. Then $X - M \in \iota_\alpha \tau$ and so $X - M = \nu^{-1}(\alpha, 1]$ for some $\nu \in \tau$. Hence, again from the previous lemma, $M = (1-\nu)^{-1}[1-\alpha, 1]$, and so, from Remark 2.14, $M = k_\alpha(M)$. □

THEOREM 2.17

Let $\alpha \in [0,1)$ *and let* $(X, \tau) \in \mathbf{FTS}$*. The following are equivalent:*

(a) (X, τ) *is* k_α*-connected;*

(b) there does not exist a family $\{u_j\}_{j \in J} \subseteq \tau$ *of cardinality greater than one with the following properties:*

- *(i)* $u_j^{-1}(\alpha, 1] \neq \emptyset$ *for all* $j \in J$;
- *(ii)* $u_j^{-1}(\alpha, 1] \cap u_k^{-1}(\alpha, 1] = \emptyset$ *if and only if* $j \neq k$;
- *(iii)* $\bigcup_{j \in J} u_j^{-1}(\alpha, 1] = X$;
- *(iv)* $\{x, y\} \subseteq u_j^{-1}(\alpha, 1]$ *for some* $j \in J$ *implies that* $u_k(x) = u_k(y)$ *for all* $k \in J$.

Proof:

(a) $\Rightarrow$ (b). Suppose that there exists a family $\{u_j\}_{j \in J}$ with the stated properties. Define an equivalence relation $\sim$ on X as follows:

$$x \sim y \Leftrightarrow \{x, y\} \subseteq u_j^{-1}(\alpha, 1] \text{ for some } j \in J$$

.

Let $X \xrightarrow{q} X/\!\sim$ denote the canonical surjective function. For each $j \in J$ define $X/\!\sim \xrightarrow{v_j} I$ by $v_j([x]) = u_j(x)$. v_j is well defined by (iv). Let σ denote the fuzzy topology on $X/\!\sim$ with $\{\gamma : \gamma \in I\} \cup \{v_j : j \in J\}$ as a subbasis. Then $(X,\tau) \xrightarrow{q} (X/\!\sim, \sigma)$ is fuzzy continuous, surjective and $(X/\!\sim, \sigma)$ is k_α-discrete. Since $X/\!\sim$ has cardinality geater than one, it follows that q is not constant.

(b) $\Rightarrow$ (a). Suppose that (X,τ) is not k_α-connected. Then there exists a surjective fuzzy continuous function $(X,\tau) \xrightarrow{f} (D,\delta')$ with $|D| > 1$ and (D,δ') k_α-discrete. If we chose any $d \in D$, then $\{d\} \in \iota_\alpha \delta'$ and so $\{d\} = v_d^{-1}(\alpha, 1]$ for some $v_d \in \delta'$. Since f is fuzzy continuous we have that $u_d = f^{-1}(v_d) \in \tau$. Then, $\{u_d\}_{d \in D}$ has the required properties. □

PROPOSITION 2.18

*Let $\alpha \in (0,1]$. The function d_α that to each subset M of $(X,\tau) \in$ **FTS**, associates the subset $d_\alpha(M) = (fcl(1_M \wedge \alpha))^{-1}(0,1]$ is a closure operator on* **FTS**.

Proof:

We need to show that the conditions of Definition 1.1 are satisfied.

(a). This is clear.

(b). Let $M \subseteq N$ be subsets of (X,τ). Then, $1_M \wedge \alpha \leq 1_N \wedge \alpha$ and so $fcl(1_M \wedge \alpha) \leq fcl(1_N \wedge \alpha)$. Consequently, $d_\alpha(M) = (fcl(1_M \wedge \alpha))^{-1}(0,1] \subseteq (fcl(1_N \wedge \alpha))^{-1}(0,1] = d_\alpha(N)$.

(c). Let $(X,\tau) \xrightarrow{f} (Y,\sigma)$ be fuzzy continuous and let $N \subseteq Y$. Now first observe that if $Y \xrightarrow{\gamma} I$ is such that $1_N \wedge \alpha \leq \gamma$ and $1-\gamma \in \sigma$, then $\omega = \gamma \circ f = f^{-1}(\gamma)$ satisfies $1-\omega \in \tau$, since f is fuzzy continuous. Moreover, $1_{f^{-1}(N)} \wedge \alpha \leq \omega$. In fact, if $z \in f^{-1}(N)$, then $(1_{f^{-1}(N)} \wedge \alpha)(z) = \alpha$. Since $f(z) \in N$, $(1_N \wedge \alpha)(f(z)) = \alpha$ and so $\omega(z) = \gamma(f(z)) \geq \alpha$. On the other side, if $z \notin f^{-1}(N)$ then $1_{f^{-1}(N)}(z) = 0$. Thus, $(1_{f^{-1}(N)} \wedge \alpha)(z) = 0$ and clearly $\omega(z) \geq 0$. So, $(1_{f^{-1}(N)} \wedge \alpha) \leq \omega$.

Now we obtain that $fcl(1_{f^{-1}(N)} \wedge \alpha) = \wedge\{X \xrightarrow{\omega} I : 1_{f^{-1}(N)} \wedge \alpha \leq \omega \text{ and } 1-\omega \in \tau\} \leq \wedge\{\omega = \gamma \circ f \text{ with } Y \xrightarrow{\gamma} I : 1_N \wedge \alpha \leq \gamma \text{ and } 1-\gamma \in \sigma\} = \wedge\{f^{-1}(\gamma) : Y \xrightarrow{\gamma} I \text{ with } 1_N \wedge \alpha \leq \gamma \text{ and } 1-\gamma \in \sigma\} = f^{-1}(\wedge\{Y \xrightarrow{\gamma} I : 1_N \wedge \alpha \leq \gamma \text{ and } 1-\gamma \in \sigma\}) = f^{-1}(fcl(1_N \wedge \alpha))$. This clearly implies that $d_\alpha(f^{-1}(N)) \leq f^{-1}(d_\alpha(N))$. □

REMARK 2.19

Notice that the closure operator d_α is not idempotent for any $\alpha \in (0,1]$. As a matter of fact, let $X = \{a,b,c\}$, let $\alpha \in (0,1]$ and choose $\gamma \in (0,1)$ with $1-\alpha < \gamma$. Let τ be the fuzzy topology on X which has the collection $\{\beta : \beta \in I\} \cup \{u,v\}$ as a subbasis, where $u(a) = 1-\alpha$, $u(b) = \gamma$, $u(c) = 1$; $v(a) = v(b) = 1-\alpha$, $v(c) = \gamma$. Now, $fcl(1_{\{a\}} \wedge \alpha) = 1-u$, so $d_\alpha(\{a\}) = \{a,b\}$. Also, $fcl(1_{\{a,b\}} \wedge \alpha) = 1-v$, so $d_\alpha(\{a,b\}) = X$.

LEMMA 2.20

Let $\alpha \in (0,1]$ and $(X,\tau) \in \mathbf{FTS}$. If $M \subseteq X$, then $M = d_\alpha(M) \Leftrightarrow 1_{X-M} \vee (1-\alpha) \in \tau$.

Proof:

$(\Rightarrow)$. Suppose that $M = d_\alpha(M)$. Then $(fcl(1_M \wedge \alpha))(x) = 0$ if $x \notin M$. Now, $1_M \wedge \alpha \leq (fcl(1_M \wedge \alpha)) \leq \alpha$, and since $(1_M \wedge \alpha)(x) = \alpha$ if $x \in M$, it follows that $(fcl(1_M \wedge \alpha))(x) = \alpha$ if $x \in M$. Hence $1_M \wedge \alpha = (fcl(1_M \wedge \alpha))$. Consequently, $1 - (1_M \wedge \alpha) = 1_{X-M} \vee (1-\alpha) \in \tau$.

$(\Leftarrow)$. If $1_{X-M} \vee (1-\alpha) \in \tau$ then $1_M \wedge \alpha = (fcl(1_M \wedge \alpha))$. Hence $M = (1_M \wedge \alpha)^{-1}(0,1] = (fcl(1_M \wedge \alpha))^{-1}(0,1] = d_\alpha(M)$. □

REMARK 2.21

Clearly we have that $d_1 = c_0$.

DEFINITION 2.22

For $\alpha \in (0,1]$, we define a concrete functor $\mathbf{FTS} \xrightarrow{\mathbf{G}_\alpha} \mathbf{Top}$ as follows: if $(X,\tau) \in \mathbf{FTS}$, then $G_\alpha(X,\tau) = (X, G_\alpha\tau)$, where $G_\alpha\tau = \{M \subseteq X : 1_M \vee \alpha \in \tau\}$.

PROPOSITION 2.23

Let $\alpha \in (0,1]$ and let $(X,\tau) \in \mathbf{FTS}$. The following are equivalent:

(a) (X,τ) is d_α-discrete;

(b) $(X, G_{1-\alpha}\tau)$ is discrete.

Proof:

(a) $\Rightarrow$ (b). If $\emptyset \neq M \subseteq X$, then $X - M = d_\alpha(X-M)$. So, from Lemma 2.20, $1_M \vee (1-\alpha) \in \tau$, and then $M \in G_{1-\alpha}\tau$.

(b) $\Rightarrow$ (a). If $\emptyset \neq M \subseteq X$, then M is closed in $(X, G_{1-\alpha}\tau)$, so $1_{X-M} \vee (1-\alpha) \in \tau$, which means that $M = d_\alpha(M)$. □

PROPOSITION 2.24

Let $\alpha \in (0,1]$ and let $(X,\tau) \in \mathbf{FTS}$. The following are equivalent:

(a) (X,τ) is d_α-connected;

(b) $(X, G_{1-\alpha}\tau)$ is connected.

Proof:

(a) $\Rightarrow$ (b). Given that (X,τ) is d_α-connected, suppose that $(X, G_{1-\alpha}\tau) \xrightarrow{f} (D,\delta)$ is continuous, where $D = \{0,1\}$ and δ is the discrete topology on D. Let δ' be the fuzzy topology on D that has $\{\nu : \nu \in I\} \cup \{u_0, u_1\}$ as a subbasis, where $u_0(0) = 1$, $u_0(1) = 1-\alpha$, $u_1(0) = 1-\alpha$

and $u_1(1) = 1$. Now, (D, δ') is $d_{1-\alpha}$-discrete, for $\{i\} \in G_{1-\alpha}\delta'$ since $1_{\{i\}} \vee (1-\alpha) = u_i \in \delta'$, for $i = 0, 1$ (cf. Proposition 2.23). Also, $(X, \tau) \xrightarrow{f} (D, \delta')$ is fuzzy continuous, since for $i = 0, 1$ we have that $f^{-1}(\{i\}) \in G_{1-\alpha}\tau$ and so $f^{-1}(u_i) = 1_{f^{-1}(\{i\})} \vee (1-\alpha) \in \tau$. Hence f is constant.

(b) $\Rightarrow$ (a). If $(X, G_{1-\alpha}\tau)$ is connected and $(X, \tau) \xrightarrow{f} (D, \delta)$ is fuzzy continuous, with (D, δ) being $d_{1-\alpha}$-discrete, then $(X, G_{1-\alpha}\tau) \xrightarrow{f} (D, G_{1-\alpha}\delta)$ is continuous and $(D, G_{1-\alpha}\delta)$ is discrete, and so f is constant. □

Hence we obtain the following characterization of d_α-connectedness:

THEOREM 2.25

*Let $\alpha \in (0, 1]$ and let $(X, \tau) \in$ **FTS**. The following are equivalent:*

(a) (X, τ) is d_α-connected;

(b) there does not exist a nonempty proper subset M of X with $1_M \vee (1-\alpha) \in \tau$ and $1_{X-M} \vee (1-\alpha) \in \tau$. □

PROPOSITION 2.26

*Let $\alpha \in [0, 1)$. The function e_α that to each subset M of $(X, \tau) \in$ **FTS**, associates the subset $e_\alpha(M) = \cap\{v^{-1}(\alpha, 1] : v$ is closed in (X, τ) and $M \subseteq v^{-1}(\alpha, 1]\}$ is a closure operator on **FTS**.*

Proof:

We need to show that the conditions of Definition 1.1 are satisfied.

(a). This is straightforward.

(b). Let $M \subseteq N$ be subsets of (X, τ). Notice that every $X \xrightarrow{\omega} I$ that occurs in the construction of $e_\alpha(N)$ also occurs in the construction of $e_\alpha(M)$. So, by taking the intersection we obtain that $e_\alpha(M) \subseteq e_\alpha(N)$.

(c). Let $(X, \tau) \xrightarrow{f} (Y, \sigma)$ be fuzzy continuous and let $N \subseteq Y$. Suppose that $x \notin f^{-1}(e_\alpha(N))$. Then $f(x) \notin e_\alpha(N)$ and so there is a function $Y \xrightarrow{\omega} I$, closed in (Y, σ), such that $N \subseteq \omega^{-1}(\alpha, 1]$ and $f(x) \notin \omega^{-1}(\alpha, 1]$, i.e., $\omega(f(x)) < \alpha$. Now set $\nu = \omega \circ f$. Notice that ν is closed in (X, τ) since $\nu = f^{-1}(\omega)$. Now, the fact that $\nu(x) = \omega(f(x)) < \alpha$ implies that $x \notin \nu^{-1}(\alpha, 1]$. However, $f^{-1}(N) \subseteq \nu^{-1}(\alpha, 1]$, in fact if $x \in f^{-1}(N)$ then $f(x) \in N \subseteq \omega^{-1}(\alpha, 1]$ and so $\nu(x) = \omega(f(x)) > \alpha$. Thus, $x \notin \cap\{\nu^{-1}(\alpha, 1] : \nu$ is closed in (X, τ) and $f^{-1}(N) \subseteq \nu^{-1}(\alpha, 1]\} = e_\alpha(f^{-1}(N))$. □

REMARK 2.27

It is easy to verify that the closure operator e_α is idempotent.

DEFINITION 2.28

For $\alpha \in [0, 1)$, we define a concrete functor $\mathbf{FTS} \xrightarrow{\mathbf{H}_\alpha} \mathbf{Top}$ as follows: if $(X, \tau) \in \mathbf{FTS}$, then $H_\alpha(X, \tau) = (X, H_\alpha\tau)$, where $H_\alpha\tau$ has as a basis the set $\{u^{-1}[1 - \alpha, 1] : u \in \tau\}$.

PROPOSITION 2.29

Let $\alpha \in [0, 1)$ and let $(X, \tau) \in \mathbf{FTS}$. The following are equivalent:

(a) (X, τ) *is e_α-discrete;*

(b) $(X, H_\alpha\tau)$ *is discrete.*

Proof:

(a) $\Rightarrow$ (b). If $M \subseteq X$, then $X - M = e_\alpha(X - M)$, so $X - M = \cap\{v^{-1}(\alpha, 1] : v$ is closed in (X, τ) and $X - M \subseteq v^{-1}(\alpha, 1]\}$. Hence $M = \cup\{(1 - v)^{-1}[1 - \alpha, 1] : v$ is closed in (X, τ) and $X - M \subseteq v^{-1}(\alpha, 1]\} \in H_\alpha\tau$.

(b) $\Rightarrow$ (a). If $M \subseteq X$, then $X - M \in H_\alpha\tau$ and so $X - M = \cup u_i^{-1}[1 - \alpha, 1]$ for some $u_i \in \tau$, $i \in I$. Hence $M = \cap(1 - u_i)^{-1}(\alpha, 1]$ and so $M = e_\alpha(M)$. □

The proof of the following result is very similar to the one of Proposition 2.17, so we omit it.

THEOREM 2.30

Let $\alpha \in [0, 1)$ and let $(X, \tau) \in \mathbf{FTS}$. The following are equivalent:

(a) (X, τ) *is e_α-connected;*

(b) there does not exist a family $\{u_j\}_{j \in J} \subseteq \tau$ of cardinality greater than one with the following properties:

  (i) $u_j^{-1}[1 - \alpha, 1] \neq \emptyset$ *for all* $j \in J$;

  (ii) $u_j^{-1}[1 - \alpha, 1] \cap u_k^{-1}[1 - \alpha, 1] = \emptyset$ *if and only if* $j \neq k$;

  (iii) $\bigcup_{j \in J} u_j^{-1}[1 - \alpha, 1] = X$;

  (iv) $\{x, y\} \subseteq u_j^{-1}[1 - \alpha, 1]$ *for some* $j \in J$ *implies that* $u_k(x) = u_k(y)$ *for all* $k \in J$. □

PROPOSITION 2.31

Let $\alpha \in (0, 1]$. The function l_α that to each subset M of $(X, \tau) \in \mathbf{FTS}$, associates the subset $l_\alpha(M) = (fcl(1_M))^{-1}[\alpha, 1]$ is a closure operator on $\mathbf{FTS}$.

Proof:

We need to show that the conditions of Definition 1.1 are satisfied.

(a). This is clear.

(b). Let $M \subseteq N$ be subsets of (X, τ). Then, $1_M \leq 1_N$ and so $fcl(1_M) \leq fcl(1_N)$. Consequently, $l_\alpha(M) = (fcl(1_M))^{-1}[\alpha, 1] \subseteq (fcl(1_N))^{-1}[\alpha, 1] = l_\alpha(N)$.

(c). Let $(X, \tau) \xrightarrow{f} (Y, \sigma)$ be fuzzy continuous and let $N \subseteq Y$. Now first observe that if $Y \xrightarrow{\gamma} I$ is such that $1_N \leq \gamma$ and $1 - \gamma \in \sigma$, then $\omega = \gamma \circ f = f^{-1}(\gamma)$ satisfies $1 - \omega \in \tau$ and $1_{f^{-1}(N)} \leq \omega$. In fact, if $z \in f^{-1}(N)$, then $(1_{f^{-1}(N)})(z) = 1$. Since $f(z) \in N$, $(1_N)(f(z)) = 1$ and so $\omega(z) = \gamma(f(z)) = 1$. On the other side, if $z \notin f^{-1}(N)$ then $1_{f^{-1}(N)}(z) = 0$ and clearly $\omega(z) \geq 0$. So, $1_{f^{-1}(N)} \leq \omega$. Now we obtain that $fcl(1_{f^{-1}(N)}) = \wedge\{X \xrightarrow{\omega} I : 1_{f^{-1}(N)} \leq \omega \text{ and } 1 - \omega \in \tau\} \leq \wedge\{\omega = \gamma \circ f \text{ with } Y \xrightarrow{\gamma} I : 1_N \leq \gamma \text{ and } 1 - \gamma \in \sigma\} = \wedge\{f^{-1}(\gamma) : Y \xrightarrow{\gamma} I \text{ with } 1_N \leq \gamma \text{ and } 1 - \gamma \in \sigma\} = f^{-1}(\wedge\{Y \xrightarrow{\gamma} I : 1_N \leq \gamma \text{ and } 1 - \gamma \in \sigma\}) = f^{-1}(fcl(1_N))$. This clearly implies that $l_\alpha(f^{-1}(N)) \leq f^{-1}(l_\alpha(N))$. □

REMARK 2.32

The same example in Remark 2.4(a) shows that the closure operator l_α is not idempotent. In fact, $l_\alpha(\{a\}) = \{a, b\}$ and $l_\alpha(\{a, b\}) = X$.

PROPOSITION 2.33

For $\alpha \in (0, 1]$, the function **FTS** $\xrightarrow{\mathbf{P}_\alpha}$ **Top** *defined as follows: if $(X, \tau) \in$* **FTS***, then $P_\alpha(X, \tau) = (X, P_\alpha\tau)$, where the topology $P_\alpha\tau$ has as a basis the set $\{M \subseteq X : (1_M \wedge (1 - \gamma)) \in \tau \text{ for some } \gamma < \alpha\}$, is a concrete functor.*

Proof:

We check the action of P_α on morphisms. Suppose that $(X, \tau) \xrightarrow{f} (Y, \sigma)$ is fuzzy continuous. It must be shown that $(X, P_\alpha\tau) \xrightarrow{f} (Y, P_\alpha\sigma)$ is continuous. Suppose that $N \subseteq Y$ is such that $(1_N \wedge (1 - \gamma)) \in \sigma$ for some $\gamma < \alpha$. Now, $1_{f^{-1}(N)} \wedge (1 - \gamma) = f^{-1}(1_N) \wedge f^{-1}(1 - \gamma) = f^{-1}(1_N \wedge (1 - \gamma))$. This clearly belongs to τ by fuzzy continuity. Thus $f^{-1}(N) \in P_\alpha\tau$. □

PROPOSITION 2.34

Let $\alpha \in (0, 1]$, $(X, \tau) \in$ **FTS** *and $M \subseteq X$. The following are equivalent:*

(a) $M = l_\alpha(M)$;

(b) $1_M \vee \gamma$ *is fuzzy closed for some* $\gamma < \alpha$;

(c) $1_{X-M} \wedge (1 - \gamma) \in \tau$ *for some* $\gamma < \alpha$;

(d) $X - M \in P_\alpha\tau$.

Proof:

(a)⇒(b). Suppose that $M = (fcl(1_M))^{-1}[\alpha, 1]$. It is claimed that $1_M \vee \gamma = fcl(1_M) \vee \gamma$ for some $\gamma < \alpha$. If $x \in M$ then $(1_M \vee \beta)(x) = (fcl(1_M) \vee \beta)(x) = 1$ for all $\beta \in I$. If $x \notin M$, then $(1_M \vee \beta)(x) = \beta$ for all $\beta \in I$. It follows from the initial assumption that $fcl(1_M)(x) < \alpha$, so taking $(fcl(1_M))(x) = \gamma$ we get $(fcl(1_M) \vee \gamma)(x) = \gamma$.

(b)⇒(a). If $1_M \vee \gamma$ is fuzzy closed for some $\gamma < \alpha$, then $1_M \vee \gamma = fcl(1_M \vee \gamma) = fcl(1_M) \vee \gamma$. It just needs to be shown that $l_\alpha(M) \subseteq M$. If $x \in (fcl(1_M))^{-1}[\alpha, 1]$ then $(fcl(1_M))(x) \geq \alpha > \gamma$. Hence $(fcl(1_M) \vee \gamma)(x) = (1_M \vee \gamma)(x) > \gamma$. Consequently $1_M(x) = 1$, i.e., $x \in M$.

(b)⇔(c). Clear.

(c)⇔(d). This follows from the definition of $P_\alpha \tau$. □

PROPOSITION 2.35

*Let $\alpha \in (0, 1]$ and let $(X, \tau) \in$ **FTS**. The following are equivalent:*

(a) (X, τ) *is l_α-discrete;*

(b) $(X, P_\alpha \tau)$ *is discrete.*

Proof:

This follows easily from the equivalence of (a) and (d) of the previous proposition. □

PROPOSITION 2.36

*Let $\alpha \in (0, 1]$ and let $(X, \tau) \in$ **FTS**. The following are equivalent:*

(a) (X, τ) *is l_α-connected;*

(b) $(X, P_\alpha \tau)$ *is connected.*

Proof:

(a)⇒(b). Suppose that $(X, P_\alpha \tau) \xrightarrow{f} (D, \delta)$ is continuous, where $D = \{0, 1\}$ and δ is the discrete topology on D. Now, $f^{-1}(0) \in P_\alpha \tau$, so $1_{f^{-1}(0)} \wedge \beta \in \tau$ for some $\beta > 1 - \alpha$. Similarly, $1_{f^{-1}(1)} \wedge \lambda \in \tau$ for some $\lambda > 1 - \alpha$. Let δ' be the fuzzy topology on D which has as a subbasis $\{\gamma : \gamma \in I\} \cup \{1_{\{0\}} \wedge \lambda, 1_{\{1\}} \wedge \beta\}$. (D, δ') is l_α-discrete because $P_\alpha \delta' = \delta$. Also, $(X, \tau) \xrightarrow{f} (D, \delta)$ is fuzzy continuous since $f^{-1}(1_{\{0\}} \wedge \lambda) = 1_{f^{-1}(0)} \wedge \lambda$ and $f^{-1}(1_{\{1\}} \wedge \beta) = 1_{f^{-1}(1)} \wedge \beta$. Thus f is constant.

(b)⇒(a). Suppose that $(X, \tau) \xrightarrow{f} (Y, \sigma)$ is fuzzy continuous, where (Y, σ) is l_α-discrete. Then, $(X, P_\alpha \tau) \xrightarrow{f} (Y, P_\alpha \sigma)$ is continuous and $(Y, P_\alpha \sigma)$ is discrete, and so f is constant. □

As an easy consequence of Propositions 2.36 we obtain the following:

THEOREM 2.37

*Let $\alpha \in (0, 1]$ and let $(X, \tau) \in$ **FTS**. The following are equivalent:*

(a) (X, τ) *is l_α-connected;*

(b) There exists no nonempty proper subset M of X with $1_M \wedge \lambda \in \tau$ for some $\lambda > 1 - \alpha$, and $1_{X-M} \wedge \beta \in \tau$ for some $\beta > 1 - \alpha$;

(c) There exists no nonempty proper subset M of X with $1_M \wedge \mu \in \tau$ and $1_{X-M} \wedge \mu \in \tau$ for some $\mu > 1-\alpha$. □

REMARK 2.38

We recall that in [19] a fuzzy topological space (X,τ) is called **D**-connected if it is 2_α-connected for each $\alpha \in (0,1]$. As already observed in Remark 2.10, each closure operator c_α defined in Proposition 2.3 yields the notion of $2_{1-\alpha}$-connectedness. It may be also worth noticing that $cl_2 = \bigcap_{\alpha\in[0,1)} c_\alpha$. However, if we take $\alpha = 1$ in Proposition 2.31, then we also obtain that $cl_2 = l_1$. Consequently, from the above theorem we obtain that the closure operator cl_2 yields exactly the notion of **D**-connectedness.

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I. W. Alderton, Department of Mathematics, University of South Africa, P.O. Box 392, Pretoria 0003, South Africa.

G. Castellini, Department of Mathematics, Univerisy of Puerto Rico, Mayagüez campus, P.O. Box 9018, Mayagüez, PR 00681-9018.