EPIMORPHISMS IN CATEGORIES OF SEPARATED FUZZY TOPOLOGICAL SPACES

I. Alderton[1] G. Castellini

ABSTRACT: The categorical theory of closure operators is used to characterize the epimorphisms in certain categories of separated fuzzy topological spaces (in the sense of Lowen). These include the 0^*-T_0-spaces of Wuyts and Lowen; the FT_S-spaces of Ghanim, Kerre and Mashhour; and the α-T_2-spaces of Rodabaugh.

KEY WORDS: Closure operator, fuzzy topological space, epimorphism, equalizer, T_0-axiom, T_1-axiom, α-T_2-axiom.

1991 AMS SUBJECT CLASSIFICATION: 54A40, 18A20, 54B30, 18A30, 18A99.

0 INTRODUCTION

Categorical closure operators have been used to generalize many classical topological notions to arbitrary categories. A special type of closure operator was introduced by Salbany in [20]; and has been used to characterize the epimorphisms in subcategories of an arbitrary category under mild conditions. For specific examples see [12] and [4].

In this paper we use the Salbany-type closure operator to characterize the epimorphisms in three categories of separated fuzzy topological spaces. The separation axioms thus considered are the 0^*-T_0-axiom of Wuyts and Lowen [22], the FT_S-axiom of Ghanim, Kerre and Mashhour [11], and the α-T_2-axiom of Rodabaugh [19]. These axioms are analogues of the T_0-, T_1- and T_2-axioms, respectively, in ordinary topology.

Because of the very weak nature of the 0^*-T_0-axiom, the characterization of the epimorphisms in the corresponding category was by no means straightforward; and the relationship of this result to that of Baron (cf. [3]) is not an obvious one.

The FT_S case was a rather easy extension of the corresponding result for T_1-topological spaces (cf. [12]).

The task of characterizing the epimorphisms in the α-T_2 case was simplified by the fact that some of the necessary machinary was already available in [19].

---

[1] The first author acknowledges a grant from the Research and Bursaries Committee of the University of South Africa. He also would like to thank the Mayagüez campus of the University of Puerto Rico for its hospitality while working on this paper.

1 PRELIMINARIES

As usual I denotes the closed unit interval $[0,1]$. If X is any set and $\alpha \in I$, then α will also be used to denote the constant function from X into I with value α. The notion of a fuzzy topological space used in this paper is the one due to Lowen [14]:

A pair (X,τ), where X is a set and τ is contained in I^X, is called a *fuzzy topological space* (and τ is called a *fuzzy topology* on X) if:

(i) $\alpha \in \tau$ for each $\alpha \in I$

(ii) $\{W_i : i \in I\} \subseteq \tau \Rightarrow \vee\{W_i : i \in I\} \in \tau$

(iii) $W_1, W_2 \in \tau \Rightarrow W_1 \wedge W_2 \in \tau$.

If $X \xrightarrow{f} Y$ is a function, and $X \xrightarrow{U} I$ and $Y \xrightarrow{V} I$ are fuzzy sets, then the fuzzy set $f^{-1}(V) : X \longrightarrow I$ is defined by $f^{-1}(V) = V \circ f$, and $f(U) : Y \longrightarrow I$ is defined as follows:

$$f(U)(y) = \begin{cases} sup\{U(x) : x \in f^{-1}(y)\}, & \text{if } f^{-1}(y) \neq \emptyset; \\ 0, & \text{if } f^{-1}(y) = \emptyset. \end{cases}$$

If (X,τ) and (Y,σ) are fuzzy topological spaces, then a function $(X,\tau) \xrightarrow{f} (Y,\sigma)$ is said to be fuzzy continuous provided that $f^{-1}(V) \in \tau$ whenever $V \in \sigma$. The notation **FTS** will denote the category of fuzzy topological spaces and fuzzy continuous functions. (We shall use the categorical terminology of [1].)

The category **FTS** has initial structures [15]: if $\{(X_i,\tau_i)\}_I$ is a family of fuzzy topological spaces, and for each $i \in I$ we have a function $X \xrightarrow{f_i} X_i$, then the fuzzy topology τ on X which is initial with respect to $(X \xrightarrow{f_i} (X_i,\tau_i))_I$ has as subbasis $\{f_i^{-1}(U_i) : i \in I, U_i \in \tau_i\}$. (The notion of subbasis for a fuzzy topology is analogous to the corresponding notion in ordinary topology.)

If $\{(X_i,\tau_i)\}_I$ is a family of fuzzy topological spaces, then their product is the fuzzy topological space $\Pi(X_i,\tau_i) = (\Pi X_i, \tau)$, where ΠX_i is the ordinary cartesian product of the sets X_i, and τ is the initial fuzzy topology with respect to the family of projections $(\Pi X_i \xrightarrow{\pi_i} (X_i,\tau_i))_I$. Note that $\Pi(X_i,\tau_i)$ is actually the categorical product in **FTS**.

For more information about fuzzy sets and the category **FTS**, the reader could consult [16].

The following definition of a closure operator is a specialization to **FTS** of a notion for more general categories, studied in e.g., [4], [5], [6], [8], [9], [10] and [13].

DEFINITION 1.1

A closure operator C on **FTS** is a family $\{[\ ]_C^{(X,\tau)}\}_{(X,\tau)\in\mathbf{FTS}}$ of functions on the subset lattices with the following properties which hold for every $(X,\tau) \in \mathbf{FTS}$:

(a) $M \subseteq [M]_C^{(X,\tau)}$ for every subset M of (X,τ)

(b) $M \subseteq N \Rightarrow [M]_C^{(X,\tau)} \subseteq [N]_C^{(X,\tau)}$ for every pair of subsets M, N of (X,τ)

(c) If P is the inverse image of a subset M of (Y,σ) under some **FTS**-morphism $(X,\tau) \xrightarrow{f} (Y,\sigma)$, and Q is the inverse image of $[M]_C^{(Y,\sigma)}$ under f, then $[P]_C^{(X,\tau)} \subseteq Q$.

We say that the subset M of (X,τ) is *C-closed* if $M \simeq [M]_C^{(X,\tau)}$. $[M]_C^{(X,\tau)}$ is called the *C-closure* of M. We call C *idempotent* provided that for every $(X,\tau) \in$ **FTS**, $[M]_C^{(X,\tau)}$ is C-closed for every subset M of (X,τ).

The subscripts or superscripts in $[\ ]_C^{(X,\tau)}$ will be omitted when no confusion is possible.

REMARK 1.2

The reader who is more categorically inclined could make use of the following equivalent definition. A *closure operator* on a category $\mathcal{X}$ (with respect to a class of monomorphisms $\mathcal{M}$) is a pair $C = (\gamma, F)$, where F is an endofunctor on $\mathcal{M}$ that satisfies $UF = U$, and γ is a natural transformation from $id_{\mathcal{M}}$ to F that satisfies $(id_U)\gamma = id_U$ (cf. [9]).

Note that in all the categories of fuzzy topological spaces studied in this paper, the equalizer of a pair of fuzzy continuous functions $(X,\tau) \overset{f}{\underset{g}{\rightrightarrows}} (Y,\sigma)$ can be identified with the set A given by $\{x \in X : f(x) = g(x)\}$. (The corresponding fuzzy topology τ' on A and the corresponding morphism $(A,\tau') \longrightarrow (X,\tau)$ are the obvious ones and may be omitted from the discussion.)

DEFINITION 1.3

A special case of an idempotent closure operator arises in the following way. Given any class $\mathcal{A}$ of **FTS**-objects and any subset M of $(X,\tau) \in$ **FTS**, define $[M]_{\mathcal{A}}$ to be the intersection of all equalizers of pairs of **FTS**-morphisms r, s from (X,τ) to some $\mathcal{A}$-object (Y,σ) that agree on M. It is easy to see that such a construction defines an idempotent closure operator. This was first introduced by Salbany in the category of topological spaces (cf. [20]).

2 MAIN RESULTS

We will make use of the following theorem which is a special case of Theorem 1.11 of [4]. We therefore state it without proof.

THEOREM 2.1

Let $\mathcal{A}$ be a subcategory of **FTS** *and let $(X,\tau) \xrightarrow{f} (Y,\sigma)$ be a morphism in $\mathcal{A}$. f is an epimorphism in $\mathcal{A}$ iff $[f(X)]_{\mathcal{A}} = Y$.* □

The following separation axiom is the 0^*-T_0-axiom of Wuyts and Lowen [22], but is here called the T_0-axiom since it appears to be the categorically "right" concept of this axiom (cf. [17]

and [2]).

DEFINITION 2.2

A fuzzy topological space (X, τ) is said to satisfy the T_0-axiom if for every pair of distinct points $x, y \in X$, there exists $V \in \tau$ such that $V(x) \neq V(y)$.

The full subcategory of **FTS** whose objects are all those satisfying the above condition will be denoted by $\mathbf{FTS}_0$.

REMARK 2.3

An important ingredient in the proof of the following proposition is the Sierpinski object (I, Δ_S), studied in [21] and in [17], where Δ_S is the fuzzy topology on I generated by $I \xrightarrow{id} I$, i.e.,

$$\Delta_S = \{(\alpha \wedge id) \vee \beta : \alpha, \beta \in I\}.$$

It should be noted that , given any $(X, \tau) \in \mathbf{FTS}$ and $W \in \tau$, it holds that $(X, \tau) \xrightarrow{W} (I, \Delta_S)$ is fuzzy continuous, because $W^{-1}((\alpha \wedge id) \vee \beta)) = (\alpha \wedge W) \vee \beta$ for every choice of α and β in I. Also note that (I, Δ_S) satisfies the T_0-axiom.

PROPOSITION 2.4

Let $(X, \tau) \in \mathbf{FTS}$ and let M be a subset of X. Consider the set

$$b(M) = \{x \in X : \forall V \in \tau, \forall \epsilon > 0, \exists a \in M, \exists \delta > 0 \text{ such that } \forall W \in \tau, V(a) - W(a) < \delta \Rightarrow V(x) - W(x) < \epsilon\}.$$

Then, M is an equalizer in $\mathbf{FTS}_0$ iff $M = b(M)$.

Proof:

$(\Rightarrow)$. Suppose that $M = equ(f, g)$, $(X, \tau) \overset{f}{\underset{g}{\rightrightarrows}} (Y, \sigma)$ with $(X, \tau), (Y, \sigma) \in \mathbf{FTS_0}$. Choose $x \in X - M$. Then, $f(x) \neq g(x)$. There exists $U \in \sigma$ such that, say, $0 \leq U(f(x)) < U(g(x)) \leq 1$. Let $V = g^{-1}(U) \in \tau$. Now, take $\epsilon_0 = (U(g(x)) - U(f(x)))/2$. Consider $W_0 = f^{-1}(U) \in \tau$. For every $a \in M$ we have that $V(a) = U(g(a)) = U(f(a)) = W_0(a)$. Therefore, it trivially occurs that for every choice of $\delta > 0$, $V(a) - W_0(a) < \delta$. However, $V(x) - W_0(x) = U(g(x)) - U(f(x)) = 2\epsilon_0$. Hence $x \notin b(M)$.

$(\Leftarrow)$. Suppose that $b(M) = M$, and let $x \in X - M$. Since $x \notin b(M)$, we can choose $N \in \tau$ with the property that there exists $\epsilon > 0$ such that for each $a \in M$ and each $\delta > 0$ there exists $W_{a,\delta} \in \tau$ such that $N(a) - W_{a,\delta}(a) < \delta$ and $N(x) - W_{a,\delta}(x) \geq \epsilon$.

Set $F_x = \vee\{W_{a,\delta} : a \in M, \delta > 0\}$ and let $G_x = N \vee F_x$. The aim is to show that F_x and G_x differ at x and agree on the whole of M. Hence it must be shown that $N(x) > F_x(x)$, and $F_x(a) \geq N(a)$ for each $a \in M$.

Now, $N(x) - W_{a,\delta}(x) \geq \epsilon \Rightarrow N(x) - \epsilon \geq W_{a,\delta}(x)$ for each $a \in M$ and $\delta > 0 \Rightarrow N(x) - \epsilon \geq F_x(x) \Rightarrow N(x) \geq F_x(x) + \epsilon > F_x(x)$.

Let $a \in M$. Then for each $\delta > 0$ it holds that $N(a) - F_x(a) \leq N(a) - W_{a,\delta}(a) < \delta$. Since this holds for each $\delta > 0$, we have that $N(a) - F_x(a) \leq 0$. Hence $F_x(a) \geq N(a)$ for every $a \in M$.

Hence, for each $x \in X - M$ we have two fuzzy continuous functions $(X, \tau) \xrightarrow{F_x} (I, \Delta_S)$ and $(X, \tau) \xrightarrow{G_x} (I, \Delta_S)$ in $\mathbf{FTS_0}$ which agree at every element of M but differ at x (see Remark 2.3). Consider $\Pi(I, \Delta_S)$, where the product is taken over the index set $X - M$. By definition of product, we obtain two fuzzy continuous functions $(X, \tau) \xrightarrow{F} \Pi(I, \Delta_S)$ and $(X, \tau) \xrightarrow{G} \Pi(I, \Delta_S)$ that agree precisely on M. Observe that the functions F and G are morphisms in $\mathbf{FTS_0}$ because $\mathbf{FTS_0}$ is closed under the formation of products in $\mathbf{FTS}$. $\square$

COROLLARY 2.5

For every subset M of $(X, \tau) \in \mathbf{FTS_0}$, $[M]_{\mathbf{FTS_0}} = b(M)$.

Proof:

Since $M \subseteq b(M)$ and $b(b(M)) = b(M)$, it follows from Proposition 2.4 that $b(M)$ is an equalizer in $\mathbf{FTS_0}$ containing M. Therefore, by definition $[M]_{\mathbf{FTS_0}} \subseteq b(M)$. Now, $M \subseteq [M]_{\mathbf{FTS_0}}$ implies that $b(M) \subseteq b([M]_{\mathbf{FTS_0}})$. Since $\mathbf{FTS_0}$ is closed under products, $[M]_{\mathbf{FTS_0}}$ is an equalizer in $\mathbf{FTS_0}$ (cf. [4, Proposition 1.6]). Hence, from the above proposition, $b([M]_{\mathbf{FTS_0}}) = [M]_{\mathbf{FTS_0}}$. Thus, $b(M) = [M]_{\mathbf{FTS_0}}$. $\square$

By applying Theorem 2.1 we obtain

THEOREM 2.6

A morphism $(X, \tau) \xrightarrow{f} (Y, \sigma)$ in $\mathbf{FTS_0}$ is an epimorphism in $\mathbf{FTS_0}$ iff $b(f(X)) = Y$. $\square$

EXAMPLE 2.7

Let (I, Δ_S) be the Sierpinski object described in Remark 2.3. Then it can be easily verified that $[[0,1)]_{\mathbf{FTS_0}} = I$. Therefore the inclusion of $[0, 1)$ into I is an epimorphism in $\mathbf{FTS_0}$.

REMARK 2.8

(a) The definition of $b(A)$ in Proposition 2.4 has been written for fuzzy topological spaces in the sense of Lowen, but could also apply to fuzzy topological spaces in the sense of Chang [7]. In this case the reader should notice that this definition, when restricted to ordinary T_0-topological spaces, need not give the b-closure of Baron (cf. [3], [18]). However, to obtain such a closure one can define $b(M)$ as follows:

$b(M) = \{x \in X : \forall V \in \tau, \forall \epsilon > 0, \exists a \in M, \exists \delta > 0$ *such that* $(i) V(x) - V(a) < \epsilon$ *and*

$$(ii)\forall W \in \tau, V(a) - W(a) < \delta \Rightarrow V(x) - W(x) < \epsilon\}.$$

The extra condition (i) is redundant for fuzzy topological spaces in the sense of Lowen because in this case it follows from (ii) and the fact that τ contains all the constant functions.

DEFINITION 2.9

Let X be a set. A fuzzy singleton of X is a function $\chi_x^\alpha : X \to I$ such that $\chi_x^\alpha(x) = \alpha$ and $\chi_x^\alpha(y) = 0$ for every $y \neq x$.

DEFINITION 2.10

A fuzzy topological space (X, τ) is said to satisfy the T_1-axiom if every fuzzy singleton is closed. (In [11] this was termed the FT_S-axiom.)

The full subcategory of **FTS** whose objects are all those satisfying the above condition will be denoted by $\mathbf{FTS_1}$.

Given a function $X \xrightarrow{U} I$, $Supp(U)$ denotes the subset of X consisting of all $x \in X$ such that $U(x) \neq 0$; coU denotes the complement of U, i.e., the function $1 - U$.

Given a set X, the family τ consisting of all functions $X \xrightarrow{U} I$ such that $Supp(coU)$ is finite, together with all constant functions forms a fuzzy topology on X called the cofinite fuzzy topology ([11]). Notice that any cofinite fuzzy topological space satisfies the T_1-axiom.

The following result is simply an extension to $\mathbf{FTS_1}$ of a similar result for T_1-topological spaces (cf. [12]).

PROPOSITION 2.11

Let M be a subset of $(X, \tau) \in \mathbf{FTS_1}$. Then $[M]_{\mathbf{FTS_1}} = M$.

Proof:

Since M is always contained in $[M]_{\mathbf{FTS_1}}$, we just need to show the other inclusion.

Let $x \notin M$. Consider a fuzzy topological space (Y, γ) such that $|Y| > |X|$ and where γ is the cofinite fuzzy topology on Y. Let $(X, \tau) \xrightarrow{f} (Y, \gamma)$ be an injective function. If $coU \in \gamma$, then $Supp(U)$ is finite. $Supp(f^{-1}(U)) = Supp(U \circ f) = \{x \in X : U(f(x)) \neq 0\}$ is finite. Thus $f^{-1}(coU) = cof^{-1}(U)$ belongs to the cofinite fuzzy topology on X and hence to τ, since any T_1-fuzzy topology on X contains the cofinite one (cf. [11, Theorem 4.2]). Consequently f is fuzzy continuous.

Take $y \notin Im(f)$. Define $X \xrightarrow{g} Y$ by $g(z) = f(z)$ for every $z \in X - \{x\}$ and $g(x) = y$. As above, g is injective and therefore fuzzy continuous. f and g agree on M but $f(x) \neq g(x)$. Thus $x \notin [M]_{FTS_1}$. $\square$

As a consequence of the above proposition and Theorem 2.1, we obtain

THEOREM 2.12

The epimorphisms in $\mathbf{FTS_1}$ *are surjective.* □

DEFINITION 2.13 (cf. [19])

Let $\alpha \in [0,1)$. A fuzzy topological space (X,τ) is said to be α-T_2 if for each pair of distinct points $x, y \in X$, there exist $U, V \in \tau$ with $U(x) > \alpha$, $V(y) > \alpha$ and $U \wedge V = 0$.

The full subcategory consisting of all the objects satisfying the above condition will be denoted by $\mathbf{FTS}_{\alpha\mathbf{2}}$

DEFINITION 2.14 (cf. [19])

(a) Let $(X,\tau) \in \mathbf{FTS}$, let $M \subseteq X$ and choose $\alpha \in [0,1)$. Define

$$Cl_\alpha(M) = \{x \in X : U \in \tau, U(x) > \alpha \Rightarrow \exists a \in M \textit{ with } U(a) > 0\}.$$

(This is precisely the α-closure of Rodabaugh.)

(b) The set M is said to be α-closed if $Cl_\alpha(M) = M$.

REMARK 2.15 (cf. [19])

The set M is α-closed iff for each $x \in X - M$ there exists $U \in \tau$ such that $U(x) > \alpha$ and $U(a) = 0$ for every $a \in M$.

Notice that since I^b in [19] is equal to I, the following result holds

PROPOSITION 2.16 (cf. [19], Theorem 5.3)

Let $(X,\tau) \underset{g}{\overset{f}{\rightrightarrows}} (Y,\sigma)$ *be fuzzy continuous with* $(X,\tau), (Y,\sigma) \in \mathbf{FTS}_{\alpha\mathbf{2}}$. *Then the subset* $\{x : f(x) = g(x)\}$ *is* α*-closed in* X. □

PROPOSITION 2.17 ([19])

Arbitrary intersections of α*-closed sets are* α*-closed.* □

COROLLARY 2.18

Let M *be a subset of* $(X,\tau) \in \mathbf{FTS}_{\alpha\mathbf{2}}$. *We have that* $Cl_\alpha(M) \subseteq [M]_{\mathbf{FTS}_{\alpha\mathbf{2}}}$.

Proof:

¿From Proposition 2.16 and Proposition 2.17, we obtain that $[M]_{\mathbf{FTS}_{\alpha\mathbf{2}}}$ is α-closed and so

$M \subseteq [M]_{\mathbf{FTS}_{\alpha 2}} \Rightarrow Cl_\alpha(M) \subseteq Cl_\alpha([M]_{\mathbf{FTS}_{\alpha 2}}) = [M]_{\mathbf{FTS}_{\alpha 2}}$. □

Final fuzzy topologies are required in the proof of the next proposition. If $\{(X_i, \tau_i)\}_I$ is a family of fuzzy topological spaces, and for each $i \in I$ we have a function $X_i \xrightarrow{f_i} X$, then the fuzzy topology τ on X which is final with respect to $((X_i, \tau_i) \xrightarrow{f_i} X)_I$ is $\{X \xrightarrow{V} I : f_i^{-1}(V) \in \tau_i, \forall i \in I\}$ (see [15]). In particular, given $(Y, \sigma) \in \mathbf{FTS}$, we shall need the coproduct (or sum) $(Y \amalg Y, \sigma \amalg \sigma)$, where $Y \amalg Y$ is the usual disjoint union, and $\sigma \amalg \sigma$ is the fuzzy topology on $Y \amalg Y$ which is final with respect to the family of canonical injections $((Y, \sigma) \xrightarrow{\mu_1} Y \amalg Y, (Y, \sigma) \xrightarrow{\mu_2} Y \amalg Y)$. It is easy to show that $\sigma \amalg \sigma = \{U \amalg V : U, V \in \sigma\}$, where, given any fuzzy sets $U, V \in \sigma$, the fuzzy set $U \amalg V$ is defined by

$$(U \amalg V)(y, i) = \begin{cases} U(y), & \text{if } i = 1; \\ V(y), & \text{if } i = 2. \end{cases}$$

Let $M \subseteq Y$. Define an equivalence relation on $Y \amalg Y$ by

$$(x, i) \sim (y, j) \Leftrightarrow \begin{cases} (x, i) = (y, j); \\ \textit{or} \\ x = y \in M. \end{cases}$$

Let Q denote the quotient of the set $Y \amalg Y$ with respect to the above relation and endowed with the final fuzzy topology μ induced by the natural quotient function $(Y \amalg Y, \sigma \amalg \sigma) \xrightarrow{q} Q$. Hence

$$q((x, i)) = \begin{cases} \{(x, 1), (x, 2)\}, & \text{if } x \in M; \\ \{(x, i)\}, & \text{if } x \notin M; \end{cases}$$

and $\mu = \{Q \xrightarrow{W} I : q^{-1}(W) \in \sigma \amalg \sigma\} = \{Q \xrightarrow{W} I : W \circ q = V_1 \amalg V_2 \text{ for some } V_1, V_2 \in \sigma\}$.

LEMMA 2.19

Let U_1 and U_2 be arbitrary functions from Y into I. Then $q^{-1}(q(U_1 \amalg U_2)) = U_1 \amalg U_2$ iff $U_1(a) = U_2(a)$ for every $a \in M$. □

PROPOSITION 2.20

For each α-T_2-space (Y, σ) and each α-closed subset M of Y, there exists an α-T_2-space (Q, μ) and fuzzy continuous functions $(Y, \sigma) \overset{r}{\underset{s}{\rightrightarrows}} (Q, \mu)$ such that $M = \{y \in Y : r(y) = s(y)\}$.

Proof:

Let (Q, μ) be the above defined quotient fuzzy topological space with M α-closed. First we wish to show that $(Q, \mu) \in \mathbf{FTS}_{\alpha 2}$.

Suppose that $q((x, i)) \neq q((y, j))$.

Case I. $x \neq y$

Since (Y, σ) is α-T_2, there exist $U_1, U_2 \in \sigma$ with $U_1(x) > \alpha$ and $U_2(y) > \alpha$ and $U_1 \wedge U_2 = 0$. Let $U'_1 = q(U_1 \amalg U_1)$ and $U'_2 = q(U_2 \amalg U_2)$. Then U'_1 and U'_2 belong to μ, by the previous lemma.

$U'_1(q((x, i))) = U_1(x) > \alpha$; $U'_2(q((y, j))) = U_2(y) > \alpha$. Also, we have $(U'_1 \wedge U'_2)(q((z, i))) = min[U'_1(q((z, i))), U'_2(q((z, i)))] = min[U_1(z), U_2(z)] = 0$.

Case II. $x = y$

Set $z = x = y$. Then $i \neq j$. Note that $z \notin M$. Since M is α-closed there exists $U \in \sigma$ with $U(z) > \alpha$, and $U(a) = 0$ for every $a \in M$. Set $U'_1 = q((U \amalg 0))$ and $U'_2 = q((0 \amalg U))$. By the previous lemma, U'_1 and U'_2 belong to μ. We have that $U'_1(q((z, 1))) = U(z) > \alpha$; $U'_2(q((z, 2))) = U(z) > \alpha$.

For $d \notin M$ we have $(U'_1 \wedge U'_2)(q((d, 1))) = min[U'_1(q((d, 1))), U'_2(q((d, 1)))] = min[U(d), 0] = 0$ and $(U'_1 \wedge U'_2)(q((d, 2))) = min[U'_1(q((d, 2))), U'_2(q((d, 2)))] = min[0, U(d)] = 0$. If $d \in M$ then we have $(U'_1 \wedge U'_2)(q((d, i))) = min[U'_1(q((d, i))), U'_2(q((d, i)))] = min[max[U(d), 0], max[U(d), 0]] = U(d) = 0$. Hence $(U'_1 \wedge U'_2) = 0$.

Now it can be easily checked that $M = \{y \in Y : r(y) = s(y)\}$ with $r = q \circ \mu_1$ and $s = q \circ \mu_2$.

$\square$

COROLLARY 2.21

Let M be a subset of $(X, \tau) \in \mathbf{FTS_{02}}$. Then $[M]_{\mathbf{FTS_{02}}} = Cl_0(M)$.

Proof:

Since $Cl_0(Cl_0(M)) = Cl_0(M)$, from Proposition 2.20 we obtain that $Cl_0(M)$ is an equalizer in $\mathbf{FTS_{02}}$. Therefore we have that $[M]_{\mathbf{FTS_{02}}} \subseteq Cl_0(M)$. This together with Corollary 2.18 gives that $[M]_{\mathbf{FTS_{02}}} = Cl_0(M)$. $\square$

Now, by applying Theorem 2.1 we obtain the following

THEOREM 2.22

A morphism $(X, \tau) \xrightarrow{f} (Y, \sigma)$ in $\mathbf{FTS_{02}}$ is an epimorphism in $\mathbf{FTS_{02}}$ iff $Cl_0(f(X)) = Y$.

$\square$

REMARK 2.23

If $\alpha > 0$, the operator Cl_α need not be idempotent. However, if for $M \subseteq (X, \tau) \in \mathbf{FTS}_{\alpha 2}$ we define

$$C_\alpha(M) = \cap\{A \subseteq X : M \subseteq A \text{ and } Cl_\alpha(A) = A\},$$

then the resulting operator C_α is idempotent and results analogous to those above allow us to conclude that $[M]_{\mathbf{FTS_{\alpha 2}}} = C_\alpha(M)$. Consequently a morphism $(X, \tau) \xrightarrow{f} (Y, \sigma)$ in $\mathbf{FTS_{\alpha 2}}$ is an epimorphism in $\mathbf{FTS}_{\alpha\mathbf{2}}$ iff $C_\alpha(f(X)) = Y$.

EXAMPLE 2.24

Consider the fuzzy topological space (I, ω), where ω is the collection of all lower semicontinuous functions from I with the usual topology into itself (cf. [14]). This space is α-T_2 for each $\alpha \in [0, 1)$. Now, if $\mathcal{Q}$ is the set of rational numbers, then $Cl_\alpha(\mathcal{Q} \cap I) = Cl_\alpha([0, 1)) = I$. So, the inclusions of $\mathcal{Q} \cap I$ and $[0, 1)$ into I are both epimorphisms in $\mathbf{FTS}_{\alpha\mathbf{2}}$.

REFERENCES

[1] J. Adámek, H. Herrlich, G.E. Strecker, *Abstract and Concrete Categories*, Wiley, New York, 1990.

[2] I. W. Alderton, "Function spaces in fuzzy topology," *Fuzzy Sets and Systems* **32** (1989), 115-124.

[3] S. Baron, "A note on epi in T_0," *Canad. Math. Bull.* **11** (1968), 503-504.

[4] G. Castellini, "Closure operators, monomorphisms and epimorphisms in categories of groups," *Cahiers Topologie Géom. Différentielle Catégoriques* **27(2)** (1986), 151-167.

[5] G. Castellini, D. Hajek, "Closure operators and connectedness," preprint.

[6] G. Castellini, J. Koslowski, G.E. Strecker, "Closure operators and polarities," in progress.

[7] C. L. Chang, "Fuzzy topological spaces," *J. Math. Anal. Appl.* **24** (1968), 182-190.

[8] D. Dikranjan, E. Giuli, "Closure operators induced by topological epireflections," *Coll. Math. Soc. J. Bolyai* **41** (1983), 233-246.

[9] D. Dikranjan, E. Giuli, "Closure operators I," *Topology and its Appl.* **27** (1987), 129-143.

[10] D. Dikranjan, E. Giuli, W. Tholen, "Closure operators II," *Proceedings of the Conference in Categorical Topology*, (Prague, 1988), World Scientific (1989), 297-335.

[11] M. H. Ghanim, E. E. Kerre, A. S. Mashhour, "Separation axioms, subspaces and sums in fuzzy topology," *J. Math. Anal. Appl.* **102** (1984), 189-202.

[12] E. Giuli, "Bases of topological epireflections," *Topology and its Appl.*, **27** (1980), 265-273.

[13] J. Koslowski, "Closure operators with prescribed properties," *Category Theory and its Applications* (Louvain-la-Neuve, 1987) Springer L.N.M. **1248** (1988), 208–220.

[14] R. Lowen, "Fuzzy topological spaces and fuzzy compactness," *J. Math. Anal. Appl.* **56** (1976), 621-633.

[15] R. Lowen, "Initial and final fuzzy topologies and the fuzzy Tychonoff theorem," *J. Math. Anal. Appl.* **58** (1977), 11-21.

[16] R. Lowen, *On the existence of natural non-topological fuzzy topological spaces*, Heldermann, Berlin, 1985.

[17] R. Lowen, A. K. Srivastava, "FTS_0: The epireflective hull of the Sierpinski object in FTS," *Fuzzy Sets and Systems* **29** (1989), 171-176.

[18] L. D. Nel, R. G. Wilson, "Epireflections in the category of T_0 spaces," *Fund. Math.* **75** (1972), 69-74.

[19] S. E. Rodabaugh, "The Hausdorff separation axiom for fuzzy topological spaces," *Topology and its Appl.* **11** (1980), 319-334.

[20] S. Salbany, "Reflective subcategories and closure operators," *Proceedings of the Conference in Categorical Topology* (Mannheim, 1975), Springer L.N.M. **540** (1976), 548-565.

[21] A. K. Srivastava, R. Srivastava, "Fuzzy Sierpinski space: another note," *Fuzzy Math.* **3** (1985), 99-103.

[22] P. Wuyts, R. Lowen, "On local and global measures of separation in fuzzy topological spaces," *Fuzzy Sets and Systems* **19** (1986), 51-80.

Ian W. Alderton
Department of Mathematics
University of South Africa
P.O. Box 392
0001 Pretoria
Republic of South Africa

Gabriele Castellini
Department of Mathematics
University of Puerto Rico
Mayagüez campus
P.O. Box 5000
Mayagüez, PR 00681-5000