A LINK BETWEEN TWO CONNECTEDNESS NOTIONS

G. Castellini[1] D. Holgate[2]

ABSTRACT: The composition of two previously introduced Galois connections is used to provide a wider perspective on the Clementino-Tholen connectedness versus separation Galois connection. Moreover, a link between this and Castellini's connectedness-disconnectednes Galois connection is also presented.

KEY WORDS: Closure operator, Galois connection, connectedness, morphism orthogonality.

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0 INTRODUCTION

The development of a general theory of topological connectedness was started by Preuß (cf. [Pr_{1-3}]). Afterwards, a considerable number of papers were published on this subject and on possible generalizations of it. Most of them used the common approach of first defining a notion of constant morphism and then using it to introduce a notion of connectedness and disconnectedness, accordingly. The categorical notion of closure operator that in the meantime was introduced and developed provided two further approaches to the above problem. Two different notions of connectedness with respect to a closure operator in an arbitrary category were introduced and studied by Clementino and Tholen ([CT], [Cl]) and independently by Castellini ([C_{1-6}]). The main aim of this paper is to provide a link between these two approaches.

The setting for this paper is a category $\mathcal{X}$ with an $(\mathbf{E}, \mathcal{M})$-factorization structure for sinks. Let $S(\mathcal{X})$ denote the collection of all subcategories of $\mathcal{X}$, ordered by inclusion and let $\mathcal{N}$ be a fixed subclass of $\mathcal{M}$. For a closure operator C on the category $\mathcal{X}$, the two assignments: $A_{\mathcal{N}}(C) = \{X \in \mathcal{X} : \text{ every } m \in \mathcal{N} \text{ with domain } X \text{ is } C\text{-closed}\}$ and $B_{\mathcal{N}}(C) = \{X \in \mathcal{X} : \text{ every } m \in \mathcal{N} \text{ with domain } X \text{ is } C\text{-dense}\}$, were proven in [CH] to preserve suprema and in-

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fima, respectively. Consequently, they yield two Galois connections $CL(\mathcal{X},\mathcal{M}) \underset{P_{\mathcal{N}}}{\overset{A_{\mathcal{N}}}{\rightleftarrows}} S(\mathcal{X})^{\mathbf{op}}$ and $S(\mathcal{X}) \underset{B_{\mathcal{N}}}{\overset{Q_{\mathcal{N}}}{\rightleftarrows}} CL(\mathcal{X},\mathcal{M})$. The general properties of these Galois connections are studied in [CH]. In this paper, their composition is described in terms of morphism orthogonality. In the case that $\mathcal{N}$ consists of diagonal morphisms, this yields a more general description of the connectedness-separation Galois connection introduced in [CT]. Using a similar but dual approach, the connectedness-disconnectedness Galois connection introduced in [C_{1-4}] is also described in terms of morphism orthogonality.

As a consequence, this allows us to construct a "butterfly" of Galois connections that puts into perspective a long sought relationship between the Clementino-Tholen notion of connectedness ([CT]) and the one of Castellini ([C_4]).

We use the terminology of [AHS] throughout the paper[3].

1 PRELIMINARIES

Throughout we consider a category $\mathcal{X}$ and a fixed class $\mathcal{M}$ of $\mathcal{X}$-monomorphisms, which contains all $\mathcal{X}$-isomorphisms. It is assumed that $\mathcal{X}$ is an $(\mathbf{E},\mathcal{M})$-category for sinks.

This implies the following features of $\mathcal{M}$ and $\mathbf{E}$ (cf. [AHS] for the dual case):

PROPOSITION 1.1

(1) Every isomorphism is in both $\mathcal{M}$ and $\mathbf{E}$ (as a singleton sink).

(2) $\mathcal{M}$ is closed under $\mathcal{M}$-relative first factors, i.e. if $n \circ m \in \mathcal{M}$, and $n \in \mathcal{M}$, then $m \in \mathcal{M}$.

(3) $\mathcal{M}$ is closed under composition.

(4) Pullbacks of $\mathcal{X}$-morphisms in $\mathcal{M}$ exist and belong to $\mathcal{M}$.

(5) The $\mathcal{M}$-subobjects of every $\mathcal{X}$-object form a (possibly large) complete lattice; suprema are formed via $(\mathbf{E},\mathcal{M})$-factorizations and infima are formed via intersections. □

If $X \xrightarrow{f} Y$ is an $\mathcal{X}$-morphism and $M \xrightarrow{m} X$ is an $\mathcal{M}$-subobject, then $M \xrightarrow{e_{f\circ m}} M_f \xrightarrow{m_f} Y$ will denote the $(\mathbf{E},\mathcal{M})$-factorization of $f \circ m$. $M_f \xrightarrow{m_f} Y$ will be called the direct image of m along f and $M \xrightarrow{e_{f\circ m}} M_f$ will be called the restriction of the morphism f to the $\mathcal{M}$-subobject m. If $N \xrightarrow{n} Y$ is an $\mathcal{M}$-subobject, then the pullback $f^{-1}(N) \xrightarrow{f^{-1}(n)} X$ of n along f will be called the inverse image of n along f. Whenever no confusion is likely to arise, to simplify the notation we will denote the morphism $e_{f\circ m}$ simply e_f.

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[3] Paul Taylor's Commutative Diagrams in TEX macro package was used to typeset most of the diagrams in this paper.

DEFINITION 1.2

A *closure operator* C on $\mathcal{X}$ (with respect to $\mathcal{M}$) is a family $\{(\ )_X^C\}_{X \in \mathcal{X}}$ of functions on the $\mathcal{M}$-subobject lattices of $\mathcal{X}$ with the following properties that hold for each $X \in \mathcal{X}$:

(a) $m \leq (m)_X^C$, for every $\mathcal{M}$-subobject $M \xrightarrow{m} X$;

(b) $m \leq n \Rightarrow (m)_X^C \leq (n)_X^C$ for every pair of $\mathcal{M}$-subobjects of X;

(c) If p is the pullback of the $\mathcal{M}$-subobject $M \xrightarrow{m} Y$ along some $\mathcal{X}$-morphism $X \xrightarrow{f} Y$ and q is the pullback of $(m)_Y^C$ along f, then $(p)_X^C \leq q$, i.e., the closure of the inverse image of m is less than or equal to the inverse image of the closure of m.

Condition (a) implies that for every closure operator C on $\mathcal{X}$, every $\mathcal{M}$-subobject $M \xrightarrow{m} X$ has a canonical factorization

$$\begin{array}{ccc} M & \xrightarrow{t} & (M)_X^C \\ & {\scriptstyle m}\searrow & \downarrow {\scriptstyle (m)_X^C} \\ & & X \end{array}$$

where $(m)_X^C$ is called the *C-closure* of the subobject m.

When no confusion is likely we will write m^C rather than $(m)_X^C$ and for notational symmetry we will denote the morphism t by m_C.

DEFINITION 1.3

Given a closure operator C, we say that $m \in \mathcal{M}$ is *C-closed* if m_C is an isomorphism. An $\mathcal{X}$-morphism f is called *C-dense* if for every $(\mathbf{E}, \mathcal{M})$-factorization (e, m) of f we have that m^C is an isomorphism. We call C *idempotent* provided that m^C is C-closed for every $m \in \mathcal{M}$. C is called *weakly hereditary* if m_C is C-dense for every $m \in \mathcal{M}$.

Notice that Definition 1.2(c) implies that pullbacks of C-closed $\mathcal{M}$-subobjects are C-closed.

We denote the collection of all closure operators on $\mathcal{M}$ by $\mathbf{CL}(\mathcal{X}, \mathcal{M})$ pre-ordered as follows: $C \sqsubseteq D$ if $m^C \leq m^D$ for all $m \in \mathcal{M}$ (where $\leq$ is the usual order on subobjects). Notice that arbitrary suprema and infima exist in $\mathbf{CL}(\mathcal{X}, \mathcal{M})$, they are formed pointwise in the $\mathcal{M}$-subobject fibers.

For more background on closure operators see, e.g., [CKS$_{1-2}$], [DG] and [DGT]. For a recent survey on the same topic, one could check [C$_7$]. Detailed proofs can be found in [H] and [DT].

DEFINITION 1.4

For pre-ordered classes $\mathcal{X} = (\mathbf{X}, \leq)$ and $\mathcal{Y} = (\mathbf{Y}, \leq)$, a *Galois connection* $\mathcal{X} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{Y}$ consists of order preserving functions F and G that satisfy $F \dashv G$, i.e. $x \leq GF(x)$ for every $x \in \mathbf{X}$ and

$FG(y) \leq y$ for every $y \in \mathbf{Y}$. (G is adjoint and has F as coadjoint).

If $x \in \mathcal{X}$ and $y \in \mathcal{Y}$ are such that $F(x) = y$ and $G(y) = x$, then x and y are said to be corresponding fixed points of the Galois connection $(\mathcal{X}, F, G, \mathcal{Y})$.

Properties and many examples of Galois connections can be found in [EKMS].

2 A DESCRIPTION VIA ORTHOGONALITY

The main aim of this Section is to provide a description of the composition of two Galois connections introduced in [CH] by means of the notion of morphism orthogonality. Then, this will be used to provide an alternative description of the composition of two other Galois connections studied in [CT].

We begin by recalling the Galois connections from [CH].

Let $S(\mathcal{X})$ denote the collection of all subcategories of $\mathcal{X}$, ordered by inclusion. Throughout, $\mathcal{N}$ will be a fixed subclass of $\mathcal{M}$. Then, we have the following:

PROPOSITION 2.1

Let $CL(\mathcal{X}, \mathcal{M}) \xrightarrow{A_{\mathcal{N}}} S(\mathcal{X})^{\mathbf{op}}$ *and* $S(\mathcal{X})^{\mathbf{op}} \xrightarrow{P_{\mathcal{N}}} CL(\mathcal{X}, \mathcal{M})$ *be defined by:*

$A_{\mathcal{N}}(C) = \{X \in \mathcal{X} :$ *every* $m \in \mathcal{N}$ *with domain* X *is* C*-closed*$\}$

$P_{\mathcal{N}}(\mathcal{A}) = Sup\{C \in CL(\mathcal{X}, \mathcal{M}) : A_{\mathcal{N}}(C) \supseteq \mathcal{A}\}$.

Then, $CL(\mathcal{X}, \mathcal{M}) \underset{P_{\mathcal{N}}}{\overset{A_{\mathcal{N}}}{\rightleftarrows}} S(\mathcal{X})^{\mathbf{op}}$ *is a Galois connection.* □

PROPOSITION 2.2

Let $CL(\mathcal{X}, \mathcal{M}) \xrightarrow{B_{\mathcal{N}}} S(\mathcal{X})$ *and* $S(\mathcal{X}) \xrightarrow{Q_{\mathcal{N}}} CL(\mathcal{X}, \mathcal{M})$ *be defined by:*

$B_{\mathcal{N}}(C) = \{X \in \mathcal{X} :$ *every* $m \in \mathcal{N}$ *with domain* X *is* C*-dense*$\}$

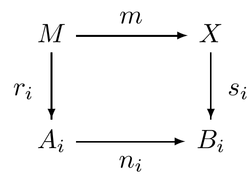
$Q_{\mathcal{N}}(\mathcal{B}) = Inf\{C \in CL(\mathcal{X}, \mathcal{M}) : B_{\mathcal{N}}(C) \supseteq \mathcal{B}\}$.

Then, $S(\mathcal{X}) \underset{B_{\mathcal{N}}}{\overset{Q_{\mathcal{N}}}{\rightleftarrows}} CL(\mathcal{X}, \mathcal{M})$ *is a Galois connection.* □

The following characterizations of the closure operators $P_{\mathcal{N}}(\mathcal{B})$ and $Q_{\mathcal{N}}(\mathcal{A})$ for $\mathcal{A} \in S(\mathcal{X})^{\mathbf{op}}$ and $\mathcal{B} \in S(\mathcal{X})$, were also obtained in [CH]. We report them here, since they will be useful later.

PROPOSITION 2.3

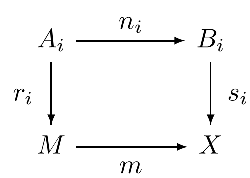
Let $\mathcal{A} \in S(\mathcal{X})^{\mathbf{op}}$. *For every* $X \in \mathcal{X}$ *and for every* $\mathcal{M}$*-subobject* $M \xrightarrow{m} X$, *consider all commutative squares of the form*



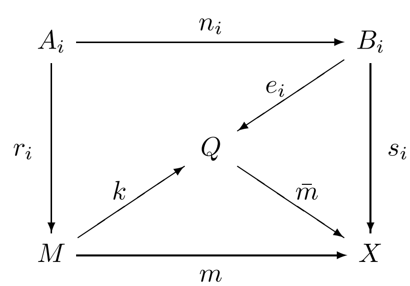
with $A_i \in \mathcal{A}$ and $n_i \in \mathcal{N}$, indexed by I. If we form all pullbacks m_i of n_i along s_i, then we have that $m^{P_{\mathcal{N}}(\mathcal{A})} \simeq \bigwedge_{i \in I} m_i$. Moreover, $P_{\mathcal{N}}(\mathcal{A})$ is idempotent. $\square$

PROPOSITION 2.4

Let $\mathcal{B} \in S(\mathcal{X})$. For every $Y \in \mathcal{X}$ and for every $\mathcal{M}$-subobject $M \xrightarrow{m} Y$, consider all commutative squares of the form



with $A_i \in \mathcal{B}$ and $n_i \in \mathcal{N}$, indexed by I. Take the $(\mathbf{E}, \mathcal{M})$-factorization of the sink $(s_i)_{i \in I} \cup \{m\}$. Thus we obtain the following commutative diagram

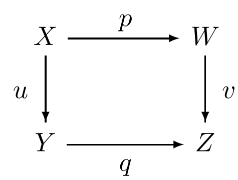


where $(e_i)_{i \in I} \cup \{k\} \in \mathbf{E}$ and $\bar{m} \in \mathcal{M}$ satisfy $\bar{m} \circ e_i = s_i$ for every $i \in I$ and $\bar{m} \circ k = m$. Then, we have that $m^{Q_{\mathcal{N}}(\mathcal{B})} \simeq \bar{m}$. Moreover $Q_{\mathcal{N}}(\mathcal{B})$ is weakly hereditary. $\square$

The two Galois connections introduced in Propositions 2.1 and 2.2 yield via composition a third Galois connection between $S(\mathcal{X})$ and $S(\mathcal{X})^{\mathbf{op}}$. In this Section we present a direct description of it by means of a familiar concept.

For $X \in \mathcal{X}$ and $\mathcal{P} \subseteq Mor\mathcal{X}$ we set $\mathcal{P}^X = \{f \in \mathcal{P} : dom(f) = X\}$ and dually $\mathcal{P}_X = \{f \in \mathcal{P} : cod(f) = X\}$.

For $\mathcal{P}, \mathcal{Q} \subseteq Mor\mathcal{X}$, we write $\mathcal{P} \perp \mathcal{Q}$ if every commutative diagram:



with $p \in \mathcal{P}$ and $q \in \mathcal{Q}$, has a unique diagonal, that is, there is a unique morphism $W \xrightarrow{d} Y$ such that $d \circ p = u$ and $q \circ d = v$. Using the classical terminology, this means that every element of $\mathcal{P}$ is "left orthogonal" to every element of $\mathcal{Q}$ (or equivalently, every element of $\mathcal{Q}$ is "right orthogonal" to every element of $\mathcal{P}$).

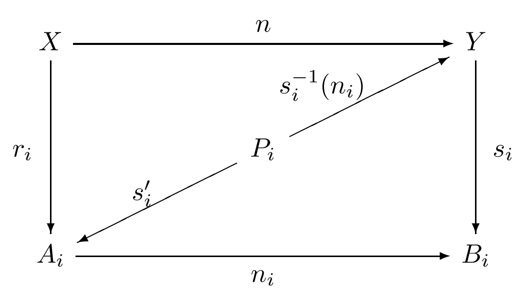
The above notion of orthogonality yields a Galois connection $S(\mathcal{X}) \underset{\lambda_{\mathcal{N}}}{\overset{\rho_{\mathcal{N}}}{\rightleftarrows}} S(\mathcal{X})^{\mathbf{op}}$ where for $\mathcal{B} \in S(\mathcal{X})$, $\rho_{\mathcal{N}}(\mathcal{B}) = \{Y \in \mathcal{X} : \forall X \in \mathcal{B}, \mathcal{N}^X \perp \mathcal{N}^Y\}$ and for $\mathcal{A} \in S(\mathcal{X})^{\mathbf{op}}$, $\lambda_{\mathcal{N}}(\mathcal{A}) = \{X \in \mathcal{X} : \forall Y \in \mathcal{A}, \mathcal{N}^X \perp \mathcal{N}^Y\}$. This newly obtained Galois connection is not really new, as the following theorem shows.

THEOREM 2.5

For any subclass $\mathcal{N}$ of $\mathcal{M}$, the Galois connection $S(\mathcal{X}) \underset{\lambda_{\mathcal{N}}}{\overset{\rho_{\mathcal{N}}}{\rightleftarrows}} S(\mathcal{X})^{\mathbf{op}}$ factors through $CL(\mathcal{X}, \mathcal{M})$ via the Galois connections $S(\mathcal{X}) \underset{B_{\mathcal{N}}}{\overset{Q_{\mathcal{N}}}{\rightleftarrows}} CL(\mathcal{X}, \mathcal{M})$ and $CL(\mathcal{X}, \mathcal{M}) \underset{P_{\mathcal{N}}}{\overset{A_{\mathcal{N}}}{\rightleftarrows}} S(\mathcal{X})^{\mathbf{op}}$.

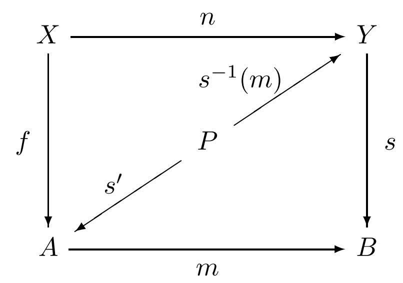
Proof:

Let $\mathcal{A} \in S(\mathcal{X})^{\mathbf{op}}$ and let $X \in \lambda_{\mathcal{N}}(\mathcal{A})$. Take all commutative squares $n_i \circ r_i = s_i \circ n$, where $n \in \mathcal{N}$, $n_i \in \mathcal{N}$ and $A_i \in \mathcal{A}$, indexed by I. Form the inverse images $s_i^{-1}(n_i)$.



Since $\mathcal{N}^X \perp \mathcal{N}^{A_i}$, for every $i \in I$, we have that the above diagram has a diagonal. As it is easily checked, this implies that $s_i^{-1}(n_i) \simeq id_Y$, for every $i \in I$. Hence, $n^{P_{\mathcal{N}}(\mathcal{A})} \simeq id_Y$ and so, $X \in B_{\mathcal{N}}(P_{\mathcal{N}}(\mathcal{A}))$.

Now let $X \in B_{\mathcal{N}}(P_{\mathcal{N}}(\mathcal{A}))$ and let us consider the following commutative diagram:



where $m, n \in \mathcal{N}$ and $A \in \mathcal{A}$. Clearly, we have that $id_Y \simeq n^{P_{\mathcal{N}}(\mathcal{A})} \leq s^{-1}(m) \leq id_Y$, which implies that $s^{-1}(m) \simeq id_Y$. Hence $d = s' \circ (s^{-1}(m))^{-1}$ is the sought diagonal. Uniqueness is implied by the fact that m is a monomorphism. Hence, $X \in \lambda_{\mathcal{N}}(\mathcal{A})$. Thus, we conclude that $\lambda_{\mathcal{N}} = B_{\mathcal{N}} \circ P_{\mathcal{N}}$.

Since, from the general properties of Galois connections, $\rho_{\mathcal{N}}$ and $\lambda_{\mathcal{N}}$ uniquely determine each other (cf. [EKMS]), then we can conclude that $\rho_{\mathcal{N}} = A_{\mathcal{N}} \circ Q_{\mathcal{N}}$. □

At this point we observe that under the assumption of $\mathcal{M}$ containing all regular monomorphisms, if $\mathcal{N}$ consists of all diagonal morphisms then the Galois connections in Propositions 2.1 and 2.2 reduce to the Galois connections used by Clementino and Tholen ([CT]) to study the notions of separation and connectedness in an arbitrary category. As a matter of fact the above theorem yields an interesting description of their composition. In this case we have that for $\mathcal{B} \in S(\mathcal{X})$ and $\mathcal{A} \in S(\mathcal{X})^{\mathbf{op}}$:

$$\rho_{\mathcal{N}}(\mathcal{B}) = \{Y \in \mathcal{X} : \forall X \in \mathcal{B}, \delta_X \perp \delta_Y\}$$
$$\lambda_{\mathcal{N}}(\mathcal{A}) = \{X \in \mathcal{X} : \forall Y \in \mathcal{A}, \delta_X \perp \delta_Y\}.$$

In [CT], the composition of the two Galois connections arising from the case of $\mathcal{N}$ consisting of diagonal morphisms was given in terms of the "left-right constant" Galois connection. In what follows we show that under appropriate hypotheses the Galois connection $S(\mathcal{X}) \underset{\lambda_{\mathcal{N}}}{\overset{\rho_{\mathcal{N}}}{\rightleftarrows}} S(\mathcal{X})^{\mathbf{op}}$ reduces to that one.

First we recall the following concepts from [CT].

DEFINITION 2.6

(a) An $\mathcal{X}$-object P is called *preterminal* if $|\mathcal{X}(X, P)| \leq 1$, for each $X \in \mathcal{X}$.

(b) An $\mathcal{X}$-morphism $X \xrightarrow{f} Y$ is called *constant* if in its $(\mathbf{E}, \mathcal{M})$-factorization $X \xrightarrow{e} P \xrightarrow{m} Y$, P is preterminal.

(c) An $\mathcal{M}$-subobject $P \xrightarrow{p} X$ of $X \in \mathcal{X}$, with P preterminal, is called a *quasipoint* of X if P is isomorphic to the middle object of the $(\mathbf{E}, \mathcal{M})$-factorization of the unique morphism $X \longrightarrow T$, with T denoting the terminal object.

(d) An object $X \in \mathcal{X}$ is said to *have enough quasipoints* if the supremum of all quasipoints of

X is isomorphic to id_X. If this is true for every $X \in \mathcal{X}$, then the category $\mathcal{X}$ is said to have enough quasipoints.

We write $X||Y$ if and only if every morphism $X \xrightarrow{f} Y$ is constant. The relation $||$ gives rise to the "left-right constant" Galois connection: $S(\mathcal{X}) \underset{l}{\overset{r}{\rightleftarrows}} S(\mathcal{X})^{\mathbf{op}}$, where for $\mathcal{B} \in S(\mathcal{X})$, $r(\mathcal{B}) = \{Y \in \mathcal{X} : \forall X \in \mathcal{B}, X||Y\}$ and for $\mathcal{A} \in S(\mathcal{X})^{\mathbf{op}}$, $l(\mathcal{A}) = \{X \in \mathcal{X} : \forall Y \in \mathcal{A}, X||Y\}$.

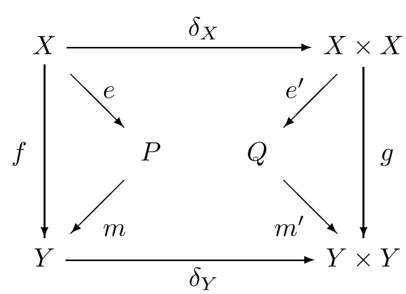
From here on we assume that $\mathcal{X}$ has squares and that $\mathcal{M}$ contains all regular monomorphisms.

LEMMA 2.7

For every $X, Y \in \mathcal{X}$, we have that $X^2||Y^2$ implies that $\delta_X \perp \delta_Y$.

Proof:

Let us consider the following diagram:



where $f = m \circ e$ and $g = m' \circ e'$ are $(\mathbf{E}, \mathcal{M})$-factorizations.

If $g \circ \delta_X = \delta_Y \circ f$, then the $(\mathbf{E}, \mathcal{M})$-diagonalization property yields a morphism $P \xrightarrow{h} Q$ such that $h \circ e = e' \circ \delta_X$ and $m' \circ h = \delta_Y \circ m$. Since Q is preterminal, we have that $h \circ e \circ p_1 = e'$, where $X \times X \xrightarrow{p_1} X$ denotes the first projection. Consequently, $\delta_Y \circ m \circ e \circ p_1 = m' \circ h \circ e \circ p_1 = m' \circ e' = g$ and $m \circ e \circ p_1 \circ \delta_X = m \circ e \circ id_X = f$. Thus, since δ_Y is a monomorphism, $m \circ e \circ p_1$ is the sought unique diagonal and so $\delta_X \perp \delta_Y$. $\square$

We observe that since the morphism h in the above proof is a monomorphism , it also shows that $X^2||Y^2$ implies that $X||Y$.

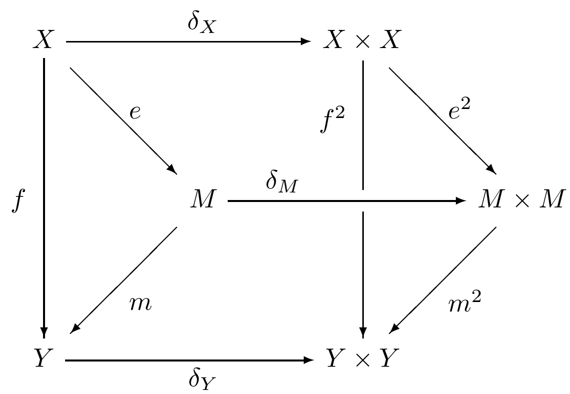
Let $\mathcal{E}$ denote the class of singleton $\mathbf{E}$-sinks.

LEMMA 2.8

If $\mathcal{E}$ is closed under the formation of squares, then for every $X, Y \in \mathcal{X}$, we have that $\delta_X \perp \delta_Y$ implies that $X||Y$.

Proof:

Let us consider the following commutative diagram:



where (e, m) is the $(\mathbf{E}, \mathcal{M})$-factorization of f.

By assumption there is a unique diagonal d satisfying $d \circ \delta_X = f$ and $\delta_Y \circ d = f \times f$. Let $X \times X \xrightarrow{p_1} X$, $M \times M \xrightarrow{q_1} M$ and $Y \times Y \xrightarrow{r_1} Y$ denote first projections. Then we have that $m^2 \circ \delta_M \circ q_1 \circ e^2 = \delta_Y \circ m \circ q_1 \circ e^2 = \delta_Y \circ m \circ e \circ p_1 = \delta_Y \circ f \circ p_1 = \delta_Y \circ r_1 \circ f^2 = \delta_Y \circ r_1 \circ \delta_Y \circ d = \delta_Y \circ d = f^2 = m^2 \circ e^2$. Since $m^2 \in \mathcal{M}$ and by assumption $e^2 \in \mathcal{E}$, it follows that $\delta_M \circ q_1 \circ e^2 = e^2 \in \mathcal{E}$. Hence, the $\mathcal{E}$-morphism e^2 factors through the $\mathcal{M}$-morphism δ_M, which from the general properties of $(\mathbf{E}, \mathcal{M})$-factorization structures implies that δ_M is an isomorphism. Thus, $M \simeq M^2$, which, as it is easily seen, implies that M is preterminal and so f is constant. $\square$

REMARK 2.9

It is easy to verify that for $\mathcal{B} \in S(\mathcal{X})$, $r(\mathcal{B})$ is closed under the formation of monosources and so in particular under the formation of products. Consequently, $A||B$ implies that $A||B^2$. However, in general $l(\mathcal{A})$ for $\mathcal{A} \in S(\mathcal{X})^{\mathbf{op}}$ may fail to be closed under the formation of squares. A result in this direction was presented in [CT, Corollary 6.2].

As a consequence of the previous two lemmas and of the above remark, we obtain the following:

PROPOSITION 2.10

If $\mathcal{E}$ is closed under the formation of squares and $X||Y$ implies $X^2||Y$ for every $X, Y \in \mathcal{X}$, then $X||Y$ if and only if $\delta_X \perp \delta_Y$. $\square$

COROLLARY 2.11

If $\mathcal{E}$ is closed under the formation of squares and if $l(\{Y\})$ is closed under the formation of squares for every $\mathcal{X}$-object Y, then $X||Y$ if and only if $\delta_X \perp \delta_Y$. $\square$

There are several regularly used properties of the category $\mathcal{X}$ under which the conditions in the above corollary hold. For instance, if $\mathcal{E}$ is pullback stable, then as a consequence it is closed under squares. This last condition implies that preterminal objects are closed under $\mathcal{E}$-images, so by using [CT, Corollary 6.2] we can conclude that under appropriate hypotheses our description in terms of morphism orthogonality of the composition of the two Galois connections arising from the case of $\mathcal{N}$ consisting of diagonal morphisms turns out to agree with the one given in terms of constant morphisms, which yields as a special case what was already shown in [CT]. We formalize these thoughts under the following:

THEOREM 2.12

Assume that $\mathcal{E}$ is pullback stable and that $\mathcal{X}$ has squares and enough quasipoints. If $\mathcal{N}$ consists of diagonal morphisms, then the two Galois connections $S(\mathcal{X}) \underset{\lambda_{\mathcal{N}}}{\overset{\rho_{\mathcal{N}}}{\rightleftarrows}} S(\mathcal{X})^{\mathbf{op}}$ *and* $S(\mathcal{X}) \underset{l}{\overset{r}{\rightleftarrows}} S(\mathcal{X})^{\mathbf{op}}$ *coincide.* □

3 A BUTTERFLY OF GALOIS CONNECTIONS

The special case of $\mathcal{N}$ consisting of diagonal morphisms in the Galois connections of the previous Section was used by Clementino and Tholen ([CT]) to introduce a notion of connectedness with respect to a closure operator. Independently, another approach to a general theory of connectedness and disconnectedness was presented by Castellini ([C_{1-6}]). In this Section we provide the appropriate link between the two theories.

We begin by recalling the main results in Castellini's approach. Let $\mathcal{N}$ be a subclass of $\mathcal{M}$. The following two propositions were proved in [C_1].

PROPOSITION 3.1

Let $CL(\mathcal{X},\mathcal{M}) \xrightarrow{D_{\mathcal{N}}} S(\mathcal{X})^{\mathbf{op}}$ *and* $S(\mathcal{X})^{\mathbf{op}} \xrightarrow{T_{\mathcal{N}}} CL(\mathcal{X},\mathcal{M})$ *be defined by:*

$D_{\mathcal{N}}(C) = \{X \in \mathcal{X} :$ *every* $n \in \mathcal{N}_X$ *is* C*-closed*$\}$

$T_{\mathcal{N}}(\mathcal{A}) = Sup\{C \in CL(\mathcal{X},\mathcal{M}) : D_{\mathcal{N}}(C) \supseteq \mathcal{A}\}$.

Then, $CL(\mathcal{X},\mathcal{M}) \underset{T_{\mathcal{N}}}{\overset{D_{\mathcal{N}}}{\rightleftarrows}} S(\mathcal{X})^{\mathbf{op}}$ *is a Galois connection.* □

PROPOSITION 3.2

Let $CL(\mathcal{X},\mathcal{M}) \xrightarrow{I_{\mathcal{N}}} S(\mathcal{X})$ *and* $S(\mathcal{X}) \xrightarrow{J_{\mathcal{N}}} CL(\mathcal{X},\mathcal{M})$ *be defined by:*

$I_{\mathcal{N}}(C) = \{X \in \mathcal{X} :$ *every* $n \in \mathcal{N}_X$ *is* C*-dense*$\}$

$J_{\mathcal{N}}(\mathcal{B}) = Inf\{C \in CL(\mathcal{X},\mathcal{M}) : I_{\mathcal{N}}(C) \supseteq \mathcal{B}\}$.

Then, $S(\mathcal{X}) \underset{I_{\mathcal{N}}}{\overset{J_{\mathcal{N}}}{\rightleftarrows}} CL(\mathcal{X},\mathcal{M})$ *is a Galois connection.* □

In [C$_1$], some characterizations of the functions $T_{\mathcal{N}}$ and $J_{\mathcal{N}}$ were also presented. For reference purposes we collect them under the following:

PROPOSITION 3.3

For every $\mathcal{A} \in S(\mathcal{X})^{\mathbf{op}}$ *and* $\mathcal{M}$*-subobject* $M \xrightarrow{m} X$, *with* $X \in \mathcal{X}$, *we have that*

$$m^{T_{\mathcal{N}}(\mathcal{A})} = inf\{f^{-1}(n) : Y \in \mathcal{A}, X \xrightarrow{f} Y, N \xrightarrow{n} Y \in \mathcal{N}_Y \text{and } m \leq f^{-1}(n)\}.$$

Moreover, for every $\mathcal{B} \in S(\mathcal{X})$ *and* $\mathcal{M}$*-subobject* $M \xrightarrow{m} Y$, *with* $Y \in \mathcal{X}$, *we have that*

$$m^{J_{\mathcal{N}}(\mathcal{B})} = sup\Big(\{m\} \cup \{(id_X)_f : X \in \mathcal{B}, X \xrightarrow{f} Y \text{ and } \exists n \in \mathcal{N}_X \text{ with } n_f \leq m\}\Big).$$

□

DEFINITION 3.4 ([C$_5$])

A morphism $X \xrightarrow{f} Y$ is $\mathcal{N}$-dependent if for every $n \in \mathcal{N}_X$ and every $p \in \mathcal{N}_Y$, $n_f \leq p$ implies $f^{-1}(p) \simeq id_X$.

Clearly, the above definition yields the following relation among $\mathcal{X}$-objects: $X\mathcal{R}_{\mathcal{N}}Y$ if and only if every morphism $X \xrightarrow{f} Y$ is $\mathcal{N}$-dependent. This in turn yields a Galois connection $S(\mathcal{X}) \underset{\nabla_{\mathcal{N}}}{\overset{\Delta_{\mathcal{N}}}{\rightleftarrows}} S(\mathcal{X})^{\mathbf{op}}$ where for $\mathcal{A} \in S(\mathcal{X})$, $\Delta_{\mathcal{N}}(\mathcal{A}) = \{Y \in \mathcal{X} : \forall X \in \mathcal{A}, X \xrightarrow{f} Y \text{ is } \mathcal{N}\text{-dependent}\}$ and for $\mathcal{B} \in S(\mathcal{X})^{\mathbf{op}}$, $\nabla_{\mathcal{N}}(\mathcal{B}) = \{X \in \mathcal{X} : \forall Y \in \mathcal{B}, X \xrightarrow{f} Y \text{ is } \mathcal{N}\text{-dependent}\}$.

In [C$_5$] the following result was proved:

THEOREM 3.5

Let $\mathcal{N}$ *be a subclass of* $\mathcal{M}$. *Then the Galois connection* $S(\mathcal{X}) \underset{\nabla_{\mathcal{N}}}{\overset{\Delta_{\mathcal{N}}}{\rightleftarrows}} S(\mathcal{X})^{\mathbf{op}}$ *factors through* $CL(\mathcal{X},\mathcal{M})$ *via the Galois connections* $S(\mathcal{X}) \underset{I_{\mathcal{N}}}{\overset{J_{\mathcal{N}}}{\rightleftarrows}} CL(\mathcal{X},\mathcal{M})$ *and* $CL(\mathcal{X},\mathcal{M}) \underset{T_{\mathcal{N}}}{\overset{D_{\mathcal{N}}}{\rightleftarrows}} S(\mathcal{X})^{\mathbf{op}}$. □

We next recall the two definitions of C-connectedness at stake.

DEFINITION 3.6 ([C$_4$])

An $\mathcal{X}$-object X is called $(C,\mathcal{N})$-connected if $X \in I_{\mathcal{N}}(T_{\mathcal{N}}(D_{\mathcal{N}}(C))) = \nabla_{\mathcal{N}}(D_{\mathcal{N}}(C))$.

DEFINITION 3.7 ([CT])

An $\mathcal{X}$-object X is called C-connected if the diagonal morphism $X \xrightarrow{\delta_X} X \times X$ is C-dense.

REMARK 3.8

It is an easy consequence of Proposition 3.3 that for $\mathcal{A} = \mathcal{B} = \mathcal{X}$, the functions $T_{\mathcal{N}}$ and $J_{\mathcal{N}}$ agree with the functions $P_{\mathcal{N}}$ and $Q_{\mathcal{N}}$, respectively. However, for arbitrary subcategories, the definitions are different but in a symmetric way. Precisely, while in the definition of $P_{\mathcal{N}}$ and $Q_{\mathcal{N}}$, the dependence on the subcategory applies to *domains* of $\mathcal{N}$-subobjects, in the case of $T_{\mathcal{N}}$ and $J_{\mathcal{N}}$ this dependence applies to *codomains* of $\mathcal{N}$-subobjects. This very difference, dependence on domain versus codomain, is the main factor that distinguishes the two approaches to connectedness. This is already evident in Definitions 3.6 and 3.7 and we expand thereon below.

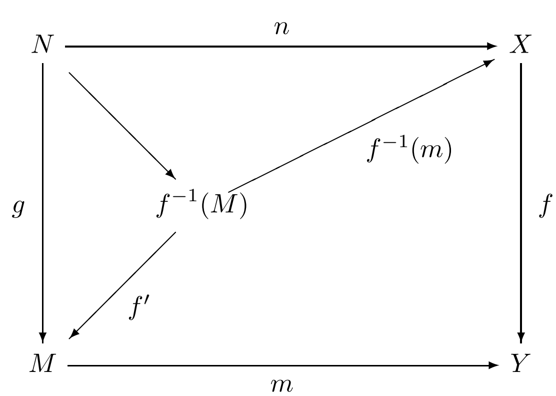
In order to find a more detailed relation between the two definitions, we reinterpret the notion of $\mathcal{N}$-dependent morphism in terms of morphism orthogonality.

PROPOSITION 3.9

Any morphism $X \xrightarrow{f} Y$ is $\mathcal{N}$-dependent if and only if every commutative square $f \circ n = m \circ g$, with $n, m \in \mathcal{N}$ has a diagonal. Thus, for $X, Y \in \mathcal{X}$, $X \mathcal{R}_{\mathcal{N}} Y$ if and only if $\mathcal{N}_X \perp \mathcal{N}_Y$.

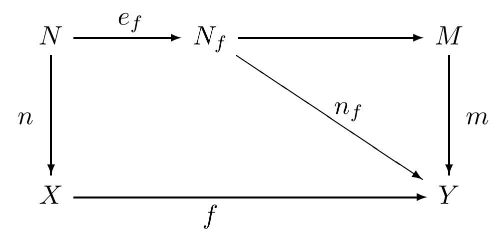
Proof:

($\Rightarrow$). Let us consider the following commutative diagram:

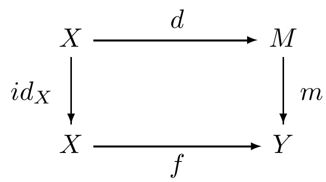


where $n, m \in \mathcal{N}$. Using the $(\mathbf{E}, \mathcal{M})$-diagonalization property, it is easily seen that $n_f \leq m$ and so $X \mathcal{R}_{\mathcal{N}} Y$ implies that $f^{-1}(m)$ is an isomorphism. Hence, $d = f' \circ (f^{-1}(m))^{-1}$ is the sought diagonal.

($\Leftarrow$). Let $X \xrightarrow{f} Y$ be a morphism and let $n \in \mathcal{N}_X$ and $m \in \mathcal{N}_Y$ be such that $n_f \leq m$. Hence, we have the following commutative diagram:



Since $\mathcal{N}_X \perp \mathcal{N}_Y$, there is a diagonal $X \xrightarrow{d} M$. The fact that m is a monomorphism implies that the following diagram is a pullback:



Thus, we conclude that $X\mathcal{R}_{\mathcal{N}}Y$. ☐

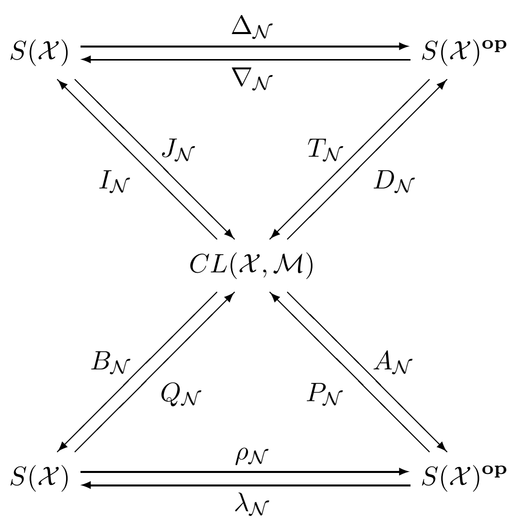
As a consequence of the above proposition, the Galois connection $S(\mathcal{X}) \underset{\nabla_{\mathcal{N}}}{\overset{\Delta_{\mathcal{N}}}{\rightleftarrows}} S(\mathcal{X})^{\mathbf{op}}$ can be described in terms of orthogonality, that is for $\mathcal{A} \in S(\mathcal{X})$ and $\mathcal{B} \in S(\mathcal{X})^{\mathbf{op}}$ we have:

$$\Delta_{\mathcal{N}}(\mathcal{A}) = \{Y \in \mathcal{X} : \forall X \in \mathcal{A}, \mathcal{N}_X \perp \mathcal{N}_Y\}$$

$$\nabla_{\mathcal{N}}(\mathcal{B}) = \{X \in \mathcal{X} : \forall Y \in \mathcal{B}, \mathcal{N}_X \perp \mathcal{N}_Y\}.$$

This contrasts with the definitions of $\rho_{\mathcal{N}}$ and $\lambda_{\mathcal{N}}$, and again bears out the difference of domain versus codomain.

The following "butterfly" illustrates the symmetry that arises from Theorems 2.5 and 3.5.



REMARK 3.10

(a) The upper triangle summarizes Castellini's approach to connectedness and the lower one extends the approach taken by Clementino and Tholen.

(b) Theorem 2.12 reveals how the lower triangle with $\mathcal{N}$ the class of diagonal morphisms reduces to the classical approach to connectedness, using constant morphisms into certain subcategories of $\mathcal{X}$. However, if $\mathcal{N} = \{T \xrightarrow{m} X : T \text{ is terminal }, m \in \mathcal{M}\}$ then we have that $X||Y$ if and only if $\mathcal{N}_X \perp \mathcal{N}_Y$, provided that $\mathcal{N}_X \neq \emptyset$. (This can be obtained by putting together Proposition 3.9 and [C$_5$, Lemma 3.4], where an even larger class $\mathcal{N}$ is used.) Thus, for this choice of $\mathcal{N}$, the upper triangle also reduces to the classical approach to connectedness.

(c) The above diagram also puts into perspective the difference in philosophy between the two connetedness notions. The Clementino-Tholen approach defines connectedness as a right fixed point of the Galois connection $S(\mathcal{X}) \underset{B_\mathcal{N}}{\overset{Q_\mathcal{N}}{\rightleftarrows}} CL(\mathcal{X}, \mathcal{M})$ in the case that $\mathcal{N}$ consists of diagonal morphisms. However, Castellini's approach does not use the symmetric counterpart $I_\mathcal{N}(C)$ for the definition of connectedness but uses $I_\mathcal{N}(T_\mathcal{N}(D_\mathcal{N}(C)))$ instead. This is a fixed point of the Galois connection $S(\mathcal{X}) \underset{\nabla_\mathcal{N}}{\overset{\Delta_\mathcal{N}}{\rightleftarrows}} S(\mathcal{X})^{\mathbf{op}}$ and so necessarily a fixed point of $S(\mathcal{X}) \underset{I_\mathcal{N}}{\overset{J_\mathcal{N}}{\rightleftarrows}} CL(\mathcal{X}, \mathcal{M})$, while $I_\mathcal{N}(C)$ need not be a fixed point of $S(\mathcal{X}) \underset{\nabla_\mathcal{N}}{\overset{\Delta_\mathcal{N}}{\rightleftarrows}} S(\mathcal{X})^{\mathbf{op}}$.

(d) We also observe that the diagonals of the above diagram, when composed, automatically yield two new Galois connections that can still be described via orthogonality and that not surprisingly, switch from domains to codomains. Precisely, we have that for every $\mathcal{A} \in S(\mathcal{X})^{\mathbf{op}}$, $B_\mathcal{N}(T_\mathcal{N}(\mathcal{A})) = \{X : \mathcal{N}^X \perp \mathcal{N}_A, \forall A \in \mathcal{A}\}$ and $I_\mathcal{N}(P_\mathcal{N}(\mathcal{A})) = \{X : \mathcal{N}_X \perp \mathcal{N}^A, \forall A \in \mathcal{A}\}$. Moreover, for every $\mathcal{B} \in S(\mathcal{X})$, $D_\mathcal{N}(Q_\mathcal{N}(\mathcal{B})) = \{X : \mathcal{N}^B \perp \mathcal{N}_X, \forall B \in \mathcal{B}\}$ and $A_\mathcal{N}(J_\mathcal{N}(\mathcal{B})) = \{X : \mathcal{N}_B \perp \mathcal{N}^X, \forall B \in \mathcal{B}\}$.

Other interesting compositions can be considered, although they do not (automatically) yield Galois connections. For instance, $B_\mathcal{N}(T_\mathcal{N}(D_\mathcal{N}(J_\mathcal{N}(\mathcal{B})))) = \{X : \mathcal{N}^X \perp \mathcal{N}_Y, \forall Y \in D_\mathcal{N}(J_\mathcal{N}(\mathcal{B}))\} = \{X : \mathcal{N}^X \perp \mathcal{N}_Y, \forall Y : \mathcal{N}_Y \perp \mathcal{N}_B, \forall B \in \mathcal{B}\} = \{X : \mathcal{N}^X \subseteq (\mathcal{N}_B^\perp \cap \mathcal{N})_\perp, \forall B \in \mathcal{B}\}$. Notice that here we have used the notations: $\mathcal{N}^\perp = \{f : f \perp n, \forall n \in \mathcal{N}\}$ and symmetrically $\mathcal{N}_\perp = \{f : n \perp f, \forall n \in \mathcal{N}\}$.

To the reader who is interested in seeing examples of the above mentioned connectedness notions we recommend [CT] and the sequence [C$_{1-6}$].

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Gabriele Castellini, Department of Mathematics, University of Puerto Rico, Mayagüez campus, P.O. Box 9018, Mayagüez, PR 00681-9018, U.S.A. e-mail: castell@cs.upr.clu.edu

David Holgate, Department of Mathematics, University of Stellenbosch, Stellenbosch 7600, South Africa. e-mail: dh2@land.sun.ac.za