REGULAR CLOSURE OPERATORS AND COMPACTNESS

G. Castellini

Résumé. Une légère modification de la notion de compacité relativement à un opérateur de fermeture permet d'étendre à la catégorie TOP des espaces topologiques divers résultats sur les opérateurs de fermeture réguliers obtenus pour la catégorie AB des groups abéliens. Ainsi les épimorphismes dans les sous-catégories des objets compacts ou compacts-séparés pour un opérateur de fermeture régulier additif sont surjectifs. L'auteur montre aussi que sous certaines conditions sur une sous-catégorie $\mathcal{A}$ de TOP, la sous-catégorie engendrée par les objets compacts-séparés pour l'opérateur de fermeture régulier induit sur $\mathcal{A}$ a plusieurs bonnes propriétés normalement obtenues dans des catégories algébriques.

INTRODUCTION

Let $\mathcal{A}$ be a subcategory of a given category $\mathcal{X}$. The notion of compactness with respect to a closure operator introduced in [2] (cf. also [6] and [7]) seems to yield more interesting results if, in the case of a regular closure operator induced by $\mathcal{A}$, we restrict our attention to objects of the subcategory only. This allows us to prove that in **AB** and **TOP** the epimorphisms in subcategories of compact and compact-separated objects with respect to a regular closure operator are surjective. Moreover, we are able to extend Theorem 2.6 of [2], in a modified form, to the categories **TOP**, **GR** (groups) and **TG** (topological groups).

Let $\mathcal{A}$ be a subcategory of **TOP**. The behavior of compact Hausdorff topological spaces gives rise to the question of whether the subcategory $\mathrm{Comp}_{\mathcal{X}}(\mathcal{A})\cap\mathcal{A}$ of compact-separated objects with respect to $[\ ]_{\mathcal{A}}$ might form an algebraic category in the sense of [9]. Unfortunately, the answer in general is no and the subcategory $\mathbf{TOP_1}$ of T_1 topological spaces provides the needed counterexample. As a matter of fact, $\mathrm{Comp}_{\mathbf{TOP}}(\mathbf{TOP_1})\cap\mathbf{TOP_1} = \mathbf{TOP_1}$, which is not an algebraic category. However, such a category has coequalizers and the forgetful functor $U\colon \mathbf{TOP_1} \to \mathbf{SET}$ has a left adjoint and preserves regular epimorphisms. In the last section of the paper we show that, under certain assumptions on the subcategory $\mathcal{A}$, the above mentioned properties of $\mathbf{TOP_1}$ are normally satisfied by any subcategory of **TOP** of the form $\mathrm{Comp}_{\mathcal{X}}(\mathcal{A})\cap\mathcal{A}$.

All the subcategories will be full and isomorphism closed.

We use the terminology of [9] throughout.

1 PRELIMINARIES

Throughout we consider a category $\mathcal{X}$ and a fixed class $\mathcal{M}$ of $\mathcal{X}$-monomorphisms, which contains all $\mathcal{X}$-isomorphisms. It is assumed that:

(1) $\mathcal{M}$ is closed under composition

(2) Pullbacks of $\mathcal{M}$-morphisms exist and belong to $\mathcal{M}$, and multiple pullbacks of (possibly large) families of $\mathcal{M}$-morphisms with common codomain exist and belong to $\mathcal{M}$.

In addition, we require $\mathcal{X}$ to have equalizers and $\mathcal{M}$ to contain all regular monomorphisms.

One of the consequences of the above assumptions is that there is a uniquely determined class $\mathcal{E}$ of morphisms in $\mathcal{X}$ such that ($\mathcal{E}$,$\mathcal{M}$) is a factorization structure for morphisms in $\mathcal{X}$ (cf. [5]).

We regard $\mathcal{M}$ as a full subcategory of the arrow category of $\mathcal{X}$, with the codomain functor from $\mathcal{M}$ to $\mathcal{X}$ denoted by U. Since U is faithful, $\mathcal{M}$ is concrete over $\mathcal{X}$.

As in [5], by a *closure operator* on $\mathcal{X}$ (with respect to $\mathcal{M}$) we mean a pair $C = (\gamma, [\ ]_C)$, where $[\ ]_C$ is an endofunctor on $\mathcal{M}$ that satisfies $U[\ ]_C = U$, and γ is a natural transformation from $id_{\mathcal{M}}$ to $[\ ]_C$ that satisfies $(id_U)\gamma = id_U$.

Thus, given a closure operator $C = (\gamma, [\ ]_C)$, every member m of $\mathcal{M}$ has a canonical factorization

$$\begin{array}{ccc} M & \xrightarrow{]m[_C^X} & [M]_C^X \\ & {\scriptstyle m}\searrow & \downarrow{\scriptstyle [m]_C^X} \\ & & X \end{array}$$

where $[m]_C^X = F(m)$ is called the C-*closure* of m, and $]m[_C^X$ is the domain of the m-component of γ. Subscripts and superscripts will be omitted when not necessary. Notice that, in particular, $[\ ]_C$ induces an order-preserving increasing function on the $\mathcal{M}$-subobject lattice of every $\mathcal{X}$-object. Also, these functions are related in the following sense: if p is the pullback of a morphism $m \in \mathcal{M}$ along some $\mathcal{X}$-morphism f, and q is the pullback of $[m]_C$ along f, then $[p]_C \leq q$. Conversely, every family of functions on the $\mathcal{M}$-subobject lattices that has the above properties uniquely determines a closure operator.

Given a closure operator C, we say that $m \in \mathcal{M}$ is C-*closed* if $]m[_C$ is an isomorphism. An $\mathcal{X}$-morphism f is called C-*dense* if for every $(\mathcal{E}, \mathcal{M})$-factorization (e, m) of f we have that $[m]_C$ is an isomorphism. We call C *idempotent* provided that $[m]_C$ is C-closed for every $m \in \mathcal{M}$. C is called *weakly hereditary* if $]m[_C$ is C-dense for every $m \in \mathcal{M}$. The class of all C-closed $\mathcal{M}$-subobjects and the class of all C-dense $\mathcal{X}$-morphisms will be denoted by $\mathcal{M}^C$ and $\mathcal{E}^C$, respectively. If m and n are $\mathcal{M}$-subobjects of the same object X, with $m \leq n$ and m_n denotes the morphism such that $n \circ m_n = m$, then C is called *hereditary* if $n \circ [m_n]_C \simeq n \cap [m]_C$ holds for every $X \in \mathcal{X}$ and for every pair of $\mathcal{M}$-subobjects of X, m and n with $m \leq n$. C is called *additive* if it preserves finite suprema, i.e., $sup([m]_C^X, [n]_C^X) \simeq [sup(m,n)]_C^X$ for every pair m,n of $\mathcal{M}$-subobjects of the same object X.

For more background on closure operators see, e.g., , [1], [3], [4], [5], [8] and [10].

For every (idempotent) closure operator F let $D(F)$ be the class of all $\mathcal{X}$-objects A that satisfy the following condition: whenever $M \xrightarrow{m} X$ belongs to $\mathcal{M}$ and $X \underset{s}{\overset{r}{\rightrightarrows}} A$ satisfy $r \circ m = s \circ m$, then $r \circ [m]_F = s \circ [m]_F$. If $\mathcal{X}$ has squares, this is equivalent to requiring the diagonal $A \xrightarrow{\Delta_A} A \times A$ to be F-closed. D(F) is called the class of F-separated objects of $\mathcal{X}$.

A special case of an idempotent closure operator arises in the following way. Given any class $\mathcal{A}$ of $\mathcal{X}$-objects and $M \xrightarrow{m} X$ in $\mathcal{M}$, define $[m]_{\mathcal{A}}$ to be the intersection of all equalizers of pairs of $\mathcal{X}$-morphisms r, s from X to some $\mathcal{A}$-object Y that satisfy $r \circ m = s \circ m$, and let $]m[_{\mathcal{A}} \in \mathcal{M}$ be the unique $\mathcal{X}$-morphism by which m factors through $[m]_{\mathcal{A}}$. It is easy to see that $(] \, [_{\mathcal{A}}, [\,]_{\mathcal{A}})$ forms an idempotent closure operator. Since this generalizes the Salbany construction of closure operators induced by classes of topological spaces, cf. [11], we will often refer to it as the Salbany-type closure operator induced by $\mathcal{A}$. In [5] such a type of closure operator was called *regular*. To simplify the notation, instead of "$[\,]_{\mathcal{A}}$-dense" and "$[\,]_{\mathcal{A}}$-closed" we usually write "$\mathcal{A}$-dense" and "$\mathcal{A}$-closed", respectively.

Notice that the objects of $\mathcal{A}$ are always $[\,]_{\mathcal{A}}$-separated (cf. [3]).

iCL$(\mathcal{X}, \mathcal{M})$ will denote the collection of all idempotent closure operators on $\mathcal{M}$, pre-ordered as follows: $C \sqsubseteq D$ if $[m]_C \leq [m]_D$ for all $m \in \mathcal{M}$ (where $\leq$ is the usual order on subobjects).

2 BASIC DEFINITIONS AND PRELIMINARY RESULTS

In what follows $\mathcal{X}$ will be a category with finite products and $\mathcal{A}$ will be one of its full and isomorphism-closed subcategories.

Definition 2.1. An $\mathcal{X}$-morphism $X \xrightarrow{f} Y$ is said to be $\mathcal{A}$-*closed preserving*, if for every $\mathcal{A}$-closed $\mathcal{M}$-subobject $M \xrightarrow{m} X$, in the $(\mathcal{E},\mathcal{M})$-factorization $m_1 \circ e_1 = f \circ m$, m_1 is $\mathcal{A}$-closed.

Definition 2.2. We say that an $\mathcal{X}$-object X is $\mathcal{A}$-*compact with respect to* $\mathcal{A}$ if for every $\mathcal{A}$-object Z, the projection $X \times Z \xrightarrow{\pi_Z} Z$ is $\mathcal{A}$-closed preserving.

$\mathrm{Comp}_{\mathcal{X}}(\mathcal{A})$ will denote the subcategory of all $\mathcal{A}$-compact objects with respect to $\mathcal{A}$ and $\mathrm{Comp}_{\mathcal{X}}(\mathcal{A}) \cap \mathcal{A}$ will be called the subcategory of compact-separated objects with respect to $\mathcal{A}$.

Notice that a more general version of Definition 2.2 has been recently introduced by Dikranjan and Giuli in [7]. However, in the context of this paper, since we are only dealing with idempotent closure operators, Definition 2.2 can be seen as a special case of the notion of (C,$\mathcal{A}$)-compactness that appears in [7]. In our case C is the Salbany-type closure operator induced by the subcategory $\mathcal{A}$.

A relevant number of examples of (C,$\mathcal{A}$)-compactness can be found in [6] and [7]. At the end of this section, we will only list some examples where C is the Salbany-type closure operator induced by the subcategory $\mathcal{A}$.

Notice also that Definition 2.2 only slightly differs from our previous definition of compactness with respect to a closure operator that appeared in [2]. The difference being that we now require that only the projections onto objects of the subcategory $\mathcal{A}$ be $\mathcal{A}$-closed preserving.

The proofs of the following four results are very similar to the ones in [2], so we omit them.

Proposition 2.3. *If $M \in Comp_{\mathcal{X}}(\mathcal{A})$ and M is an $\mathcal{M}$-subobject of $X \in \mathcal{A}$, then M is $\mathcal{A}$-closed.* □

Proposition 2.4.

(a) Let $\mathcal{A}$ be a subcategory of $\mathcal{X}$ such that $[\ ]_{\mathcal{A}}$ is weakly hereditary. Then $Comp_{\mathcal{X}}(\mathcal{A})$ is closed under $\mathcal{A}$-closed $\mathcal{M}$-subobjects.

(b) Let $\mathcal{A}$ be a subcategory of $\mathcal{X}$ closed under finite products and $\mathcal{M}$-subobjects. If $[\ ]_{\mathcal{A}}$ is weakly hereditary in $\mathcal{A}$, then $Comp_{\mathcal{X}}(\mathcal{A}) \cap \mathcal{A}$ is closed under $\mathcal{A}$-closed $\mathcal{M}$-subobjects. □

Proposition 2.5. *Suppose that for $e \in \mathcal{E}$, the pullback of $e \times 1$ along any $\mathcal{A}$-closed subobject belongs to $\mathcal{E}$. If $X \xrightarrow{f} Y$ is an $\mathcal{X}$-morphism and (e, m) is its $(\mathcal{E}, \mathcal{M})$-factorization, then if $X \in Comp_{\mathcal{X}}(\mathcal{A})$, so does $f(X)$ (where $f(X)$ is the middle object of the $(\mathcal{E}, \mathcal{M})$-factorization).* □

For the next result we assume that $\mathcal{X}$ has arbitrary products and that in the $(\mathcal{E}, \mathcal{M})$-factorization structure of $\mathcal{X}$, $\mathcal{E}$ is a class of epimorphisms.

Definition 2.6. (Cf. [2, Definition 3.2]). Let $\mathcal{A}$ be a subcategory of $\mathcal{X}$. The closure operator $[\ ]_{\mathcal{A}}$ is called compactly productive iff $\mathrm{Comp}_{\mathcal{X}}(\mathcal{A})$ is closed under products.

Proposition 2.7. *(Cf. [2, Proposition 3.4]). Let $\mathcal{A}$ be an extremal epireflective and co-well powered subcategory of $\mathcal{X}$, such that $[\ ]_{\mathcal{A}}$ is weakly hereditary in $\mathcal{A}$. If $[\ ]_{\mathcal{A}}$ is compactly productive, then $Comp_{\mathcal{X}}(\mathcal{A}) \cap \mathcal{A}$ is epireflective in $\mathcal{A}$.* □

We recall the following result from [2]

Proposition 2.8. *(Cf. [2, Proposition 1.16]). Let $\mathcal{X}$ be a regular well-powered category with products and let $\mathcal{A}$ be a subcategory of $\mathcal{X}$ closed under the formation of products and $\mathcal{M}$-subobjects. Then $[\ ]_{\mathcal{A}}$ is weakly hereditary in $\mathcal{A}$ iff the regular monomorphisms in $\mathcal{A}$ are closed under composition.* □

In the following examples we see that some nice and well known categories can be seen as compact-separated objects with respect to a regular closure operator. Notice that we take $(\mathcal{E},\mathcal{M})$ = (epimorphisms,extremal monomorphisms).

Examples 2.9.

(a) Let $\mathcal{X} = \mathbf{TOP}$ and let $\mathcal{A} = \mathbf{TOP_2}$. Then $\mathrm{Comp}_{\mathbf{TOP}}(\mathbf{TOP_2}) \cap \mathbf{TOP_2} = \mathbf{COMP_2}$ (compact Hausdorff topological spaces).
(b) Let $\mathcal{X} = \mathbf{TOP}$ and let $\mathcal{A} = \mathbf{TOP}$. Then $\mathrm{Comp}_{\mathbf{TOP}}(\mathbf{TOP}) \cap \mathbf{TOP} = \mathbf{TOP}$.
(c) Let $\mathcal{X} = \mathbf{TOP}$. For any bireflective subcategory $\mathcal{A}$ of **TOP**, we have that $\mathrm{Comp}_{\mathcal{X}}(\mathcal{A}) \cap \mathcal{A} = \mathcal{A}$.
(d) Let $\mathcal{X} = \mathbf{TOP}$ and let $\mathcal{A} = \mathbf{TOP_0}$. Then $\mathrm{Comp}_{\mathbf{TOP}}(\mathbf{TOP_0}) \cap \mathbf{TOP_0} = \{\text{b-compact topological spaces}\} \cap \mathbf{TOP_0}$ (cf. [6, Example 3.2]).
(e) Let $\mathcal{X} = \mathbf{TOP}$ and let $\mathcal{A} = \mathbf{TOP_1}$. Then $\mathrm{Comp}_{\mathbf{TOP}}(\mathbf{TOP_1}) \cap \mathbf{TOP_1} = \mathbf{TOP_1}$ (cf. Theorem 3.03).
(f) Let $\mathcal{X} = \mathbf{TOP}$ and let $\mathcal{A} = \mathbf{TOP_3}$ or **TYCH**. Then $\mathrm{Comp}_{\mathcal{X}}(\mathcal{A}) \cap \mathcal{A} = \mathbf{COMP_2}$.
(g) Let $\mathcal{X} = \mathbf{GR}$ and let $\mathcal{A} = \mathbf{AB}$. Then $\mathrm{Comp}_{\mathbf{AB}}(\mathbf{AB}) \cap \mathbf{AB} = \mathbf{AB}$.

3 A-COMPACTNESS AND EPIMORPHISMS

In this section, we will be working in the categories **AB**, **TOP**, **GR** and **TG**. In each of these categories $\mathcal{M}$ will be the class of all extremal monomorphisms. Therefore $(\mathcal{E},\mathcal{M})$ = (epimorphisms,extremal monomorphisms).

We start by recalling a result that in a slightly modified form can be found in [2]. The only changes we made were to replace $\mathrm{Comp}(\mathcal{A})$ by $\mathrm{Comp}_{\mathcal{X}}(\mathcal{A})$ and to add the cases $\mathcal{X} = \mathbf{GR}$ and $\mathcal{X} = \mathbf{TG}$. Its proof is not affected by such changes.

Proposition 3.1. *(Cf. [2, Proposition 2.5]). If $\mathcal{A}$ is a subcategory of* **AB**, **TOP**, **GR** *or* **TG** *and $\mathcal{A}$ is contained in $Comp_{\mathcal{X}}(\mathcal{A})$, then the epimorphisms in $\mathcal{A}$ are surjective.* □

A different version of the following theorem for epireflective subcategories of **AB** was proved in [2]. This weakened form for subcategories that are not necessarily epireflective yields an interesting consequence in **AB** and **TOP**.

Theorem 3.2. *Let $\mathcal{A}$ be a subcategory of* **AB** (**TOP**). *Let us consider the following statements:*

(a) $\mathcal{A}$ is closed under quotients

(b) Each $\mathcal{M}$-subobject of an $\mathcal{A}$-object is $\mathcal{A}$-closed

(c) The projections onto objects of $\mathcal{A}$ are $\mathcal{A}$-closed preserving

(d) $Comp_{\mathcal{X}}(\mathcal{A}) = \mathbf{AB}$ ($= \mathbf{TOP}$)

(e) $\mathcal{A} \subseteq Comp_{\mathcal{X}}(\mathcal{A})$

(f) The epimorphisms in $\mathcal{A}$ are surjective.

We have that (a)$\Rightarrow$(b)$\Rightarrow$(c)$\Rightarrow$(d)$\Rightarrow$(e)$\Rightarrow$(f). (f)$\not\Rightarrow$(a).

Proof: (a)$\Rightarrow$(b). Let $M \xrightarrow{m} X$ be an $\mathcal{M}$-subobject of $X \in \mathcal{A}$.

For $\mathcal{A} \subseteq \mathbf{AB}$, consider the pair of morphisms $X \underset{0}{\overset{q}{\rightrightarrows}} X/M$, where q and 0 denote the quotient and the zero-homomorphism, respectively. Clearly $m \simeq equ(q, 0)$.

For $\mathcal{A} \subseteq \mathbf{TOP}$, consider the pair of continuous functions $X \underset{c_M}{\overset{q}{\rightrightarrows}} X/M$ where q is the canonical function onto the quotient set X/M, c_M is the constant morphism into $\{M\}$ and X/M has the quotient topology induced by q. We have that $m \simeq equ(q, c_M)$.

Since $X/M \in \mathcal{A}$ in both cases, we obtain that m is $\mathcal{A}$-closed.

(b)$\Rightarrow$(c). Straightforward.

(c)$\Rightarrow$(d). Straightforward.

(d)$\Rightarrow$(e). Obvious.

(e)$\Rightarrow$(f). It follows from Proposition 3.1.

(f)$\not\Rightarrow$(a). In **AB** take $\mathcal{A} = \mathbf{AC}$ = algebraically compact abelian groups and in **TOP** take $\mathcal{A} = \mathbf{TOP_1}$. □

Corollary 3.3. *If $\mathcal{A}$ is a subcategory of* **AB** *or* **TOP**, *then the epimorphisms in $Comp_{\mathcal{X}}(\mathcal{A})$ are surjective.*

Proof: ¿From Proposition 2.5, $\mathrm{Comp}_{\mathcal{X}}(\mathcal{A})$ is closed under quotients and by applying Theorem 3.2, we get that the epimorphisms in $\mathrm{Comp}_{\mathcal{X}}(\mathcal{A})$ are surjective. □

Notice that the above corollary is not a consequence of Proposition 2.9 of [2], since for $F =[\ ]_{\mathcal{A}}$, $\mathrm{Comp}_{\mathcal{X}}(\mathcal{A})$ is usually larger than Comp(F).

Also notice that if we remove item (a) in Theorem 3.2, the implications (b) trough (e) hold for subcategories of **GR** and **TG** as well.

Furthermore, the notion of compactness presented in this paper allows us to extend the equivalence of some items in Theorem 2.06 of [2] to epireflective subcategories of **TOP**, **GR** and **TG**, as the following theorem shows.

Theorem 3.4. *Let $\mathcal{A}$ be an epireflective subcategory in either* **TOP**, **GR** *or* **TG**. *The following are equivalent:*

(a) $Comp_{\mathcal{X}}(\mathcal{A}) \cap \mathcal{A} = \mathcal{A}$ *and the regular monomorphisms in* $\mathcal{A}$ *are closed under composition*

(b) The epimorphisms in $\mathcal{A}$ *are surjective and* $[\ ]_{\mathcal{A}}$ *is weakly hereditary in* $\mathcal{A}$

(c) Each $\mathcal{M}$*-subobject of an* $\mathcal{A}$*-object is* $\mathcal{A}$*-closed.*

Proof: (a)⇒(b). It follows from Propositions 3.1 and 2.8.

(b)⇒(c). The same proof of e) ⇒ f) in Theorem 2.6 of [2] applies here.

(c)⇒(a). Let us consider the commutative diagram

$$\begin{array}{ccc} X \times Z & \xrightarrow{\pi_Z} & Z \\ {\scriptstyle m}\uparrow & & \uparrow{\scriptstyle m_1} \\ M & \xrightarrow[e_1]{} & M_1 \end{array}$$

where X, $Z \in \mathcal{A}$, (e_1, m_1) is the $(\mathcal{E},\mathcal{M})$-factorization of $\pi_Z \circ m$ and M is $\mathcal{A}$-closed. Clearly, M_1 is $\mathcal{A}$-closed by hypothesis, so π_Z is $\mathcal{A}$-closed preserving, i.e., X is $\mathcal{A}$-compact with respect to $\mathcal{A}$. Since $[\ ]_{\mathcal{A}}$ is weakly hereditary in $\mathcal{A}$, from Proposition 2.8 we get that the regular monomorphisms in $\mathcal{A}$ are closed under composition. □

We next extend, under certain assumptions, the result in Corollary 3.3 to subcategories of the form $\text{Comp}_{\mathcal{X}}(\mathcal{A}) \cap \mathcal{A}$. This generalizes the fact that the epimorphisms in the category of compact Hausdorff topological spaces are surjective.

Proposition 3.5. *Let* $\mathcal{A}$ *be an epireflective subcategory of* **AB**. *Then, the epimorphisms in* $Comp_{\mathcal{X}}(\mathcal{A}) \cap \mathcal{A}$ *are surjective.*

Proof: Let $X \xrightarrow{f} Y$ be an epimorphism in $\text{Comp}_{\mathcal{X}}(\mathcal{A})\cap\mathcal{A}$. Then, from Proposition 2.5, $f(X) \in \text{Comp}_{\mathcal{X}}(\mathcal{A})$. Since $Y \in \mathcal{A}$, $f(X) \xrightarrow{i} Y$ is $\mathcal{A}$-closed (cf. Proposition 2.3). So, $i \simeq equ(f, g)$, with $Y \overset{f}{\underset{g}{\rightrightarrows}} Z$, $Z \in \mathcal{A}$. This implies that $Y/f(X) \in \mathcal{A}$ and again from Proposition 2.5, $Y/f(X) \in \text{Comp}_{\mathcal{X}}(\mathcal{A})$. Let us consider $Y \overset{q}{\underset{0}{\rightrightarrows}} Y/f(X)$. If $f(X) \neq Y$ we would have that $q \circ f = 0 \circ f$ with $q \neq 0$, which contradicts the fact that f is an epimorphism in $\text{Comp}_{\mathcal{X}}(\mathcal{A}) \cap \mathcal{A}$. Therefore f is surjective. □

To show a similar result in **TOP** is a bit more laborious.

Let $Y + Y$ denote the topological sum (coproduct) of two copies of the topological space Y. If M is an extremal subobject of Y, we denote by $Y +_M Y$ the quotient of $Y + Y$ with respect to the equivalence relation $(x, i) \sim (y, j)$, i, $j = 1$, 2 iff either $i \neq j$ and $x = y \in M$ or $(x, i) = (y, j)$ (cf. [4, Definition 1.11]).

Proposition 3.6. *(Cf. [4, Proposition 1.12]). Let* $\mathcal{A}$ *be an extremal epireflective subcategory of* **TOP**. *For every* $Y \in \mathcal{A}$ *and for every extremal subobject* M *of* Y*, the following are equivalent*

(a) $Y +_{M} Y \in \mathcal{A}$
(b) $M = [M]_{\mathcal{A}}$ □

Corollary 3.7. *Let $\mathcal{A}$ be an extremal epireflective subcategory of* **TOP***, let $X \xrightarrow{f} Y$ be a $\mathcal{A}$-morphism and let $M = [f(X)]_{\mathcal{A}}$. Then, $Y +_{M} Y$ belongs to $\mathcal{A}$.*

Proof: It follows directly from Proposition 3.6. □

Lemma 3.8. *Let $\mathcal{A}$ be an extremal epireflective subcategory of* **TOP** *such that $[\ ]_{\mathcal{A}}$ is additive in $\mathcal{A}$. Then, if $Y \in Comp_{\mathcal{X}}(\mathcal{A})$, so does $Y + Y$.*

Proof: Let $Z \in \mathcal{A}$ and let $M \xrightarrow{m} (Y+Y) \times Z$ be $\mathcal{A}$-closed. Notice that $(Y+Y)\times Z$ is homeomorphic to $(Y \times Z) + (Y \times Z)$. Let us call such a homeomorphism i. Thus, $i \circ m$ is the equalizer of two morphisms $(Y \times Z) + (Y \times Z) \overset{f}{\underset{g}{\rightrightarrows}} T$, $T \in \mathcal{A}$. Let f_1, g_1 and f_2, g_2 denote the restrictions of f and g to the first and the second addend of $(Y \times Z) + (Y \times Z)$, respectively. Let $M_1 \xrightarrow{m_1} Y \times Z$ and $M_2 \xrightarrow{m_2} Y \times Z$ be two morphisms such that $m_1 = equ(f_1, g_1)$ and $m_2 = equ(f_2, g_2)$. Then $(i \circ m)(M) = m_1(M_1) + m_2(M_2)$. Let π^1_Z and π^2_Z denote the projections onto Z of the first and the second addend of $(Y \times Z) + (Y \times Z)$ and let $[\pi^1_Z, \pi^2_Z]\colon (Y \times Z) + (Y \times Z) \to Z$ denote the induced continuous function. If π_Z is the usual projection of $(Y+Y)\times Z$ onto Z, then $([\pi^1_Z, \pi^2_Z]) \circ i = \pi_Z$. Now, $(\pi_Z \circ m)(M) = (([\pi^1_Z, \pi^2_Z]) \circ i \circ m)(M) = ([\pi^1_Z, \pi^2_Z])(m_1(M_1)+m_2(M_2)) = \pi^1_Z(m_1(M_1)) \cup \pi^2_Z(m_2(M_2))$. Since $Y \in$Comp$_{\mathcal{X}}(\mathcal{A})$, $\pi^1_Z(m_1(M_1))$ and $\pi^2_Z(m_2(M_2))$ are both $\mathcal{A}$-closed and so is their union, since $[\ ]_{\mathcal{A}}$ is additive in $\mathcal{A}$. □

Proposition 3.9. *Let $\mathcal{A}$ be an extremal epireflective subcategory of* **TOP** *such that $[\ ]_{\mathcal{A}}$ is additive in $\mathcal{A}$. Let $X \xrightarrow{f} Y$ be an $\mathcal{X}$-morphism and let $M = [f(X)]_{\mathcal{A}}$. Then, if $Y \in Comp_{\mathcal{X}}(\mathcal{A})$, so does $Y +_{M} Y$.*

Proof: ¿From Lemma 3.8, $Y + Y \in$ Comp$_{\mathcal{X}}(\mathcal{A})$ and from Proposition 2.5, so does $Y +_{M} Y$. □

Theorem 3.10. *Let $\mathcal{A}$ be an extremal epireflective subcategory of* **TOP** *such that $[\ ]_{\mathcal{A}}$ is additive in $\mathcal{A}$. Then the epimorphisms in $Comp_{\mathcal{X}}(\mathcal{A}) \cap \mathcal{A}$ are surjective.*

Proof: Let $X \xrightarrow{f} Y$ be an epimorphism in Comp$_{\mathcal{X}}(\mathcal{A}) \cap \mathcal{A}$ and let M denote the subspace $[f(X)]_{\mathcal{A}}$. We have that $Y +_{M} Y \in$ Comp$_{\mathcal{X}}(\mathcal{A}) \cap \mathcal{A}$ (cf. Corollary 3.7 and Proposition 3.9). From Proposition 2.5, $f(X) \in$Comp$_{\mathcal{X}}(\mathcal{A})$ and from Proposition 2.3, $f(X)$ is $\mathcal{A}$-closed. Thus, $Y +_{M} Y = Y +_{f(X)} Y$. Let i and j be the left and the right inclusions of Y into $Y + Y$ and let $Y + Y \xrightarrow{q} Y +_{f(X)} Y$ be the quotient map. Clearly, $q \circ i \circ f = q \circ j \circ f$. If f is not surjective, then we have that $q \circ i \neq q \circ j$. This contradicts our assumption of f being an epimorphism in Comp$_{\mathcal{X}}(\mathcal{A}) \cap \mathcal{A}$. □

4 A-COMPACTNESS AND ALGEBRAIC CATEGORIES

It is well known that $\mathbf{COMP_2}$, i.e., the category of compact Hausdorff topological spaces, forms an algebraic category in the sense that $\mathbf{COMP_2}$ has coequalizers and the forgetful functor $U\colon \mathbf{COMP_2} \to \mathbf{SET}$ has a left adjoint and preserves and reflects regular epimorphisms (cf. [9]). It is quite natural to wonder whether this result could be extended in **TOP** to categories of compact-separated objects with respect to a regular closure operator. Unfortunately the subcategory $\mathbf{TOP_1}$ shows that this is not the case. As a matter of fact, $\mathrm{Comp}_{\mathbf{TOP}}(\mathbf{TOP_1}) \cap \mathbf{TOP_1} = \mathbf{TOP_1}$ (cf. Example 2.9(e)) and $\mathbf{TOP_1}$ is not an algebraic category, since the forgetful functor $U\colon \mathbf{TOP_1} \to \mathbf{SET}$ fails to reflect regular epimorphisms. However, the remaining conditions are all satisfied. We will see that, under certain assumptions on the subcategory $\mathcal{A}$, $\mathbf{TOP_1}$ outlines the behavior of $\mathrm{Comp}_{\mathcal{X}}(\mathcal{A}) \cap \mathcal{A}$.

Proposition 4.1. *If $\mathcal{A}$ is an extremal epireflective subcategory of* **TOP***, then $\mathrm{Comp}_{\mathcal{X}}(\mathcal{A}) \cap \mathcal{A}$ has coequalizers.*

Proof: Let $X \overset{f}{\underset{g}{\rightrightarrows}} Y$ be two morphisms in $\mathrm{Comp}_{\mathcal{X}}(\mathcal{A}) \cap \mathcal{A}$ and let $Y \xrightarrow{q} Q$ be their coequalizer in **TOP**. From Proposition 2.5, $Q \in \mathrm{Comp}_{\mathcal{X}}(\mathcal{A})$. Since $\mathcal{A}$ is extremal epireflective in **TOP**, we can consider the reflection $Q \xrightarrow{r} rQ$ of Q in $\mathcal{A}$. From Proposition 2.5 $rQ \in \mathrm{Comp}_{\mathcal{X}}(\mathcal{A})$. Now, it is easily shown that $Y \xrightarrow{r \circ q} rQ$ is the coequalizer of f and g in $\mathrm{Comp}_{\mathcal{X}}(\mathcal{A}) \cap \mathcal{A}$. □

Proposition 4.2. *Let $\mathcal{A}$ be an extremal epireflective subcategory of* **TOP** *such that $[\ ]_{\mathcal{A}}$ is additive in $\mathcal{A}$. Then, the forgetful functor $U\colon \mathrm{Comp}_{\mathcal{X}}(\mathcal{A}) \cap \mathcal{A} \to \mathbf{SET}$ preserves regular epimorphisms.*

Proof: Let $X \xrightarrow{f} Y$ be a regular epimorphism in $\mathrm{Comp}_{\mathcal{X}}(\mathcal{A}) \cap \mathcal{A}$. Then, from Theorem 3.10, f is surjective. Therefore $U(f)$ is a regular epimorphism in **SET**. □

Proposition 4.3. *Let $\mathcal{A}$ be an extremal epireflective and co-well powered subcategory of* **TOP***. Suppose that $[\ ]_{\mathcal{A}}$ is weakly hereditary in $\mathcal{A}$ and compactly productive. Then, the forgetful functor $U\colon \mathrm{Comp}_{\mathcal{X}}(\mathcal{A}) \cap \mathcal{A} \to \mathbf{SET}$ has a left adjoint.*

Proof: The case $\mathcal{A} = \{x\}$ is trivial. So, Let $\mathcal{A} \neq \{x\}$. Let X be a set and let X_d be the discrete topological space with underlying set X. Clearly $X_d \in \mathcal{A}$, since $\mathcal{A}$ is an extremal epireflective subcategory of **TOP**. Let βX be the $\mathcal{A}$-dense-reflection of X_d into $\mathrm{Comp}_{\mathcal{X}}(\mathcal{A}) \cap \mathcal{A}$ (cf. Proposition 2.7) and let $X_d \xrightarrow{\beta_X} \beta X$ be the reflection morphism. If $Y \in \mathrm{Comp}_{\mathcal{X}}(\mathcal{A}) \cap \mathcal{A}$ and $X \xrightarrow{f} UY$ is a morphism in **SET**, then $X_d \xrightarrow{g} Y$ such that $U(g) = f$ is continuous. ¿From Proposition 2.7, there exists a unique $\beta X \xrightarrow{f'} Y$ such that $f' \circ \beta_x = g$ (notice that f' is unique because β_x is a $\mathcal{A}$-epimorphism). Clearly we have that $Uf' \circ U\beta_x = f$. □

The results in Propositions 4.1, 4.2 and 4.3 can be summerized in the following

Theorem 4.4. *Let $\mathcal{A}$ be an extremal epireflective and co-well powered subcategory of* **TOP** *such that $[\ ]_{\mathcal{A}}$ is compactly productive, weakly hereditary in $\mathcal{A}$ and additive in $\mathcal{A}$. Then, $Comp_{\mathcal{X}}(\mathcal{A}) \cap \mathcal{A}$ has coequalizers and the forgetful functor $U\colon Comp_{\mathcal{X}}(\mathcal{A}) \cap \mathcal{A} \to$* **SET** *has a left adjoint and preserves regular epimorphisms.*

□

REFERENCES

[1] G. Castellini, "Closure operators, monomorphisms and epimorphisms in categories of groups," *Cahiers Topologie Geom. Differentielle Categoriques* **27(2)** (1986), 151-167.

[2] G. Castellini, "Compact objects, surjectivity of epimorphisms and compactifications," *Cahiers Topologie Geom. Differentielle Categoriques* **31(1)** (1990), 53-65.

[3] G. Castellini, G. E. Strecker, "Global closure operators vs. subcategories," *Quaestiones Mathematicae* **13** (1990), 417-424.

[4] D. Dikranjan, E. Giuli, "Closure operators induced by topological epireflections," *Coll. Math. Soc. J. Bolyai* **41** (1983), 233-246.

[5] D. Dikranjan, E. Giuli, "Closure operators I," *Topology Appl.* **27** (1987), 129-143.

[6] D. Dikranjan, E. Giuli, "Compactness, minimality and closedness with respect to a closure operator I," *Proceedings of the Conference in Categorical Topology* (Prague, 1988) World Scientific (1989), 284-296.

[7] D. Dikranjan, E. Giuli, "C-perfect morphisms and C-compactness," preprint.

[8] D. Dikranjan, E. Giuli, W. Tholen, "Closure operators II," *Proceedings of the Conference in Categorical Topology* (Prague, 1988) World Scientific (1989), 297-335.

[9] H. Herrlich, G. E. Strecker, *Category Theory*, 2nd ed., Helderman Verlag, Berlin, 1979.

[10] J. Koslowski, "Closure operators with prescribed properties," *Category Theory and its Applications* (Louvain-la-Neuve, 1987) Springer L.N.M. **1248** (1988), 208–220.

[11] S. Salbany, "Reflective subcategories and closure operators," *Proceedings of the Conference in Categorical Topology* (Mannheim, 1975), Springer L.N.M. **540** (1976), 548-565.

Gabriele Castellini
Department of Mathematics
University of Puerto Rico
P.O. Box 5000
Mayagüez, PR 00709-5000
U.S.A.