REGULAR CLOSURE OPERATORS
AND COMPACTNESS

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Résumé. Une légère modification de la notion de compacité relativement à un opérateur de fermeture permet d’étendre à la catégorie TOP des espaces topologiques divers résultats sur les opérateurs de fermeture réguliers obtenus pour la catégorie AB des groupes abéliens. Ainsi les épimorphismes dans les sous-catégories des objets compacts ou compacts-séparés pour un opérateur de fermeture régulier additif sont surjectifs. L’auteur montre aussi que sous certaines conditions sur une sous-catégorie \( A \) de TOP, la sous-catégorie engendrée par les objets compacts-séparés pour l’opérateur de fermeture régulier induit sur \( A \) a plusieurs bonnes propriétés normalement obtenues dans des catégories algébriques.

INTRODUCTION

Let \( A \) be a subcategory of a given category \( \mathcal{X} \). The notion of compactness with respect to a closure operator introduced in [2] (cf. also [6] and [7]) seems to yield more interesting results if, in the case of a regular closure operator induced by \( A \), we restrict our attention to objects of the subcategory only. This allows us to prove that in \( \text{AB} \) and \( \text{TOP} \) the epimorphisms in subcategories of compact and compact-separated objects with respect to a regular closure operator are surjective. Moreover, we are able to extend Theorem 2.6 of [2], in a modified form, to the categories \( \text{TOP}, \text{GR} \) (groups) and \( \text{TG} \) (topological groups).

Let \( A \) be a subcategory of \( \text{TOP} \). The behavior of compact Hausdorff topological spaces gives rise to the question of whether the subcategory \( \text{Comp}_X(A) \cap A \) of compact-separated objects with respect to \( \lceil \cdot \rceil_A \) might form an algebraic category in the sense of [9]. Unfortunately, the answer in general is no and the subcategory \( \text{TOP}_1 \) of \( T_1 \) topological spaces provides the needed counterexample. As a matter of fact, \( \text{Comp}_{\text{TOP}}(\text{TOP}_1) \cap \text{TOP}_1 = \text{TOP}_1 \), which is not an algebraic category. However, such a category has coequalizers and the forgetful functor \( U: \text{TOP}_1 \to \text{SET} \) has a left adjoint and preserves regular epimorphisms. In the last section of the paper we show that, under certain assumptions on the subcategory \( A \), the above mentioned properties of \( \text{TOP}_1 \) are normally satisfied by any subcategory of \( \text{TOP} \) of the form \( \text{Comp}_X(A) \cap A \).

All the subcategories will be full and isomorphism closed.

We use the terminology of [9] throughout.
1 PRELIMINARIES
Throughout we consider a category $\mathcal{X}$ and a fixed class $\mathcal{M}$ of $\mathcal{X}$-monomorphisms, which contains all $\mathcal{X}$-isomorphisms. It is assumed that:

1. $\mathcal{M}$ is closed under composition
2. Pullbacks of $\mathcal{M}$-morphisms exist and belong to $\mathcal{M}$, and multiple pullbacks of (possibly large) families of $\mathcal{M}$-morphisms with common codomain exist and belong to $\mathcal{M}$.

In addition, we require $\mathcal{X}$ to have equalizers and $\mathcal{M}$ to contain all regular monomorphisms.

One of the consequences of the above assumptions is that there is a uniquely determined class $\mathcal{E}$ of morphisms in $\mathcal{X}$ such that $(\mathcal{E}, \mathcal{M})$ is a factorization structure for morphisms in $\mathcal{X}$ (cf. [5]).

We regard $\mathcal{M}$ as a full subcategory of the arrow category of $\mathcal{X}$, with the codomain functor from $\mathcal{M}$ to $\mathcal{X}$ denoted by $U$. Since $U$ is faithful, $\mathcal{M}$ is concrete over $\mathcal{X}$.

As in [5], by a closure operator on $\mathcal{X}$ (with respect to $\mathcal{M}$) we mean a pair $C = (\gamma, [\ ]_C)$, where $[\ ]_C$ is an endofunctor on $\mathcal{M}$ that satisfies $U[\ ]_C = U$, and $\gamma$ is a natural transformation from $\text{id}_\mathcal{M}$ to $[\ ]_C$ that satisfies $(\text{id}_U)\gamma = \text{id}_U$.

Thus, given a closure operator $C = (\gamma, [\ ]_C)$, every member $m$ of $\mathcal{M}$ has a canonical factorization

$$
\begin{array}{cccc}
M & \xrightarrow{m_C} & [M]^X_C \\
\downarrow & & \downarrow [m]^X_C \\
X & & [m]^X_C \\
\end{array}
$$

where $[m]^X_C = F(m)$ is called the $C$- closure of $m$, and $[m]^X_C$ is the domain of the $m$-component of $\gamma$. Subscripts and superscripts will be omitted when not necessary. Notice that, in particular, $[\ ]_C$ induces an order-preserving increasing function on the $\mathcal{M}$-subobject lattice of every $\mathcal{X}$-object. Also, these functions are related in the following sense: if $p$ is the pullback of a morphism $m \in \mathcal{M}$ along some $\mathcal{X}$-morphism $f$, and $q$ is the pullback of $[m]^X_C$ along $f$, then $[p]^X_C \leq q$. Conversely, every family of functions on the $\mathcal{M}$-subobject lattices that has the above properties uniquely determines a closure operator.

Given a closure operator $C$, we say that $m \in \mathcal{M}$ is $C$-closed if $[m]^X_C$ is an isomorphism. An $\mathcal{X}$-morphism $f$ is called $C$-dense if for every $(\mathcal{E}, \mathcal{M})$-factorization $(e, m)$ of $f$ we have that $[m]^X_C$ is an isomorphism. We call $C$ idempotent provided that $[m]^X_C$ is $C$-closed for every $m \in \mathcal{M}$. $C$ is called weakly hereditary if $[m]^X_C$ is $C$-dense for every $m \in \mathcal{M}$. The class of all $C$-closed $\mathcal{M}$-subobjects and the class of all $C$-dense $\mathcal{X}$-morphisms will be denoted by $\mathcal{M}^C$ and $\mathcal{E}^C$, respectively. If $m$ and $n$ are $\mathcal{M}$-subobjects of the same object $X$, with $m \leq n$ and $m_n$ denotes the morphism such that $n \circ m_n = m$, then $C$ is called hereditary if $n \circ [m_n]^X_C \simeq n \cap [m]^X_C$ holds for
every $X \in \mathcal{X}$ and for every pair of $\mathcal{M}$-subobjects of $X$, $m$ and $n$ with $m \leq n$. $C$ is called additive if it preserves finite suprema, i.e., $\sup([m]^X_C, [n]^X_C) \simeq [\sup(m, n)]^X_C$ for every pair $m,n$ of $\mathcal{M}$-subobjects of the same object $X$.

For more background on closure operators see, e.g., [1], [3], [4], [5], [8] and [10].

For every (idempotent) closure operator $F$ let $D(F)$ be the class of all $\mathcal{X}$-objects $A$ that satisfy the following condition: whenever $M \xrightarrow{m} X$ belongs to $\mathcal{M}$ and $X \xrightarrow{r,s} A$ satisfy $r \circ m = s \circ m$, then $r \circ [m]_F = s \circ [m]_F$. If $\mathcal{X}$ has squares, this is equivalent to requiring the diagonal $A \xrightarrow{\Delta A} A \times A$ to be $F$-closed. $D(F)$ is called the class of $F$-separated objects of $X$.

A special case of an idempotent closure operator arises in the following way. Given any class $\mathcal{A}$ of $\mathcal{X}$-objects and $M \xrightarrow{m} X$ in $\mathcal{M}$, define $[m]_\mathcal{A}$ to be the intersection of all equalizers of pairs of $\mathcal{X}$-morphisms $r,s$ from $X$ to some $\mathcal{A}$-object $Y$ that satisfy $r \circ m = s \circ m$, and let $\delta m[\mathcal{A}] \in \mathcal{M}$ be the unique $\mathcal{X}$-morphism by which $m$ factors through $[m]_\mathcal{A}$. It is easy to see that $(\mathcal{A}, [\mathcal{A}])$ forms an idempotent closure operator. Since this generalizes the Salbany construction of closure operators induced by classes of topological spaces, cf. [11], we will often refer to it as the Salbany-type closure operator induced by $\mathcal{A}$. In [5] such a type of closure operator was called regular. To simplify the notation, instead of “[ $m$]_\mathcal{A}$-dense” and “[ $m$]_\mathcal{A}$-closed” we usually write “$\mathcal{A}$-dense” and “$\mathcal{A}$-closed”, respectively.

Notice that the objects of $\mathcal{A}$ are always [ $m$]_\mathcal{A}$-separated (cf. [3]).

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IEL($\mathcal{X}, \mathcal{M}$) will denote the collection of all idempotent closure operators on $\mathcal{M}$, pre-ordered as follows: $C \subseteq D$ if $[m]_C \leq [m]_D$ for all $m \in \mathcal{M}$ (where $\leq$ is the usual order on subobjects).

2 BASIC DEFINITIONS AND PRELIMINARY RESULTS

In what follows $\mathcal{X}$ will be a category with finite products and $\mathcal{A}$ will be one of its full and isomorphism-closed subcategories.

Definition 2.1. An $\mathcal{X}$-morphism $X \xrightarrow{f} Y$ is said to be $\mathcal{A}$-closed preserving, if for every $\mathcal{A}$-closed $\mathcal{M}$-subobject $M \xrightarrow{m} X$ in the $(E,M)$-factorization $m \circ e_1 = f \circ m$, $m_1$ is $\mathcal{A}$-closed.

Definition 2.2. We say that an $\mathcal{X}$-object $X$ is $\mathcal{A}$-compact with respect to $\mathcal{A}$ if for every $\mathcal{A}$-object $Z$, the projection $X \times Z \xrightarrow{\pi_2} Z$ is $\mathcal{A}$-closed preserving.

Comp$_\mathcal{X}(\mathcal{A})$ will denote the subcategory of all $\mathcal{A}$-compact objects with respect to $\mathcal{A}$ and Comp$_\mathcal{X}(\mathcal{A}) \cap \mathcal{A}$ will be called the subcategory of compact-separated objects with respect to $\mathcal{A}$.
Notice that a more general version of Definition 2.2 has been recently introduced by Dikranjan and Giuli in [7]. However, in the context of this paper, since we are only dealing with idempotent closure operators, Definition 2.2 can be seen as a special case of the notion of $(C,A)$-compactness that appears in [7]. In our case $C$ is the Salbany-type closure operator induced by the subcategory $A$.

A relevant number of examples of $(C,A)$-compactness can be found in [6] and [7]. At the end of this section, we will only list some examples where $C$ is the Salbany-type closure operator induced by the subcategory $A$.

Notice also that Definition 2.2 only slightly differs from our previous definition of compactness with respect to a closure operator that appeared in [2]. The difference being that we now require that only the projections onto objects of the subcategory $A$ be $A$-closed preserving.

The proofs of the following four results are very similar to the ones in [2], so we omit them.

**Proposition 2.3.** If $M \in \text{Comp}_X(A)$ and $M$ is an $M$-subobject of $X \in A$, then $M$ is $A$-closed.

**Proposition 2.4.**
(a) Let $A$ be a subcategory of $X$ such that $\lfloor \rfloor_A$ is weakly hereditary. Then $\text{Comp}_X(A)$ is closed under $A$-closed $M$-subobjects.
(b) Let $A$ be a subcategory of $X$ closed under finite products and $M$-subobjects. If $\lfloor \rfloor_A$ is weakly hereditary in $A$, then $\text{Comp}_X(A) \cap A$ is closed under $A$-closed $M$-subobjects.

**Proposition 2.5.** Suppose that for $e \in E$, the pullback of $e \times 1$ along any $A$-closed subobject belongs to $E$. If $X \xrightarrow{f} Y$ is an $X$-morphism and $(e,m)$ is its $(E,M)$-factorization, then if $X \in \text{Comp}_X(A)$, so does $f(X)$ (where $f(X)$ is the middle object of the $(E,M)$-factorization).

For the next result we assume that $X$ has arbitrary products and that in the $(E,M)$-factorization structure of $X$, $E$ is a class of epimorphisms.

**Definition 2.6.** (Cf. [2, Definition 3.2]). Let $A$ be a subcategory of $X$. The closure operator $\lfloor \rfloor_A$ is called compactly productive iff $\text{Comp}_X(A)$ is closed under products.

**Proposition 2.7.** (Cf. [2, Proposition 3.4]). Let $A$ be an extremal epireflective and co-well powered subcategory of $X$, such that $\lfloor \rfloor_A$ is weakly hereditary in $A$. If $\lfloor \rfloor_A$ is compactly productive, then $\text{Comp}_X(A) \cap A$ is epireflective in $A$. 

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We recall the following result from [2]

**Proposition 2.8.** (Cf. [2, Proposition 1.16]). Let \( \mathcal{X} \) be a regular well-powered category with products and let \( \mathcal{A} \) be a subcategory of \( \mathcal{X} \) closed under the formation of products and \( \mathcal{M} \)-subobjects. Then \( \mathcal{A} \) is weakly hereditary in \( \mathcal{A} \) iff the regular monomorphisms in \( \mathcal{A} \) are closed under composition.

In the following examples we see that some nice and well known categories can be seen as compact-separated objects with respect to a regular closure operator. Notice that we take \((\mathcal{E}, \mathcal{M}) = (\text{epimorphisms}, \text{extremal monomorphisms})\).

**Examples 2.9.**

(a) Let \( \mathcal{X} = \text{TOP} \) and let \( \mathcal{A} = \text{TOP}_2 \). Then \( \text{Comp}_{\text{top}}(\text{TOP}_2) \cap \text{TOP}_2 = \text{COMP}_2 \) (compact Hausdorff topological spaces).

(b) Let \( \mathcal{X} = \text{TOP} \) and let \( \mathcal{A} = \text{TOP} \). Then \( \text{Comp}_{\text{top}}(\text{TOP}) \cap \text{TOP} = \text{TOP} \).

(c) Let \( \mathcal{X} = \text{TOP} \). For any bireflective subcategory \( \mathcal{A} \) of \( \text{TOP} \), we have that \( \text{Comp}_\mathcal{X}(\mathcal{A}) \cap \mathcal{A} = \mathcal{A} \).

(d) Let \( \mathcal{X} = \text{TOP} \) and let \( \mathcal{A} = \text{TOP}_0 \). Then \( \text{Comp}_{\text{top}}(\text{TOP}_0) \cap \text{TOP}_0 = \{\text{b-compact topological spaces}\} \cap \text{TOP}_0 \) (cf. [6, Example 3.2]).

(e) Let \( \mathcal{X} = \text{TOP} \) and let \( \mathcal{A} = \text{TOP}_1 \). Then \( \text{Comp}_{\text{top}}(\text{TOP}_1) \cap \text{TOP}_1 = \text{TOP}_1 \) (cf. Theorem 3.03).

(f) Let \( \mathcal{X} = \text{TOP} \) and let \( \mathcal{A} = \text{TOP}_3 \) or \( \text{TYCH} \). Then \( \text{Comp}_\mathcal{X}(\mathcal{A}) \cap \mathcal{A} = \text{COMP}_2 \).

(g) Let \( \mathcal{X} = \text{GR} \) and let \( \mathcal{A} = \text{AB} \). Then \( \text{Comp}_{\text{AB}}(\text{AB}) \cap \text{AB} = \text{AB} \).

3 A-COMPACTNESS AND EPIMORPHISMS

In this section, we will be working in the categories \( \text{AB}, \text{TOP}, \text{GR} \) and \( \text{TG} \). In each of these categories \( \mathcal{M} \) will be the class of all extremal monomorphisms. Therefore \((\mathcal{E}, \mathcal{M}) = (\text{epimorphisms}, \text{extremal monomorphisms})\).

We start by recalling a result that in a slightly modified form can be found in [2]. The only changes we made were to replace \( \text{Comp}(\mathcal{A}) \) by \( \text{Comp}_\mathcal{X}(\mathcal{A}) \) and to add the cases \( \mathcal{X} = \text{GR} \) and \( \mathcal{X} = \text{TG} \). Its proof is not affected by such changes.

**Proposition 3.1.** (Cf. [2, Proposition 2.5]). If \( \mathcal{A} \) is a subcategory of \( \text{AB}, \text{TOP}, \text{GR} \) or \( \text{TG} \) and \( \mathcal{A} \) is contained in \( \text{Comp}_\mathcal{X}(\mathcal{A}) \), then the epimorphisms in \( \mathcal{A} \) are surjective.

A different version of the following theorem for epireflective subcategories of \( \text{AB} \) was proved in [2]. This weakened form for subcategories that are not necessarily epireflective yields an interesting consequence in \( \text{AB} \) and \( \text{TOP} \).
Theorem 3.2. Let $\mathcal{A}$ be a subcategory of $\mathcal{AB}$ (TOP). Let us consider the following statements:

(a) $\mathcal{A}$ is closed under quotients
(b) Each $\mathcal{M}$-subobject of an $\mathcal{A}$-object is $\mathcal{A}$-closed
(c) The projections onto objects of $\mathcal{A}$ are $\mathcal{A}$-closed preserving
(d) $\text{Comp}_X(\mathcal{A}) = \mathcal{AB} (= \text{TOP})$
(e) $\mathcal{A} \subseteq \text{Comp}_X(\mathcal{A})$
(f) The epimorphisms in $\mathcal{A}$ are surjective.

We have that $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d) \Rightarrow (e) \Rightarrow (f)$. $(f) \not\Rightarrow (a)$.

Proof: $(a) \Rightarrow (b)$. Let $M \longrightarrow X$ be an $\mathcal{M}$-subobject of $X \in \mathcal{A}$.

For $\mathcal{A} \subseteq \mathcal{AB}$, consider the pair of morphisms $X \xrightarrow{q} X/M$, where $q$ and $0$ denote the quotient and the zero-homomorphism, respectively. Clearly $m \simeq \text{equ}(q, 0)$.

For $\mathcal{A} \subseteq \text{TOP}$, consider the pair of continuous functions $X \xrightarrow{q} X/M$ where $q$ is the canonical function onto the quotient set $X/M$, $c_M$ is the constant morphism into $\{M\}$ and $X/M$ has the quotient topology induced by $q$. We have that $m \simeq \text{equ}(q, c_M)$.

Since $X/M \in \mathcal{A}$ in both cases, we obtain that $m$ is $\mathcal{A}$-closed.

$(b) \Rightarrow (c)$. Straightforward.

$(c) \Rightarrow (d)$. Straightforward.

$(d) \Rightarrow (e)$. Obvious.

$(e) \Rightarrow (f)$. It follows from Proposition 3.1.

$(f) \not\Rightarrow (a)$. In $\mathcal{AB}$ take $\mathcal{A} = \mathcal{AC}$ = algebraically compact abelian groups and in $\text{TOP}$ take $\mathcal{A} = \text{TOP}_1$.

Corollary 3.3. If $\mathcal{A}$ is a subcategory of $\mathcal{AB}$ or $\text{TOP}$, then the epimorphisms in $\text{Comp}_X(\mathcal{A})$ are surjective.

Proof: From Proposition 2.5, $\text{Comp}_X(\mathcal{A})$ is closed under quotients and by applying Theorem 3.2, we get that the epimorphisms in $\text{Comp}_X(\mathcal{A})$ are surjective.

Notice that the above corollary is not a consequence of Proposition 2.9 of [2], since for $F = \lfloor \mathcal{A} \rfloor$, $\text{Comp}_X(\mathcal{A})$ is usually larger than $\text{Comp}(F)$.

Also notice that if we remove item (a) in Theorem 3.2, the implications (b) through (e) hold for subcategories of $\text{GR}$ and $\text{TG}$ as well.

Furthermore, the notion of compactness presented in this paper allows us to extend the equivalence of some items in Theorem 2.06 of [2] to epireflective subcategories of $\text{TOP}$, $\text{GR}$ and $\text{TG}$, as the following theorem shows.

Theorem 3.4. Let $\mathcal{A}$ be an epireflective subcategory in either $\text{TOP}$, $\text{GR}$ or $\text{TG}$. The following are equivalent:
(a) \( \text{Comp}_X(A) \cap A = A \) and the regular monomorphisms in \( A \) are closed under composition.

(b) The epimorphisms in \( A \) are surjective and \( \| \|_A \) is weakly hereditary in \( A \).

(c) Each \( M \)-subobject of an \( A \)-object is \( A \)-closed.

**Proof:**

(a)⇒(b). It follows from Propositions 3.1 and 2.8.

(b)⇒(c). The same proof of e) ⇒ f) in Theorem 2.6 of [2] applies here.

(c)⇒(a). Let us consider the commutative diagram

\[
\begin{array}{ccc}
X \times Z & \xrightarrow{\pi_z} & Z \\
\downarrow m & & \uparrow m_1 \\
M & \xrightarrow{e_1} & M_1
\end{array}
\]

where \( X, Z \in A \), \((e_1, m_1)\) is the \((\mathcal{E}, \mathcal{M})\)-factorization of \( \pi_z \circ m \) and \( M \) is \( A \)-closed. Clearly, \( M_1 \) is \( A \)-closed by hypothesis, so \( \pi_z \) is \( A \)-closed preserving, i.e., \( X \) is \( A \)-compact with respect to \( A \). Since \( \| \|_A \) is weakly hereditary in \( A \), from Proposition 2.8 we get that the regular monomorphisms in \( A \) are closed under composition.

We next extend, under certain assumptions, the result in Corollary 3.3 to subcategories of the form \( \text{Comp}_X(A) \cap A \). This generalizes the fact that the epimorphisms in the category of compact Hausdorff topological spaces are surjective.

**Proposition 3.5.** Let \( A \) be an epireflective subcategory of \( AB \). Then, the epimorphisms in \( \text{Comp}_X(A) \cap A \) are surjective.

**Proof:** Let \( X \xrightarrow{f} Y \) be an epimorphism in \( \text{Comp}_X(A) \cap A \). Then, from Proposition 2.5, \( f(X) \in \text{Comp}_X(A) \). Since \( Y \in A \), \( f(X) \xrightarrow{i} Y \) is \( A \)-closed (cf. Proposition 2.3). So, \( i \simeq \text{equ}(f, g) \), with \( Y \xrightarrow{g} Z \), \( Z \in A \). This implies that \( Y/f(X) \in A \) and again from Proposition 2.5, \( Y/f(X) \in \text{Comp}_X(A) \). Let us consider \( Y \xrightarrow{g} Y/f(X) \). If \( f(X) \neq Y \) we would have that \( q \circ f = 0 \circ f \) with \( q \neq 0 \), which contradicts the fact that \( f \) is an epimorphism in \( \text{Comp}_X(A) \cap A \). Therefore \( f \) is surjective.

To show a similar result in \( \text{TOP} \) is a bit more laborious.

Let \( Y + Y \) denote the topological sum (coproduct) of two copies of the topological space \( Y \). If \( M \) is an extremal subobject of \( Y \), we denote by \( Y +_MY \) the quotient of \( Y + Y \) with respect to the equivalence relation \( (x, i) \sim (y, j) \), \( i, j = 1, 2 \) iff either \( i \neq j \) and \( x = y \in M \) or \( (x, i) = (y, j) \) (cf. [4, Definition 1.11]).

**Proposition 3.6.** (Cf. [4, Proposition 1.12]). Let \( A \) be an extremal epireflective subcategory of \( \text{TOP} \). For every \( Y \in A \) and for every extremal subobject \( M \) of \( Y \), the following are equivalent.

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(a) $Y +_M Y \in \mathcal{A}$
(b) $M = [M]_A$

**Corollary 3.7.** Let $\mathcal{A}$ be an extremal epireflective subcategory of $\text{TOP}$, let $X \xrightarrow{f} Y$ be a $\mathcal{A}$-morphism and let $M = [f(X)]_\mathcal{A}$. Then, $Y +_M Y$ belongs to $\mathcal{A}$.

**Proof:** It follows directly from Proposition 3.6.

**Lemma 3.8.** Let $\mathcal{A}$ be an extremal epireflective subcategory of $\text{TOP}$ such that $[\ ]_\mathcal{A}$ is additive in $\mathcal{A}$. Then, if $Y \in \text{Comp}_X(\mathcal{A})$, so does $Y + Y$.

**Proof:** Let $Z \in \mathcal{A}$ and let $M -\xrightarrow{m} (Y + Y) \times Z$ be $\mathcal{A}$-closed. Notice that $(Y + Y) \times Z$ is homeomorphic to $(Y \times Z) + (Y \times Z)$. Let us call such a homeomorphism $i$. Thus, $i \circ m$ is the equalizer of two morphisms $(Y \times Z) + (Y \times Z) \xrightarrow{g} T$, $T \in \mathcal{A}$. Let $f_1$, $g_1$ and $f_2$, $g_2$ denote the restrictions of $f$ and $g$ to the first and the second addend of $(Y \times Z) + (Y \times Z)$, respectively. Let $M_1 -\xrightarrow{m_1} Y \times Z$ and $M_2 -\xrightarrow{m_2} Y \times Z$ be two morphisms such that $m_1 = \text{equ}(f_1, g_1)$ and $m_2 = \text{equ}(f_2, g_2)$. Then $(i \circ m)(M) = m_1(M_1) + m_2(M_2)$. Let $\pi_2$ and $\pi_2$ denote the projections onto $Z$ of the first and the second addend of $(Y \times Z) + (Y \times Z)$ and let $[\pi_2, \pi_2]: (Y \times Z) + (Y \times Z) \rightarrow Z$ denote the induced continuous function. If $\pi_2$ is the usual projection of $(Y + Y) \times Z$ onto $Z$, then $([\pi_2, \pi_2]) \circ i = \pi_2$. Now, $(\pi_2 \circ m)(M) = ([\pi_2, \pi_2]) \circ (i \circ m)(M) = ([\pi_2, \pi_2])(m_1(M_1) + m_2(M_2)) = \pi_2(m_1(M_1) \cup \pi_2(m_2(M_2)))$. Thus, $Y \in \text{Comp}_X(\mathcal{A})$, $\pi_2(m_1(M_1))$ and $\pi_2(m_2(M_2))$ are both $\mathcal{A}$-closed and so is their union, since $[\ ]_\mathcal{A}$ is additive in $\mathcal{A}$.

**Proposition 3.9.** Let $\mathcal{A}$ be an extremal epireflective subcategory of $\text{TOP}$ such that $[\ ]_\mathcal{A}$ is additive in $\mathcal{A}$. Let $X \xrightarrow{f} Y$ be an $\mathcal{X}$-morphism and let $M = [f(X)]_\mathcal{A}$. Then, if $Y \in \text{Comp}_X(\mathcal{A})$, so does $Y +_M Y$.

**Proof:** From Lemma 3.8, $Y + Y \in \text{Comp}_X(\mathcal{A})$ and from Proposition 2.5, so does $Y +_M Y$.

**Theorem 3.10.** Let $\mathcal{A}$ be an extremal epireflective subcategory of $\text{TOP}$ such that $[\ ]_\mathcal{A}$ is additive in $\mathcal{A}$. Then, the epimorphisms in $\text{Comp}_X(\mathcal{A}) \cap \mathcal{A}$ are surjective.

**Proof:** Let $X \xrightarrow{f} Y$ be an epimorphism in $\text{Comp}_X(\mathcal{A}) \cap \mathcal{A}$ and let $M$ denote the subspace $[f(X)]_\mathcal{A}$. We have that $Y +_M Y \in \text{Comp}_X(\mathcal{A}) \cap \mathcal{A}$ (cf. Corollary 3.7 and Proposition 3.9). From Proposition 2.5, $f(X) \in \text{Comp}_X(\mathcal{A})$ and from Proposition 2.3, $f(X)$ is $\mathcal{A}$-closed. Thus, $Y +_M Y = Y +_{f(X)} Y$. Let $i$ and $j$ be the left and the right inclusions of $Y$ into $Y + Y$ and let $Y + Y -\xrightarrow{q} Y +_{f(X)} Y$ be the quotient map. Clearly, $q \circ i \circ f = q \circ j \circ f$. If $f$ is not surjective, then we have that $q \circ i \neq q \circ j$. This contradicts our assumption of $f$ being an epimorphism in $\text{Comp}_X(\mathcal{A}) \cap \mathcal{A}$.
4 A-COMPACTNESS AND ALGEBRAIC CATEGORIES

It is well known that $\text{COMP}_2$, i.e., the category of compact Hausdorff topological spaces, forms an algebraic category in the sense that $\text{COMP}_2$ has coequalizers and the forgetful functor $U: \text{COMP}_2 \to \text{SET}$ has a left adjoint and preserves and reflects regular epimorphisms (cf. [9]). It is quite natural to wonder whether this result could be extended in $\text{TOP}$ to categories of compact-separated objects with respect to a regular closure operator. Unfortunately the subcategory $\text{TOP}_1$ shows that this is not the case. As a matter of fact, $\text{Comp}_{\text{TOP}}(\text{TOP}_1) \cap \text{TOP}_1 = \text{TOP}_1$ (cf. Example 2.9(e)) and $\text{TOP}_1$ is not an algebraic category, since the forgetful functor $U: \text{TOP}_1 \to \text{SET}$ fails to reflect regular epimorphisms. However, the remaining conditions are all satisfied. We will see that, under certain assumptions on the subcategory $\mathcal{A}$, $\text{TOP}_1$ outlines the behavior of $\text{Comp}_X(\mathcal{A}) \cap \mathcal{A}$.

**Proposition 4.1.** If $\mathcal{A}$ is an extremal epireflective subcategory of $\text{TOP}$, then $\text{Comp}_X(\mathcal{A}) \cap \mathcal{A}$ has coequalizers.

**Proof:** Let $X \xrightarrow{f} Y$ be two morphisms in $\text{Comp}_X(\mathcal{A}) \cap \mathcal{A}$ and let $Y \xrightarrow{q} Q$ be their coequalizer in $\text{TOP}$. From Proposition 2.5, $Q \in \text{Comp}_X(\mathcal{A})$. Since $\mathcal{A}$ is extremal epireflective in $\text{TOP}$, we can consider the reflection $Q \xrightarrow{rQ} rQ$ of $Q$ in $\mathcal{A}$. From Proposition 2.5 $rQ \in \text{Comp}_X(\mathcal{A})$. Now, it is easily shown that $Y \xrightarrow{rQ} rQ$ is the coequalizer of $f$ and $g$ in $\text{Comp}_X(\mathcal{A}) \cap \mathcal{A}$. □

**Proposition 4.2.** Let $\mathcal{A}$ be an extremal epireflective subcategory of $\text{TOP}$ such that $\lfloor \rfloor_\mathcal{A}$ is additive in $\mathcal{A}$. Then, the forgetful functor $U: \text{Comp}_X(\mathcal{A}) \cap \mathcal{A} \to \text{SET}$ preserves regular epimorphisms.

**Proof:** Let $X \xrightarrow{f} Y$ be a regular epimorphism in $\text{Comp}_X(\mathcal{A}) \cap \mathcal{A}$. Then, from Theorem 3.10, $f$ is surjective. Therefore $U(f)$ is a regular epimorphism in $\text{SET}$. □

**Proposition 4.3.** Let $\mathcal{A}$ be an extremal epireflective and co-well powered subcategory of $\text{TOP}$. Suppose that $\lfloor \rfloor_\mathcal{A}$ is weakly hereditary in $\mathcal{A}$ and compactly productive. Then, the forgetful functor $U: \text{Comp}_X(\mathcal{A}) \cap \mathcal{A} \to \text{SET}$ has a left adjoint.

**Proof:** The case $\mathcal{A} = \{x\}$ is trivial. So, Let $\mathcal{A} \neq \{x\}$. Let $X$ be a set and let $X_d$ be the discrete topological space with underlying set $X$. Clearly $X_d \in \mathcal{A}$, since $\mathcal{A}$ is an extremal epireflective subcategory of $\text{TOP}$. Let $\beta X$ be the $\mathcal{A}$-dense-reflection of $X_d$ into $\text{Comp}_X(\mathcal{A}) \cap \mathcal{A}$ (cf. Proposition 2.7) and let $X_d \xrightarrow{\beta X} \beta X$ be the reflection morphism. If $Y \in \text{Comp}_X(\mathcal{A}) \cap \mathcal{A}$ and $X \xrightarrow{f} UY$ is a morphism in $\text{SET}$, then $X_d \xrightarrow{g} Y$ such that $U(g) = f$ is continuous. From Proposition 2.7, there exists a unique $\beta X \xrightarrow{f'} Y$ such that $f' \circ \beta_x = g$ (notice that $f'$ is unique because $\beta_x$ is a $\mathcal{A}$-epimorphism). Clearly we have that $Uf' \circ U\beta_x = f$. □
The results in Propositions 4.1, 4.2 and 4.3 can be summarized in the following

**Theorem 4.4.** Let \( \mathcal{A} \) be an extremal epireflective and co-well powered subcategory of \( \text{TOP} \) such that \( \| \cdot \|_\mathcal{A} \) is compactly productive, weakly hereditary in \( \mathcal{A} \) and additive in \( \mathcal{A} \). Then, \( \text{Comp}_X(\mathcal{A}) \cap \mathcal{A} \) has coequalizers and the forgetful functor \( U : \text{Comp}_X(\mathcal{A}) \cap \mathcal{A} \to \text{SET} \) has a left adjoint and preserves regular epimorphisms.

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