A FACTORIZATION OF THE PUMPLÜN-RÖHRL CONNECTION

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ABSTRACT: The Galois connection given in 1985 by Pumplün and Röhrl between the classes of objects and the classes of morphisms in any category is shown (under ordinary circumstances) to have a "natural" factorization through the system of all idempotent closure operators over the category. Furthermore, each "component" of the factorization is a Galois connection in its own right. The first factor is obtained by using a generalization of the process, given by Salbany in 1975, that yields a closure operator for any class of topological spaces, while the second factor can be used to form the weakly hereditary core of an idempotent closure operator.

KEY WORDS: Galois connection, closure operator, separated object, dense morphism

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0 INTRODUCTION

In [PR], Pumplün and Röhrl presented for any category $\mathcal{X}$ an important Galois connection between the collection $S(\mathcal{X})$ of all classes of $\mathcal{X}$-objects, ordered by containment, and the collection $H(\mathcal{X})$ of all classes of $\mathcal{X}$-morphisms, ordered by inclusion. This connection is a polarity determined by a "separating" relation $\sigma \subseteq \mathbf{Mor}(\mathcal{X}) \times \mathbf{Ob}(\mathcal{X})$, cf. Definition 1.03. Examples of pairs of object classes and morphism classes related by this connection can be found in [HSS].

In 1975, S. Salbany, [S], introduced certain closure operators induced by classes $\mathcal{A}$ of topological spaces. The $\mathcal{A}$-closure of any subset M of a space X is obtained by intersecting the set of all those subsets of X that contain M and are precisely the set of points for which some pair of continuous functions to some space in $\mathcal{A}$ agree.

By generalizing the Salbany process to categorical situations, Castellini and Strecker ([CS]) showed that it is typically part of another Galois connection — between classes of objects in a category $\mathcal{X}$ and idempotent closure operators over $\mathcal{X}$.

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In this paper we obtain a third Galois connection, this time between the idempotent closure operators over $\mathcal{X}$ and the classes of morphisms of $\mathcal{X}$, and show that its composition with the Galois connection of Salbany type provides a factorization of the Pumplün-Röhrl connection. It is shown that the new Galois connection can be used to obtain the weakly hereditary core of an idempotent closure operator.

We use the terminology of [HS] throughout.

1 PRELIMINARIES

Throughout we assume that $\mathcal{X}$ is an ($\mathbf{E}$, $\mathcal{M}$)-category for sinks, i.e., $\mathbf{E}$ is a collection of sinks, and $\mathcal{M}$ is a class of $\mathcal{X}$-morphisms such that:

(1) each of $\mathbf{E}$ and $\mathcal{M}$ is closed under compositions with isomorphisms.

(2) $\mathcal{M}$ has $(\mathbf{E},\mathcal{M})$-factorizations (of sinks); i.e., each sink S in $\mathcal{X}$ has a factorization $S = m \circ E$ with $E \in \mathbf{E}$ and $m \in \mathcal{M}$, and

(3) $\mathcal{X}$ has the unique $(\mathbf{E},\mathcal{M})$-diagonalization property; i.e., if $B \xrightarrow{g} D$ and $C \xrightarrow{m} D$ are $\mathcal{X}$-morphisms with $m \in \mathcal{M}$, and $E = (A_i \xrightarrow{e_i} B)_I$ and $S = (A_i \xrightarrow{s_i} C)_I$ are sinks in $\mathcal{X}$ with $E \in \mathbf{E}$, such that $m \circ S = g \circ E$, then there exists a unique diagonal, i.e., a morphism $B \xrightarrow{d} C$ such that for each $i \in I$ the both triangles of the diagram

$$\begin{array}{ccc} A_i & \xrightarrow{e_i} & B \\ {\scriptstyle s_i}\downarrow & \overset{d}{\swarrow} & \downarrow{\scriptstyle g} \\ C & \xrightarrow[m]{} & D \end{array}$$

commute.

(Morphisms will always be identified with singleton sinks.)

In addition, we require $\mathcal{X}$ to have equalizers and $\mathcal{M}$ to contain all regular monomorphisms. We list some consequences of these assumptions:

(i) every m in $\mathcal{M}$ is a monomorphism;

(ii) every E in $\mathbf{E}$ is an epi-sink;

(iii) $\mathcal{M}$ is closed under composition;

(iv) $\mathcal{M}$ is closed under relative first factors, i.e., if $n \circ m \in \mathcal{M}$, and $n \in \mathcal{M}$, then $m \in \mathcal{M}$;

(v) pullbacks of morphisms in $\mathcal{M}$ exist and belong to $\mathcal{M}$;

(vi) the $\mathcal{M}$-subobjects of every $\mathcal{X}$-object form a (possibly large) complete lattice.

We regard $\mathcal{M}$ as a full subcategory of the arrow category of $\mathcal{X}$, with the codomain functor from $\mathcal{M}$ to $\mathcal{X}$ denoted by U.

1.00 DEFINITION

A *closure operator* on $\mathcal{M}$ (over $\mathcal{X}$) is a pair $F = (\gamma, C)$, where C is an endofunctor on $\mathcal{M}$ that satisfies $UC = U$, and γ is a natural transformation from $id_{\mathcal{M}}$ to C that satisfies $(id_U)\gamma = id_U$.

Thus, given a closure operator $F = (\gamma, C)$, every member m of $\mathcal{M}$ has a canonical factorization

$$\begin{array}{ccc} \bullet & \xrightarrow{\;]m[_F\;} & \bullet \\ & {\scriptstyle m}\searrow & \big\downarrow {\scriptstyle [m]_F} \\ & & \bullet \end{array} ,$$

where $[m]_F = C(m)$ is called the F-*closure* of m , and $]m[_F$ is the domain of the m-component of γ . In particular, $[\;]_F$ induces an order-preserving increasing function on the $\mathcal{M}$-subobject lattice of every $\mathcal{X}$-object. Also, these functions are related in the following sense: if p is the pullback of a morphism $m \in \mathcal{M}$ along some $\mathcal{X}$-morphism f , and q is the pullback of $[m]_F$ along f , then $[p]_F \leq q$. Conversely, every family of functions on the $\mathcal{M}$-subobject lattices that has the above properties uniquely determines a closure operator.

1.01 DEFINITION

Given a closure operator F , we say that $m \in \mathcal{M}$ is F-*closed* if $]m[_F$ is an isomorphism. A sink S in $\mathcal{X}$ is called F-*dense* if for every $(\mathbf{E}, \mathcal{M})$-factorization (E, m) of S we have that $[m]_F$ is an isomorphism. We call F *idempotent* provided that $[\;]_F \circ [\;]_F \cong [\;]_F$, i.e., provided that $[m]_F$ is F-closed for every $m \in \mathcal{M}$. F is called *weakly hereditary* if $]m[_F$ is F-dense for every $m \in \mathcal{M}$.

Notice that $]\;[_F$ may be viewed as an endofunctor on $\mathcal{M}$ that preserves domains. Then the condition that F is weakly hereditary is equivalent to $]\;[_F \circ \;]\;[_F \cong \;]\;[_F$.

For more background on closure operators see, e.g., [T], [DG$_1$], [C], [K], and [DGT].

A special case of an idempotent closure operator arises in the following way. Given any class $\mathcal{A}$ of $\mathcal{X}$-objects and $M \xrightarrow{m} X$ in $\mathcal{M}$, define $[m]^{\mathcal{A}}$ to be the intersection of all equalizers of pairs of $\mathcal{X}$-morphisms r, s from X to some $\mathcal{A}$-object A that satisfy $r \circ m = s \circ m$, and let $]m[^{\mathcal{A}} \in \mathcal{M}$ be the unique $\mathcal{X}$-morphism by which m factors through $[m]^{\mathcal{A}}$. It is easy to see that $(]\;[^{\mathcal{A}}, [\;]^{\mathcal{A}})$ forms an idempotent closure operator, which we denote by $K(\mathcal{A})$. This generalizes the Salbany construction of closure operators induced by classes of topological spaces, cf. [S]. To simplify the notation, instead of "$[\;]^{\mathcal{A}}$-dense" we usually write "$\mathcal{A}$-dense".

We denote the collection of all idempotent closure operators on $\mathcal{M}$ by $\mathbf{iCL}(\mathcal{X},\mathcal{M})$ preordered as follows: $F \sqsubseteq G$ if $[m]_F \leq [m]_G$ for all $m \in \mathcal{M}$ (where $\leq$ is the usual order on subobjects).

For every (idempotent) closure operator F let $D(F)$ be the class of all $\mathcal{X}$-objects A that satisfy the following condition: whenever $M \xrightarrow{m} X$ belongs to $\mathcal{M}$ and $X \underset{s}{\overset{r}{\rightrightarrows}} A$ satisfy $r \circ m = s \circ m$, then $r \circ [m]_F = s \circ [m]_F$. If $\mathcal{X}$ has squares, this is equivalent to requiring the diagonal $A \xrightarrow{\Delta_A} A \times A$ to be F-closed.

1.02 THEOREM (cf. [CS, Theorem 2.5])

(D,K) is an (order-preserving) Galois connection between $S(\mathcal{X})$ and $\mathbf{iCL}(\mathcal{X},\mathcal{M})$. □

Next we recall the Pumplün-Röhrl Galois connection.

1.03 DEFINITION

For any category $\mathcal{X}$, let the relation $\sigma \subseteq \mathbf{Mor}(\mathcal{X}) \times \mathbf{Ob}(\mathcal{X})$ consist of all pairs (e,Y) with the property that for any two $\mathcal{X}$-morphisms r,s from the codomain of e to Y, $r\circ e = s\circ e$ implies $r = s$.

Given a class E of $\mathcal{X}$-morphisms, $\alpha(E) = \{\, Y \in \mathbf{Ob}(\mathcal{X}) \mid e\,\sigma\,Y \text{ for every } e \in E \,\}$ is called the class of *E-separated objects* in $\mathcal{X}$. For $\mathcal{A} \subseteq \mathbf{Ob}(\mathcal{X})$ the class of *$\mathcal{A}$-epimorphisms* in $\mathcal{X}$ is given by $\beta(\mathcal{A}) = \{\, e \in \mathbf{Mor}(\mathcal{X}) \mid e\,\sigma\,Y \text{ for every } Y \in \mathcal{A} \,\}$.

1.04 THEOREM (cf. [PR, Lemma A.1])

(α,β) is an (order-preserving) Galois connection between $S(\mathcal{X})$ and $H(\mathcal{X})$. □

2 MAIN RESULTS

The following proposition and its corollary provide a link between the Galois connections presented in the previous section.

2.00 PROPOSITION (cf. [C, Theorem 1.11])

For any class $\mathcal{A}$ of $\mathcal{X}$-objects, an $\mathcal{X}$-morphism is an $\mathcal{A}$-epimorphism iff it is $\mathcal{A}$-dense.

Proof:

Let (e,m) be an $(\mathbf{E},\mathcal{M})$-factorization of an $\mathcal{A}$-epimorphism f. To show that $[m]^{\mathcal{A}}$ is an isomorphism, it suffices to show that any two morphisms r,s from the codomain of f to some object in $\mathcal{A}$ that agree on m must coincide. But $r\circ m = s\circ m$ implies that $r\circ f = s\circ f$, and since $f \in \beta(\mathcal{A})$ we have $r = s$. Thus f is $\mathcal{A}$-dense.

Conversely, let f be $\mathcal{A}$-dense, and let r, s be morphisms with codomain in $\mathcal{A}$ such that $r \circ f = s \circ f$. If (e, m) is an $(\mathbf{E}, \mathcal{M})$-factorization of f, then since e is an epimorphism we have $r \circ m = s \circ m$. But this implies that r and s agree on the isomorphism $[m]^{\mathcal{A}}$, i.e., $r = s$. Thus f is an $\mathcal{A}$-epimorphism. $\square$

2.01 COROLLARY

Let $\mathbf{iCL}(\mathcal{X}, \mathcal{M}) \xrightarrow{R} H(\mathcal{X})$ *be given by*

$$R(F) = \{ f \in \mathbf{Mor}(\mathcal{X}) \mid f \text{ is } F\text{-dense} \} .$$

Then $RK = \beta$. $\square$

We now proceed to define the operator $H(\mathcal{X}) \xrightarrow{S} \mathbf{iCL}(\mathcal{X}, \mathcal{M})$: Given $\rho \in H(\mathcal{X})$, let $\hat{\rho}$ consist of all those $t \in \mathcal{M}$ such that for all commutative squares

$$\begin{array}{ccc} \bullet & \xrightarrow{f} & \bullet \\ {\scriptstyle r}\big\downarrow & & \big\downarrow{\scriptstyle s} \\ \bullet & \xrightarrow[t]{} & \bullet \end{array}$$

with $f \in \rho$ there exists a unique diagonal d with $d \circ f = r$ and $t \circ d = s$.

2.02 PROPOSITION

For any $M \xrightarrow{m} X$ *in* $\mathcal{M}$ *let*

$$[m]_{S(\rho)} = \bigcap \{ n \in \hat{\rho} \mid N \xrightarrow{n} X \text{ and } m \leq n \} .$$

and let $]m[_{S(\rho)}$ *have the property that* $m = [m]_{S(\rho)} \circ\,]m[_{S(\rho)}$. *Then* $S(\rho)$ *is a weakly hereditary idempotent closure operator on* $\mathcal{M}$.

Proof:

Clearly, $m \leq [m]_{S(\rho)}$. If $m \leq n$, then whenever n factors through some $p \in \hat{\rho}$, so does m. Therefore $[\;]_{S(\rho)}$ is order-preserving on the $\mathcal{M}$-subobject lattices. Let p be the pullback of a morphism $m \in \mathcal{M}$ along some $\mathcal{X}$-morphism f, and let q be the pullback of $[m]_F$ along f. Since limits commute, by the construction of $[m]_{S(\rho)}$ as an intersection, q is an intersection of pullbacks of members of $\hat{\rho}$ along f, each of which is larger than or equal to p. But $\hat{\rho}$ clearly is pullback-stable, hence $[p]_{S(\rho)} \leq q$. This establishes $S(\rho)$ as a closure operator. Since $\hat{\rho}$ by construction is closed under arbitrary intersections, the idempotency of $S(\rho)$ is immediate. Notice that since $\mathcal{M}$ is closed under composition, so is $\hat{\rho}$. Hence, whenever $]m[_{S(\rho)}$ factors through $p \in \hat{\rho}$, the composition $[m]_{S(\rho)} \circ p$ belongs to $\hat{\rho}$, and therefore was used in the construction of $[m]_{S(\rho)}$. This implies that $\big[\,]m[_{S(\rho)} \big]_{S(\rho)}$ is an isomorphism. Thus $S(\rho)$ is weakly hereditary. $\square$

2.03 THEOREM

(S,R) is an (order-preserving) Galois connection.

Proof:

If $\rho \subseteq \xi$, then $\hat{\rho} \supseteq \hat{\xi}$, hence $[m]_{S(\rho)} \leq [m]_{S(\xi)}$, for all $m \in \mathcal{M}$. Thus S is order-preserving.

Given $F \in \mathbf{iCL}(\mathcal{X},\mathcal{M})$, we have $[m]_F \in \widehat{R(F)}$ for all $m \in \mathcal{M}$. So by construction, $[m]_{SR(F)} \leq [m]_F$, i.e., SR is decreasing.

Let $F \sqsubseteq G$, and let (e,m) be an $(\mathbf{E},\mathcal{M})$-factorization of $f \in R(F)$. Since $[m]_F$ is an isomorphism, and $[m]_F \leq [m]_G$, it follows that $[m]_G$ is a monomorphic retraction, and hence an isomorphism. Thus $f \in R(G)$. This shows that R is order-preserving.

Now consider an $(\mathbf{E},\mathcal{M})$-factorization (e,m) of $f \in \rho$. Since $[m]_{S(\rho)}$ belongs to $\hat{\rho}$, there exists a d with $[m]_{S(\rho)} \circ d = id$. Then $[m]_{S(\rho)}$ is a monomorphic retraction, and thus is an isomorphism. This shows that $f \in RS(\rho)$. Consequently RS is increasing. □

2.04 THEOREM

The Galois connections (S,R) and (D,K) provide a factorization of the Pumplün-Röhrl connection, i.e., $(\alpha,\beta) = (S,R) \circ (D,K)$.

Proof:

Since $\alpha \dashv \beta$, $S \dashv R$, and $D \dashv K$, this follows directly from the fact that $\beta = RK$ (Corollary 2.01). □

Next we investigate the relationship between weakly hereditary idempotent closure operators and the new Galois connection (S,R) obtained above. By the general theory on closure operators it can be seen that for every idempotent closure operator F the collection of weakly hereditary idempotent closure operators G with $G \sqsubseteq F$ has a supremum, $\check{F}$, called the *weakly hereditary core* of F, cf. [DG$_1$, Theorem 4.2] and [K, Proposition 1.13].

We now show that the operator SR obtained from the new Galois connection yields these cores. First we recall the following result:

2.05 LEMMA (cf. [K, Proposition 1.09].)

A closure operator F is weakly hereditary iff every $\mathcal{M}$-object m satisfies:

$$[m]_F = \sup\{\, p \in \mathcal{M} \mid m = p \circ n \text{ and } n \text{ is } F\text{-dense} \,\} .$$

□

2.06 THEOREM

If F is an idempotent closure operator, then $SR(F)$ is its weakly hereditary core.

Proof:

By Proposition 2.02, $SR(F)$ is weakly hereditary, and the Galois connection implies that $SR(F) \sqsubseteq F$. Thus if $\check{F}$ is the weakly hereditary core of F, we get $SR(F) \sqsubseteq \check{F} \sqsubseteq F$. Applying R yields $RSR(F) \sqsubseteq R(\check{F}) \sqsubseteq R(F) = RSR(F)$, so $SR(F)$ and $\check{F}$ have the same dense morphisms. By Lemma 2.05, $SR(F)$ and $\check{F}$ agree. □

2.07 COROLLARY

For a class $\mathcal{A}$ of $\mathcal{X}$-objects $K(\mathcal{A})$ is weakly hereditary iff $K(\mathcal{A}) = S\beta(\mathcal{A})$. □

2.08 PROPOSITION

If $\mathcal{A} \subseteq \mathbf{Ob}(\mathcal{X})$ has a coseparating class of objects each of which is injective with respect to $K(\mathcal{A})$-closed morphisms, then $K(\mathcal{A})$ is weakly hereditary.

Proof:

Let $\mathcal{C}$ be a coseparating subclass of $\mathcal{A}$ such that each $C \in \mathcal{C}$ is injective with respect to $K(\mathcal{A})$-closed morphisms. Since $\mathcal{C}$ is coseparating for $\mathcal{A}$ we have $K(\mathcal{C}) = K(\mathcal{A})$, cf. [C, Proposition 1.4].

For $m \in \mathcal{M}$ consider morphisms h, k with codomain $C \in \mathcal{C}$ such that $h \circ]m[_{K(\mathcal{A})} = k \circ]m[_{K(\mathcal{A})}$. Since C is injective with respect to $[m]_{K(\mathcal{A})}$, there exist extensions h' and k' of h and k, respectively, along $[m]_{K(\mathcal{A})}$. Now $h' \circ m = k' \circ m$ and $C \in \mathcal{A}$ implies $h' \circ [m]_{K(\mathcal{A})} = k' \circ [m]_{K(\mathcal{A})}$, and hence $h = k$. Therefore $\big[\,]m[_{K(\mathcal{A})}\big]_{K(\mathcal{A})}$ is an isomorphism. □

3 EXAMPLES

We now explore the implications of the general theory for some examples.

3.00 EXAMPLE

Let $\mathcal{X}$ be the category **Top** of topological spaces, let $\mathcal{M}$ be the class of usual topological embeddings, and let $\mathcal{A}$ be the category **Haus** of Hausdorff spaces. Then $\beta(\mathbf{Haus})$ properly contains the class of all continuous functions that are dense (in the ordinary sense), cf. [PR]. Thus $S\beta(\mathbf{Haus})$ is strictly larger than the usual closure operator T for topological spaces, even though both agree on **Haus**, and $D(T) = DS\beta(\mathbf{Haus}) = \mathbf{Haus}$. Moreover, $K(\mathbf{Haus})$ is not weakly hereditary. Thus $K(\mathbf{Haus}) \neq S\beta(\mathbf{Haus})$. In particular, this implies that **Haus** has no coseparating class of objects that are $\mathcal{M}$-injective.

3.01 EXAMPLE

Let $\mathcal{X}$ and $\mathcal{M}$ be as above, and let $\mathcal{A}$ be the category $\mathbf{Top_0}$. Then $\beta(\mathbf{Top_0})$ is the class of all b-dense continuous functions (cf. [B], [NW]), and $K(\mathbf{Top_0})$ is the b-closure operator for topological spaces. Since the Sierpinski space is injective with respect to embeddings and is a coseparator for $\mathbf{Top_0}$, the b-closure is weakly hereditary. Therefore $K(\mathbf{Top_0}) = S\beta(\mathbf{Top_0})$.

3.02 EXAMPLE

Let $\mathcal{X}$ be the category $\mathbf{Ab}$ of abelian groups, let $\mathcal{M}$ be the class of monomorphisms in $\mathbf{Ab}$, and let $\mathcal{A}$ be the category $\mathbf{TfAb}$ of torsion-free abelian groups. Then $\beta(\mathbf{TfAb})$ is the class of all homomorphisms $X \xrightarrow{f} Y$ with the property that $Y/f[X]$ is a torsion group. The closure operator $K(\mathbf{TfAb})$ can be described as follows: For any monomorphism $M \xrightarrow{m} X$ the closure $[m]_{K(\mathbf{TfAb})}$ is the smallest subgroup N of X that contains M and for which X/N is torsion-free. Since $K(\mathbf{TfAb})$ is weakly hereditary, $K(\mathbf{TfAb}) = S\beta(\mathbf{TfAb})$.

3.03 EXAMPLE

Let $\mathcal{X}$ and $\mathcal{M}$ be as in Example 3.02, and let $\mathcal{A}$ be the category $\mathbf{RdAb}$ of reduced abelian groups. Then $\beta(\mathbf{RdAb})$ is the class of all homomorphisms $X \xrightarrow{f} Y$ with the property that $Y/f[X]$ is divisible. The closure operator $K(\mathbf{RdAb})$ can be described as follows: For any monomorphism $M \xrightarrow{m} X$ the closure $[m]_{K(\mathbf{RdAb})}$ is the smallest subgroup N of X that contains M and for which X/N is reduced. Since $K(\mathbf{RdAb})$ is weakly hereditary, $K(\mathbf{RdAb}) = S\beta(\mathbf{RdAb})$.

3.04 EXAMPLE

More generally, for a fixed ring R with unity let $\mathcal{X}$ be the category R-$\mathbf{Mod}$ of left R-modules, let $\mathcal{M}$ be the class of monomorphisms in R-$\mathbf{Mod}$, and let $(\mathcal{T}, \mathcal{F})$ be a torsion theory. Then $\beta(\mathcal{F})$ is the class of all homomorphisms $X \xrightarrow{f} Y$ with the property that $Y/f[X] \in \mathcal{T}$. The $K(\mathcal{F})$-closed submodules can be described as follows: A submodule $M \xrightarrow{m} X$ is $K(\mathcal{F})$-closed iff $X/M \in \mathcal{F}$. Since $K(\mathcal{F})$ is weakly hereditary, $K(\mathcal{F}) = S\beta(\mathcal{F})$ (cf. [DG_2]).

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