CONNECTEDNESS AND DISCONNECTEDNESS: A DIFFERENT PERSPECTIVE

G. Castellini [1]

ABSTRACT: A factorization of a Galois connection investigated earlier is used to give a definition of a connectedness-disconnectedness Galois connection that is free of the notion of constant morphism. A new notion of $\mathcal{N}$-fixed morphism with respect to a class $\mathcal{N}$ of monomorphisms is presented. This is used to characterize the connectedness-disconnectedness Galois connection in the case that $\mathcal{N}$ is closed under the formation of pullbacks. Some closedness properties of these Galois connections are investigated.

KEY WORDS: Closure operator, Galois connection, connectedness, disconnectedness, constant morphism.

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0 INTRODUCTION

The development of a general theory of topological connectedness was started by Preuß ([Pr_1]) and by Herrlich ([H]). Afterwards, a considerable number of papers have been publised on this subject and on possible generalizations of it (e.g. [AW], [CC], [CT], [HP], [L], [P], [Pr_{2-3}], [SV] and [T]). However, most of these papers used the common approach of first defining a notion of constant morphism and then use it to introduce the notions of connectedness and disconnectedness, accordingly. So did we in [CH] and [C_{2-3}].

Let $\mathcal{X}$ be an arbitrary category with an $(\mathbf{E}, \mathcal{M})$-factorization structure for sinks and let $\mathcal{N} \subseteq \mathcal{M}$. In [$\text{C}_2$] an $\mathcal{X}$-morphism $X \xrightarrow{f} Y$ was called $\mathcal{N}$-constant if the direct image of X under f was isomorphic to the direct image under f of every $\mathcal{N}$-subobject of X. If $S(\mathcal{X})$ denotes the collection of all subclasses of objects of $\mathcal{X}$, ordered by inclusion, for every $\mathcal{N} \subseteq \mathcal{M}$, the relation: $X \mathcal{R}_{\mathcal{N}} Y$ if and only if every $\mathcal{X}$-morphism $X \xrightarrow{f} Y$ is $\mathcal{N}$-constant yields a Galois connection $S(\mathcal{X}) \underset{\nabla_{\mathcal{N}}}{\overset{\Delta_{\mathcal{N}}}{\rightleftarrows}} S(\mathcal{X})^{\mathbf{op}}$. For clarity we observe that for every $\mathcal{B} \in S(\mathcal{X})$, $\Delta_{\mathcal{N}}(\mathcal{B}) = \{Y \in \mathcal{X} : \forall X \in \mathcal{B}, X \xrightarrow{f} Y \textit{ is } \mathcal{N}\text{-constant}\}$ and for every $\mathcal{A} \in S(\mathcal{X})^{\mathbf{op}}$, $\nabla_{\mathcal{N}}(\mathcal{A}) = \{X \in \mathcal{X} : \forall Y \in \mathcal{A}, X \xrightarrow{f} Y \textit{ is } \mathcal{N}\text{-constant}\}$. Again in [$\text{C}_2$] it was proved that if $\mathcal{N}$ is closed under direct images, we have that this Galois connection factors through $CL(\mathcal{X}, \mathcal{M})$, i.e., the collection of all

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closure operators on $\mathcal{X}$ with respect to $\mathcal{M}$, via two Galois connections $S(\mathcal{X}) \underset{I_{\mathcal{N}}}{\overset{J_{\mathcal{N}}}{\rightleftarrows}} CL(\mathcal{X},\mathcal{M})$ and $CL(\mathcal{X},\mathcal{M}) \underset{T_{\mathcal{N}}}{\overset{D_{\mathcal{N}}}{\rightleftarrows}} S(\mathcal{X})^{\mathbf{op}}$ (see 2.1 and 2.2 below).

The main point of this paper is to free the notions of connectedness and disconnectedness from their dependence on constant morphisms. More precisely, the composition of the two Galois connections $S(\mathcal{X}) \underset{I_{\mathcal{N}}}{\overset{J_{\mathcal{N}}}{\rightleftarrows}} CL(\mathcal{X},\mathcal{M})$ and $CL(\mathcal{X},\mathcal{M}) \underset{T_{\mathcal{N}}}{\overset{D_{\mathcal{N}}}{\rightleftarrows}} S(\mathcal{X})^{\mathbf{op}}$ will be called the connectedness-disconnectedness Galois connection.

Nevertheless, it is still quite useful to have some notion of constant morphism available because it can be used, under certain circumstances, to give an alternative description of the connectedness-disconnectedness Galois connection. The notion of $\mathcal{N}$-constant morphism introduced in [C_2] can be used to describe this Galois connection in the case that $\mathcal{N}$ is closed under the formation of direct images. Here we introduce a notion of $\mathcal{N}$-fixed morphism and show that under the assumption of $\mathcal{N}$ being closed under the formation of pullbacks it provides an alternative description of the same connectedness-disconnectedness Galois connection. Examples show the advantage of having these two different decriptions available.

Furthermore, basic closedness properties of the Galois connections $S(\mathcal{X}) \underset{I_{\mathcal{N}}}{\overset{J_{\mathcal{N}}}{\rightleftarrows}} CL(\mathcal{X},\mathcal{M})$ and $CL(\mathcal{X},\mathcal{M}) \underset{T_{\mathcal{N}}}{\overset{D_{\mathcal{N}}}{\rightleftarrows}} S(\mathcal{X})^{\mathbf{op}}$ are analyzed as well as some properties of the closure operators $T_{\mathcal{N}}(\mathcal{A})$ and $J_{\mathcal{N}}(\mathcal{B})$.

We use the terminology of [AHS] throughout the paper[2].

1 PRELIMINARIES

Throughout we consider a category $\mathcal{X}$ and a fixed class $\mathcal{M}$ of $\mathcal{X}$-monomorphisms, which contains all $\mathcal{X}$-isomorphisms. It is assumed that $\mathcal{X}$ is $\mathcal{M}$-complete; i.e.,

(1) $\mathcal{M}$ is closed under composition

(2) Pullbacks of $\mathcal{M}$-morphisms exist and belong to $\mathcal{M}$, and multiple pullbacks of (possibly large) families of $\mathcal{M}$-morphisms with common codomain exist and belong to $\mathcal{M}$.

One of the consequences of the above assumptions is that there is a uniquely determined class $\mathbf{E}$ of sinks in $\mathcal{X}$ such that $\mathcal{X}$ is an $(\mathbf{E},\mathcal{M})$-*category for sinks*, that is:

(a) each of $\mathbf{E}$ and $\mathcal{M}$ is closed under compositions with isomorphisms;

(b) $\mathcal{X}$ has $(\mathbf{E},\mathcal{M})$-factorizations (of sinks); i.e., each sink $\mathbf{s}$ in $\mathcal{X}$ has a factorization $\mathbf{s} = m \circ \mathbf{e}$

[2] Paul Taylor's commutative diagrams macro package was used to typeset most of the diagrams in this paper.

with $\mathbf{e} \in \mathbf{E}$ and $m \in \mathcal{M}$, and

(c) $\mathcal{X}$ has the unique $(\mathbf{E}, \mathcal{M})$-diagonalization property; i.e., if $B \xrightarrow{g} D$ and $C \xrightarrow{m} D$ are $\mathcal{X}$-morphisms with $m \in \mathcal{M}$, and $\mathbf{e} = (A_i \xrightarrow{e_i} B)_I$ and $\mathbf{s} = (A_i \xrightarrow{s_i} C)_I$ are sinks in $\mathcal{X}$ with $\mathbf{e} \in \mathbf{E}$, such that $m \circ \mathbf{s} = g \circ \mathbf{e}$, then there exists a unique diagonal $B \xrightarrow{d} C$ such that for every $i \in I$ the following diagrams commute:

$$\begin{array}{ccc} A_i & \xrightarrow{e_i} & B \\ {\scriptstyle s_i}\downarrow & \swarrow{\scriptstyle d} & \\ C & & \end{array} \qquad \text{and} \qquad \begin{array}{ccc} & & B \\ & {\scriptstyle d}\swarrow & \downarrow{\scriptstyle g} \\ C & \xrightarrow[m]{} & D \end{array}$$

That $\mathcal{X}$ is an $(\mathbf{E}, \mathcal{M})$-category implies the following features of $\mathcal{M}$ and $\mathbf{E}$ (cf. [AHS] for the dual case):

PROPOSITION 1.1

(0) Every isomorphism is in both $\mathcal{M}$ and $\mathbf{E}$ (as a singleton sink). Moreover, every morphism that is in both $\mathcal{M}$ and $\mathbf{E}$ is an isomorphism.

(1) Every m in $\mathcal{M}$ is a monomorphism.

(2) $\mathcal{M}$ is closed under $\mathcal{M}$-relative first factors, i.e., if $n \circ m \in \mathcal{M}$, and $n \in \mathcal{M}$, then $m \in \mathcal{M}$.

(3) $\mathcal{M}$ is closed under composition.

(4) Pullbacks of $\mathcal{X}$-morphisms in $\mathcal{M}$ exist and belong to $\mathcal{M}$.

(5) The $\mathcal{M}$-subobjects of every $\mathcal{X}$-object form a (possibly large) complete lattice; suprema are formed via $(\mathbf{E}, \mathcal{M})$-factorizations and infima are formed via intersections. □

If $X \xrightarrow{f} Y$ is an $\mathcal{X}$-morphism and $M \xrightarrow{m} X$ is an $\mathcal{M}$-subobject, then $X \xrightarrow{e_{f\circ m}} M_f \xrightarrow{m_f} Y$ will denote the $(\mathbf{E}, \mathcal{M})$-factorization of $f \circ m$. $M_f \xrightarrow{m_f} Y$ will be called the direct image of m along f. If $N \xrightarrow{n} Y$ is an $\mathcal{M}$-subobject, then the pullback $f^{-1}(N) \xrightarrow{f^{-1}(n)} X$ of n along f will be called the inverse image of n along f. Whenever no confusion is likely to arise, to simplify the notation we will denote the morphism $e_{f\circ m}$ simply e_f.

DEFINITION 1.2

A *closure operator* C on $\mathcal{X}$ (with respect to $\mathcal{M}$) is a family $\{(\ )^C_X\}_{X\in\mathcal{X}}$ of functions on the $\mathcal{M}$-subobject lattices of $\mathcal{X}$ with the following properties that hold for each $X \in \mathcal{X}$:

(a) [*expansiveness*] $m \leq (m)^C_X$, for every $\mathcal{M}$-subobject $M \xrightarrow{m} X$;

(b) [*order-preservation*] $m \leq n \Rightarrow (m)^C_X \leq (n)^C_X$ for every pair of $\mathcal{M}$-subobjects of X;

(c) [*morphism-consistency*] If p is the pullback of the $\mathcal{M}$-subobject $M \xrightarrow{m} Y$ along some $\mathcal{X}$-morphism $X \xrightarrow{f} Y$ and q is the pullback of $(m)^C_Y$ along f, then $(p)^C_X \leq q$, i.e., the closure of the inverse image of m is less than or equal to the inverse image of the closure of m.

Condition (a) implies that for every closure operator C on $\mathcal{X}$, every $\mathcal{M}$-subobject $M \xrightarrow{m} X$ has a canonical factorization

$$\begin{array}{ccc} M & \xrightarrow{t} & (M)^{C}_{X} \\ & {\scriptstyle m} \searrow & \downarrow {\scriptstyle (m)^{C}_{X}} \\ & & X \end{array}$$

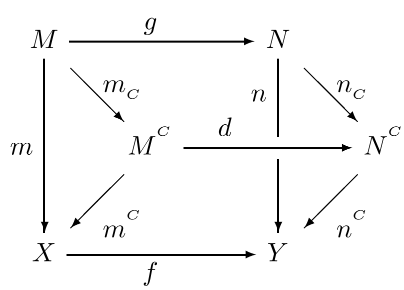
where $((M)^{C}_{X}, (m)^{C}_{X})$ is called the *C-closure* of the subobject (M, m).

When no confusion is likely we will write m^{C} rather than $(m)^{C}_{X}$ and for notational symmetry we will denote the morphism t by m_{C}.

REMARK 1.3

(1) Notice that in the above definition, under condition (b), the morphism-consistency condition (c) is equivalent to the following statement concerning direct images: if $M \xrightarrow{m} X$ is an $\mathcal{M}$-subobject and $X \xrightarrow{f} Y$ is a morphism, then $((m)^{C}_{Y})_f \leq (m_f)^{C}_{Y}$, i.e., the direct image of the closure of m is less than or equal to the closure of the direct image of m; (cf. [DG]).

(2) Under condition (a), both order-preservation and morphism-consistency, i.e., conditions (b) and (c) together are equivalent to the following: given (M, m) and (N, n) $\mathcal{M}$-subobjects of X and Y, respectively, if f and g are morphisms such that $n \circ g = f \circ m$, then there exists a unique morphism d such that the following diagram



commutes.

DEFINITION 1.4

Given a closure operator C, we say that $m \in \mathcal{M}$ is *C-closed* if m_C is an isomorphism. An $\mathcal{X}$-morphism f is called *C-dense* if for every $(\mathbf{E}, \mathcal{M})$-factorization (e, m) of f we have that m^{C} is an isomorphism. We call C *idempotent* provided that m^{C} is C-closed for every $m \in \mathcal{M}$. C is called *weakly hereditary* if $m_{_C}$ is C-dense for every $m \in \mathcal{M}$. Furthermore, if $\mathcal{M}' \subseteq \mathcal{M}$, then C is said to be *hereditary with respect to* $\mathcal{M}'$ if whenever $M \xrightarrow{m} X$, $M \xrightarrow{t} N$ and $N \xrightarrow{n} X$ are morphisms in $\mathcal{M}$ with $n \circ t = m$ and $n \in \mathcal{M}'$, we have that $t^{^C}$ is the pullback of $m^{^C}$ along n (cf. [CG]).

Notice that Definition 1.2(c) implies that pullbacks of C-closed $\mathcal{M}$-subobjects are C-closed.

We denote the collection of all closure operators on $\mathcal{M}$ by $\mathbf{CL}(\mathcal{X}, \mathcal{M})$ pre-ordered as follows: $C \sqsubseteq D$ if $m^{^C} \leq m^{^D}$ for all $m \in \mathcal{M}$ (where $\leq$ is the usual order on subobjects). Notice that arbitrary suprema and infima exist in $\mathbf{CL}(\mathcal{X}, \mathcal{M})$, they are formed pointwise in the $\mathcal{M}$-subobject fibers.

For more background on closure operators see, e.g., [C_1], [CKS_1], [CKS_2], [DG], [DGT] and [K]. For a detailed survey on the same topic, one could check [Ho].

DEFINITION 1.5

For pre-ordered classes $\mathcal{X} = (\mathbf{X}, \sqsubseteq)$ and $\mathcal{Y} = (\mathbf{Y}, \sqsubseteq)$, a *Galois connection* $\mathcal{X} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{Y}$ consists of order preserving functions F and G that satisfy $F \dashv G$, i.e., $x \sqsubseteq GF(x)$ for every $x \in \mathbf{X}$ and $FG(y) \sqsubseteq y$ for every $y \in \mathbf{Y}$. (G is adjoint and has F as coadjoint).

If $x \in \mathcal{X}$ and $y \in \mathcal{Y}$ are such that $F(x) = y$ and $G(y) = x$, then x and y are said to be corresponding fixed points of the Galois connection $(\mathcal{X}, F, G, \mathcal{Y})$.

Properties and many examples of Galois connections can be found in [EKMS].

2 GENERAL RESULTS

Throughout the paper we assume that $\mathcal{X}$ is an $(\mathbf{E}, \mathcal{M})$-category for sinks.

Let $S(\mathcal{X})$ denote the collection of all subcategories of $\mathcal{X}$, ordered by inclusion and let $\mathcal{N}$ be a fixed subclass of $\mathcal{M}$. For $X \in \mathcal{X}$, $\mathcal{N}_X$ will denote the class of all $\mathcal{N}$-subobjects of X.

We begin by recalling the following two propositions from [C_2].

PROPOSITION 2.1

Let $CL(\mathcal{X}, \mathcal{M}) \xrightarrow{D_{\mathcal{N}}} S(\mathcal{X})^{\mathbf{op}}$ *and* $S(\mathcal{X})^{\mathbf{op}} \xrightarrow{T_{\mathcal{N}}} CL(\mathcal{X}, \mathcal{M})$ *be defined by:*

$D_{\mathcal{N}}(C) = \{X \in \mathcal{X} : \text{ every } n \in \mathcal{N}_X \text{ is } C\text{-closed}\}$

$T_{\mathcal{N}}(\mathcal{A}) = Sup\{C \in CL(\mathcal{X}, \mathcal{M}) : D_{\mathcal{N}}(C) \supseteq \mathcal{A}\}$.

Then, $CL(\mathcal{X}, \mathcal{M}) \underset{T_{\mathcal{N}}}{\overset{D_{\mathcal{N}}}{\rightleftarrows}} S(\mathcal{X})^{\mathbf{op}}$ *is a Galois connection.* □

PROPOSITION 2.2

Let $CL(\mathcal{X},\mathcal{M}) \xrightarrow{I_{\mathcal{N}}} S(\mathcal{X})$ *and* $S(\mathcal{X}) \xrightarrow{J_{\mathcal{N}}} CL(\mathcal{X},\mathcal{M})$ *be defined by:*

$I_{\mathcal{N}}(C) = \{X \in \mathcal{X} : \text{ every } n \in \mathcal{N}_X \text{ is } C\text{-dense}\}$

$J_{\mathcal{N}}(\mathcal{B}) = Inf\{C \in CL(\mathcal{X},\mathcal{M}) : I_{\mathcal{N}}(C) \supseteq \mathcal{B}\}.$

Then, $S(\mathcal{X}) \underset{I_{\mathcal{N}}}{\overset{J_{\mathcal{N}}}{\rightleftarrows}} CL(\mathcal{X},\mathcal{M})$ *is a Galois connection.* □

As a consequence of the above propositions we obtain a Galois connection between $S(\mathcal{X})$ and $S(\mathcal{X})^{\mathbf{op}}$. Therefore we give the following:

DEFINITION 2.3

The Galois connection $S(\mathcal{X}) \underset{I_{\mathcal{N}} \circ T_{\mathcal{N}}}{\overset{D_{\mathcal{N}} \circ J_{\mathcal{N}}}{\rightleftarrows}} S(\mathcal{X})^{\mathbf{op}}$ will be called the connectedness-disconnectedness Galois connection.

In [C$_2$] we showed that when $\mathcal{N}$ is closed under the formation of direct images, the above Galois connection is precisely the one given by $\mathcal{N}$-constant morphisms and we presented some characterizations of the functions $T_{\mathcal{N}}$ and $J_{\mathcal{N}}$. For reference purposes we collect them under the following:

PROPOSITION 2.4

For every $\mathcal{A} \in S(\mathcal{X})^{\mathbf{op}}$ *and* $\mathcal{M}$*-subobject* $M \xrightarrow{m} X$*, with* $X \in \mathcal{X}$*, we have that*

$$m^{T_{\mathcal{N}}(\mathcal{A})} = \cap\{f^{-1}(n) : Y \in \mathcal{A}, X \xrightarrow{f} Y, N \xrightarrow{n} Y \in \mathcal{N}_Y \text{and } m \le f^{-1}(n)\}.$$

Moreover, for every $\mathcal{B} \in S(\mathcal{X})$ *and* $\mathcal{M}$*-subobject* $M \xrightarrow{m} Y$*, with* $Y \in \mathcal{X}$*, we have that*

$$m^{J_{\mathcal{N}}(\mathcal{B})} = sup\Big(\{m\} \cup \{(id_X)_f : X \in \mathcal{B}, X \xrightarrow{f} Y \text{ and } \exists n \in \mathcal{N}_X \text{ with } n_f \le m\}\Big).$$

□

Next we analyze some closedness properties of the Galois connections in Propositions 2.1 and 2.2.

PROPOSITION 2.5

If $\mathcal{N}$ *is closed under pullbacks along morphisms in* **E***, then for every closure operator* C*,* $I_{\mathcal{N}}(C)$ *is closed under* **E***-quotients.*

Proof:

Let the morphism $X \xrightarrow{q} Q$ belong to **E** with $X \in I_{\mathcal{N}}(C)$ and consider the $\mathcal{N}$-subobject $N \xrightarrow{n} Q$. By hypothesis $q^{-1}(n)$ belongs to $\mathcal{N}$ and so $(q^{-1}(n))^{C} \simeq id_X$. From the property (c) of closure operators we obtain that $(q^{-1}(n))^{C} \le q^{-1}(n^{C})$. Therefore we conclude that $q^{-1}(n^{C}) \simeq$

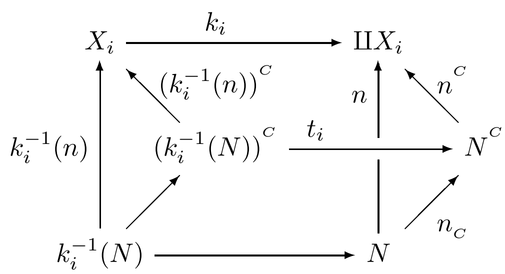
id_X. Consequently we obtain that $n^{C} \geq (q^{-1}(n^{C}))_q \simeq (id_X)_q \simeq id_Q$. Notice that the last isomorphism is a consequence of the fact that $q \in \mathbf{E}$. Hence we have that $n^{C} \simeq id_Q$, i.e., $Q \in I_{\mathcal{N}}(C)$. $\square$

PROPOSITION 2.6

Let $\mathcal{N}$ be closed under pullbacks and let C be a closure operator. If $X_i \in I_{\mathcal{N}}(C)$ for every $i \in I$ and the coproduct $\amalg X_i$ exists, then it also belongs to $I_{\mathcal{N}}(C)$.

Proof:

Consider the coproduct $(X_i \xrightarrow{k_i} \amalg X_i)_{i \in I}$ with $X_i \in I_{\mathcal{N}}(C)$ for every $i \in I$. If $N \xrightarrow{n} \amalg X_i$ belongs to $\mathcal{N}$ then, since by hypothesis $k_i^{-1}(n) \in \mathcal{N}$ for every $i \in I$, we have that each $(k_i^{-1}(n))^{C} \simeq id_{X_i}$. From the general properties of C (cf. Remark 1.3(2)), for every $i \in I$ we obtain a morphism t_i such that the following diagram commutes



To simplify the notation, let $r_i = (k_i^{-1}(n))^{C}$. Since each r_i is an isomorphism, the universal property of coproducts implies the existence of a morphism $\amalg X_i \xrightarrow{d} N^{C}$ such that $d \circ k_i = t_i \circ r_i^{-1}$, for every $i \in I$. This, together with $n^{C} \circ t_i \circ r_i^{-1} = k_i$ implies that $n^{C} \circ d \circ k_i = id_{\amalg X_i} \circ k_i$, for every $i \in I$. The fact that $(k_i)_{i \in I}$ is an epi-sink implies that $n^{C} \circ d = id_{\amalg X_i}$. Finally, this together with the fact that n^{C} is a monomorphism implies that n^{C} is an isomorphism. Thus, $\amalg X_i \in I_{\mathcal{N}}(C)$. $\square$

It may be worthwhile to observe that in the case that $\mathcal{X}$ is well-powered and has coproducts, if $\mathcal{N}$ is closed under pullbacks, Propositions 2.5 and 2.6 imply that for any closure operator C, $I_{\mathcal{N}}(C)$ is an $\mathcal{M}$-coreflective subcategory of $\mathcal{X}$ (cf. [AHS, Theorem 16.8], dual).

PROPOSITION 2.7

Let C be a closure operator. If $\mathcal{N}$ is closed under the formation of direct images along elements of $\mathcal{M}$, then $D_{\mathcal{N}}(C)$ is closed under the formation of $\mathcal{M}$-subobjects.

Proof:

Let $M \xrightarrow{m} X$ be an $\mathcal{M}$-subobject of X with $X \in D_{\mathcal{N}}(C)$ and let $N \xrightarrow{n} M$ be an $\mathcal{N}$-subobject of M. Let (e_1, m_1) be the $(\mathbf{E}, \mathcal{M})$-factorization of $m \circ n$. By hypothesis m_1 is an

$\mathcal{N}$-subobject of X and therefore it is C-closed. Since $m \circ n \in \mathcal{M}$, we obtain that e_1 is an isomorphism. Thus n is the pullback of m_1 along m and from the general theory of closure operators it is C-closed. $\square$

LEMMA 2.8

Let C be a closure operator and let $X \in \mathcal{X}$. Suppose that for every $n \in \mathcal{N}_X$, there is a source $(X \xrightarrow{f_i} Y_i)_{i \in I}$ with $Y_i \in D_{\mathcal{N}}(C)$ for every $i \in I$ and $n \simeq \cap f_i^{-1}(n_i)$, for some $n_i \in \mathcal{N}_{Y_i}$. Then $X \in D_{\mathcal{N}}(C)$.

Proof:

Consider the $\mathcal{N}$-subobject $N \xrightarrow{n} X$ and a source $(X \xrightarrow{f_i} Y_i)_{i \in I}$ such that $n \simeq \cap f_i^{-1}(n_i)$, with $n_i \in \mathcal{N}_{Y_i}$. Then we have that $n^{C} \simeq (\cap f_i^{-1}(n_i))^{C} \leq \cap (f_i^{-1}(n_i))^{C} \leq \cap f_i^{-1}(n_i^{C}) \simeq \cap f_i^{-1}(n_i) \simeq n$. Notice that in the above inequalities we have used the usual properties of closure operators (e.g., 1.3(2)). Moreover the fact that every $\mathcal{N}$-subobject of Y_i is C-closed was used to obtain the final isomorphisms. Thus, we conclude that $n \simeq n^{C}$ and so $X \in D_{\mathcal{N}}(C)$. $\square$

PROPOSITION 2.9

Let C be a closure operator. Suppose that $\mathbf{E}$ consists of episinks and that $\mathcal{X}$ has a terminal object T. Consider the class of morphisms $\mathcal{N} = \{T \xrightarrow{n} X \text{ with } X \in \mathcal{X} \text{ and } n \in \mathcal{M}\}$. Then every product of $\mathcal{X}$-objects belongs to $D_{\mathcal{N}}(C)$.

Proof:

Let $(\prod X_i \longrightarrow X_i)_{i \in I}$ be a product in $\mathcal{X}$. We need only show that it satisfies the hypothesis of Lemma 2.8.

Let $n \in \mathcal{N}$ and let (e_i, m_i) be the $(\mathbf{E}, \mathcal{M})$-factorization of $\pi_i \circ n$. Notice that since e_i is an epimorphism and T is a terminal object, we can easily conclude that e_i is an isomorphism. Therefore, $m_i \in \mathcal{N}$. Now, let us consider the pullback diagram

$$\begin{array}{ccc} \prod X_i & \xrightarrow{\pi_i} & X_i \\ {\scriptstyle \pi_i^{-1}(m_i)} \uparrow & & \uparrow {\scriptstyle m_i} \\ P_i & \xrightarrow[r_i]{} & T_{\pi_i} \end{array}$$

We would like to show that $n = \cap \pi_i^{-1}(m_i)$. Since $\pi_i \circ n = m_i \circ e_i$, we have that for every $i \in I$, there is a morphism $T \xrightarrow{t_i} P_i$ such that $\pi_i^{-1}(m_i) \circ t_i = n$ and $r_i \circ t_i = e_i$. Now, suppose that there is a morphism $D \xrightarrow{d} \prod X_i$ and a family of morphisms $(D \xrightarrow{d_i} P_i)_{i \in I}$ such that $\pi_i^{-1}(m_i) \circ d_i = d$, for every $i \in I$. Since T is a terminal object, there is a unique morphism

$D \xrightarrow{t} T$ such that $t = e_i^{-1} \circ r_i \circ d_i$. Consequently we have that $e_i \circ t = r_i \circ d_i$ and so, $\pi_i \circ d = \pi_i \circ \pi_i^{-1}(m_i) \circ d_i = m_i \circ r_i \circ d_i = m_i \circ e_i \circ t = \pi_i \circ n \circ t$ for every $i \in I$. The universal property of products implies that $n \circ t = d$. Therefore we can conclude that $n \simeq \cap \pi_i^{-1}(m_i)$. $\square$

We observe that in many concrete categories such as for instance topological spaces and groups, under the appropriate assumptions, Propositions 2.7 and 2.9 yield that $D_{\mathcal{N}}(C)$ is an epireflective subcategory (cf. [AHS, Theorem 16.8]).

Another consequence of Lemma 2.8 is the following:

PROPOSITION 2.10

Let $\mathcal{A} \in S(\mathcal{X})^{\mathbf{op}}$. An $\mathcal{X}$-object X belongs to $D_{\mathcal{N}}(T_{\mathcal{N}}(\mathcal{A}))$ if and only if for every $n \in \mathcal{N}_X$ there is a source $(X \xrightarrow{f_i} Y_i)_{i \in I}$ with $Y_i \in \mathcal{A}$ for every $i \in I$, such that $n \simeq \cap f^{-1}(n_i)$ for some $n_i \in \mathcal{N}_{Y\,i}$.

Proof:

First notice that if $X \in D_{\mathcal{N}}(T_{\mathcal{N}}(\mathcal{A}))$, then every $n \in \mathcal{N}_X$ is $T_{\mathcal{N}}(\mathcal{A})$-closed, that is $n \simeq n^{T_{\mathcal{N}}(\mathcal{A}))} = \cap\{f^{-1}(p) : Y \in \mathcal{A}, X \xrightarrow{f} Y, P \xrightarrow{p} Y \in \mathcal{N}_Y \text{and } n \leq f^{-1}(p)\}$ (cf. Proposition 2.4).

Conversely, if $X \in \mathcal{X}$ satisfies the condition in the statement, then again from the characterization of $T_{\mathcal{N}}(\mathcal{A})$ in Proposition 2.4, we obtain that every $n \in \mathcal{N}_X$ is $T_{\mathcal{N}}(\mathcal{A})$-closed and consequently $X \in D_{\mathcal{N}}(T_{\mathcal{N}}(\mathcal{A}))$. $\square$

We recall that in [C_2] the following definition of $\mathcal{N}$-constant morphism was given for any subclass $\mathcal{N}$ of $\mathcal{M}$: a morphism $X \xrightarrow{f} Y$ is $\mathcal{N}$-constant if for every $n \in \mathcal{N}_X$, $n_f \simeq (id_X)_f$. This notion was then used to obtain what was called the connectedness-disconnectedness Galois connection. Furthermore, this Galois connection was shown to factor via the collection of all closure operators on $\mathcal{X}$ with respect to $\mathcal{M}$. This result was proved under the assumption of $\mathcal{N}$ being closed under the formation of direct images. However, there are interesting subclasses of $\mathcal{M}$ that do not have this property but they are closed under the formation of pullbacks. We are going to show that a new notion of $\mathcal{N}$-fixed morphism can be formulated in order to provide a description of the connectedness-disconnectedness Galois connection in Definition 2.3, in the case that $\mathcal{N}$ is closed under the formation of pullbacks. Consequently, by putting together the results in [C_2] with the ones in this paper we obtain two different ways of describing the connectedness-disconnectedness Galois connection depending on whether the subclass $\mathcal{N}$ is closed under the formation of direct images or pullbacks.

DEFINITION 2.11

An $\mathcal{X}$-morphism $X \xrightarrow{f} Y$ is called $\mathcal{N}$-fixed if for every $n \in \mathcal{N}_Y$ we have that $f^{-1}(n) \simeq id_X$.

PROPOSITION 2.12 (cf. [H])

Let $\mathcal{N} \subseteq \mathcal{M}$. Define $S(\mathcal{X}) \xrightarrow{\hat{\Delta}_{\mathcal{N}}} S(\mathcal{X})^{\mathbf{op}}$ and $S(\mathcal{X})^{\mathbf{op}} \xrightarrow{\hat{\nabla}_{\mathcal{N}}} S(\mathcal{X})$ as follows:

$\hat{\nabla}_{\mathcal{N}}(\mathcal{A}) = \{X \in \mathcal{X} : \forall Y \in \mathcal{A}, X \xrightarrow{f} Y \text{ is } \mathcal{N}\text{-fixed}\}$

$\hat{\Delta}_{\mathcal{N}}(\mathcal{B}) = \{Y \in \mathcal{X} : \forall X \in \mathcal{B}, X \xrightarrow{f} Y \text{ is } \mathcal{N}\text{-fixed}\}$

Then, $S(\mathcal{X}) \underset{\hat{\nabla}_{\mathcal{N}}}{\overset{\hat{\Delta}_{\mathcal{N}}}{\rightleftarrows}} S(\mathcal{X})^{\mathbf{op}}$ is a Galois connection. □

THEOREM 2.13

Let $\mathcal{N}$ be a subclass of $\mathcal{M}$ closed under the formation of pullbacks. Then we have that the Galois connection $S(\mathcal{X}) \underset{\hat{\nabla}_{\mathcal{N}}}{\overset{\hat{\Delta}_{\mathcal{N}}}{\rightleftarrows}} S(\mathcal{X})^{\mathbf{op}}$ factors through $CL(\mathcal{X}, \mathcal{M})$ via the Galois connections $S(\mathcal{X}) \underset{I_{\mathcal{N}}}{\overset{J_{\mathcal{N}}}{\rightleftarrows}} CL(\mathcal{X}, \mathcal{M})$ and $CL(\mathcal{X}, \mathcal{M}) \underset{T_{\mathcal{N}}}{\overset{D_{\mathcal{N}}}{\rightleftarrows}} S(\mathcal{X})^{\mathbf{op}}$ described above; i.e., it is precisely the Galois connection of Definition 2.3.

Proof:

Let $\mathcal{A} \in S(\mathcal{X})^{\mathbf{op}}$ and let $X \in (I_{\mathcal{N}} \circ T_{\mathcal{N}})(\mathcal{A})$. Consider $X \xrightarrow{f} Y$ with $Y \in \mathcal{A}$ and let $N \xrightarrow{n} Y$ belong to $\mathcal{N}_Y$. Since n is $T_{\mathcal{N}}(\mathcal{A})$-closed, from the properties of closure operators, we have that also $f^{-1}(n)$ is $T_{\mathcal{N}}(\mathcal{A})$-closed; i.e., $f^{-1}(n)^{T_{\mathcal{N}}(\mathcal{A})} \simeq f^{-1}(n)$. Since $X \in (I_{\mathcal{N}} \circ T_{\mathcal{N}})(\mathcal{A})$ and $f^{-1}(n) \in \mathcal{N}$, we have that $f^{-1}(n)^{T_{\mathcal{N}}(\mathcal{A})} \simeq id_X$. So, $f^{-1}(n) \simeq id_X$. Thus $(I_{\mathcal{N}} \circ T_{\mathcal{N}})(\mathcal{A}) \subseteq \hat{\nabla}_{\mathcal{N}}(\mathcal{A})$.

Now, let $X \in \hat{\nabla}_{\mathcal{N}}(\mathcal{A})$. Consider $X \xrightarrow{f} Y$ with $Y \in \mathcal{A}$ and an $\mathcal{N}$-subobject $N \xrightarrow{n} X$. If $N_1 \xrightarrow{n_1} Y$ belongs to $\mathcal{N}$, then by our assumption on X, the pullback $f^{-1}(n_1) \simeq id_X$. Thus, we obtain that $n^{T_{\mathcal{N}}(\mathcal{A})} \simeq id_X$ (cf. Proposition 2.4). So, $X \in (I_{\mathcal{N}} \circ T_{\mathcal{N}})(\mathcal{A})$ and therefore $I_{\mathcal{N}} \circ T_{\mathcal{N}} = \hat{\nabla}_{\mathcal{N}}$.

Let $\mathcal{B} \in S(\mathcal{X})$ and let $Y \in (D_{\mathcal{N}} \circ J_{\mathcal{N}})(\mathcal{B})$. Consider $X \xrightarrow{f} Y$ with $X \in \mathcal{B}$. If $N \xrightarrow{n} Y$ belongs to $\mathcal{N}_Y$, then from the assumption on $\mathcal{N}$, we have that $f^{-1}(n) \in \mathcal{N}_X$. Since $X \in \mathcal{B}$, we have that $f^{-1}(n)^{J_{\mathcal{N}}(\mathcal{B})} \simeq id_X$. Since $Y \in (D_{\mathcal{N}} \circ J_{\mathcal{N}})(\mathcal{B})$, n is $J_{\mathcal{N}}(\mathcal{B})$-closed and from the properties of closure operators, so is $f^{-1}(n)$. Thus, we have that $f^{-1}(n) \simeq f^{-1}(n)^{J_{\mathcal{N}}(\mathcal{B})} \simeq id_X$. Therefore, we have that $Y \in \hat{\Delta}_{\mathcal{N}}(\mathcal{B})$ and so $(D_{\mathcal{N}} \circ J_{\mathcal{N}})(\mathcal{B}) \subseteq \hat{\Delta}_{\mathcal{N}}(\mathcal{B})$.

Now, let $Y \in \hat{\Delta}_{\mathcal{N}}(\mathcal{B})$ and let $N \xrightarrow{n} Y \in \mathcal{N}_Y$. Consider $X \xrightarrow{f} Y$ with $X \in \mathcal{B}$. By the assumption on $\mathcal{N}$, $f^{-1}(n) \in \mathcal{N}_X$. Since $(f^{-1}(n))_f \leq n$, $(id_X)_f$ occurs in the construction of $n^{J_{\mathcal{N}}(\mathcal{B})}$. Since $f^{-1}(n) \simeq id_X$, we obtain that $(id_X)_f \simeq (f^{-1}(n))_f \leq n$ and therefore $n^{J_{\mathcal{N}}(\mathcal{B})} \simeq n$. Thus, $Y \in (D_{\mathcal{N}} \circ J_{\mathcal{N}})(\mathcal{B})$. Therefore we conclude that $D_{\mathcal{N}} \circ J_{\mathcal{N}} = \hat{\Delta}_{\mathcal{N}}$. □

Notice that since the adjoint part of a Galois connection completely determines the coadjoint part, we could have omitted the second part of the above proof. However, we included it for the purpose of clarity.

The following proposition shows that there is a strong relationship between the definition of $\mathcal{N}$-constant morphism that appeared in [C_2] and the one of $\mathcal{N}$-fixed morphism in this paper.

PROPOSITION 2.14

Let $X \xrightarrow{f} Y$ be an $\mathcal{X}$-morphism. Then we have the following:

(a) If $\mathcal{N}$ is closed under the formation of pullbacks, then every $\mathcal{N}$-constant morphism is $\mathcal{N}$-fixed;

(b) If $\mathcal{N}$ is closed under the formation of direct images, then every $\mathcal{N}$-fixed morphism is $\mathcal{N}$-constant;

(c) If $\mathcal{N}$ is closed under the formation of both pullbacks and direct images, then the two notions of $\mathcal{N}$-constant and $\mathcal{N}$-fixed morphism are equivalent.

Proof:

(a). Suppose that $\mathcal{N}$ is closed under the formation of pullbacks. If $n \in \mathcal{N}_Y$, then by hypothesis $f^{-1}(n) \in \mathcal{N}_X$. Then, the fact that f is $\mathcal{N}$-constant implies that $(id_X)_f \simeq (f^{-1}(n))_f$. Thus, we have that $id_X = f^{-1}((id_X)_f) \simeq f^{-1}((f^{-1}(n))_f) \simeq f^{-1}(n)$. Therefore we conclude that $f^{-1}(n) \simeq id_X$, i.e., f is $\mathcal{N}$-fixed.

(b). Suppose now that $\mathcal{N}$ is closed under the formation of direct images. If $n \in \mathcal{N}_X$, then $n_f \in \mathcal{N}_Y$, and since f is $\mathcal{N}$-fixed, we have that $f^{-1}(n_f) \simeq id_X$. Thus we obtain that $n_f \simeq (f^{-1}(n_f))_f \simeq (id_X)_f$. So, f is $\mathcal{N}$-constant.

(c). This follows immediately from (a) and (b). $\square$

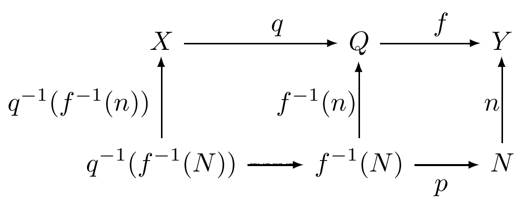
Next we analyze some closedness properties of the Galois connection in Proposition 2.12.

PROPOSITION 2.15

For every $\mathcal{A} \in S(\mathcal{X})^{\mathbf{op}}$, $\hat{\nabla}_{\mathcal{N}}(\mathcal{A})$ is closed under **E***-quotients.*

Proof:

Let us consider the following commutative diagram



where, $X \in \hat{\nabla}_{\mathcal{N}}(\mathcal{A})$, $q \in \mathbf{E}$, $Y \in \mathcal{A}$ and $n \in \mathcal{N}_Y$. Since $q^{-1}(f^{-1}(n)) \simeq (f \circ q)^{-1}(n) \simeq id_X$, we have that $(id_X)_q \simeq (q^{-1}(f^{-1}(n))_q \leq f^{-1}(n)$. Since $q \in \mathbf{E}$, we have that $(id_X)_q \simeq id_Q$, that is, $id_Q \leq f^{-1}(n)$. This implies that $f^{-1}(n) \simeq id_Q$. Thus we conclude that $Q \in \hat{\nabla}_{\mathcal{N}}(\mathcal{A})$. $\square$

PROPOSITION 2.16

Let $\mathcal{A} \in S(\mathcal{X})^{\mathbf{op}}$ and let $X_i \in \hat{\nabla}_{\mathcal{N}}(\mathcal{A})$ for every $i \in I$. If the coproduct $\amalg X_i$ exists, then it also belongs to $\hat{\nabla}_{\mathcal{N}}(\mathcal{A})$.

Proof:

Let us consider the following commutative diagram

$$\begin{array}{ccccc}
X_i & \xrightarrow{k_i} & \amalg X_i & \xrightarrow{f} & Y \\
{\scriptstyle s_i}\uparrow & & {\scriptstyle f^{-1}(n)}\uparrow & & \uparrow{\scriptstyle n} \\
k_i^{-1}(f^{-1}(N)) & \xrightarrow[t_i]{} & f^{-1}(N) & \xrightarrow[\bar{f}]{} & N
\end{array}$$

where $X_i \in \hat{\nabla}_{\mathcal{N}}(\mathcal{A})$ for every $i \in I$, $Y \in \mathcal{A}$ and $s_i = k_i^{-1}(f^{-1}(n))$ with $n \in \mathcal{N}_Y$. Notice that by hypothesis s_i is an isomorphism for every $i \in I$. The universal property of coproducts implies the existence of a unique morphism $\amalg X_i \xrightarrow{d} f^{-1}(N)$ such that $d \circ k_i = t_i \circ s_i^{-1}$, for every $i \in I$. Therefore we have that $f^{-1}(n) \circ d \circ k_i = f^{-1}(n) \circ t_i \circ s_i^{-1} = k_i \circ s_i \circ s_i^{-1} = k_i = id_{\amalg X_i} \circ k_i$, for every $i \in I$. The universal property of coproducts implies that $f^{-1}(n) \circ d = id_{\amalg X_i}$. Thus, since $f^{-1}(n)$ is a monomorphism and a retraction, we can conclude that it is an isomorphism. Hence, $\amalg X_i$ belongs to $\hat{\nabla}_{\mathcal{N}}(\mathcal{A})$. □

It is interesting to notice that if $\mathcal{X}$ is well powered and has coproducts, as a consequence of Propositions 2.15 and 2.16, we have that for every $\mathcal{A} \in S(\mathcal{X})^{\mathbf{op}}$, $\hat{\nabla}_{\mathcal{N}}(\mathcal{A})$ is an $\mathcal{M}$-coreflective subcategory of $\mathcal{X}$ (cf. [AHS, Theorem 16.8], dual).

PROPOSITION 2.17

Let $\mathcal{N}$ be closed under direct images along morphisms in $\mathcal{M}$. For every subcategory $\mathcal{B} \in S(\mathcal{X})$, $\hat{\Delta}_{\mathcal{N}}(\mathcal{B})$ is closed under $\mathcal{M}$-subobjects.

Proof:

Let $M \xrightarrow{m} Y$ be an $\mathcal{M}$-subobject with $Y \in \hat{\Delta}_{\mathcal{N}}(\mathcal{B})$, let $X \in \mathcal{B}$ and let $N \xrightarrow{n} M$ be an $\mathcal{N}$-subobject of M. Consider the $\mathcal{X}$-morphism $X \xrightarrow{f} M$. If (e_1, n_1) is the $(\mathbf{E}, \mathcal{M})$-factorization of $m \circ n$ then we obtain the following commutative diagram

$$\begin{array}{ccccc}
X & \xrightarrow{f} & M & \xrightarrow{m} & Y \\
{\scriptstyle f^{-1}(n)}\uparrow & & {\scriptstyle n}\uparrow & & {\scriptstyle n_1}\uparrow \\
f^{-1}(N) & \longrightarrow & N & \xrightarrow[e_1]{} & N_1
\end{array}$$

We first observe that since $m \circ n \in \mathcal{M}$, we have that e_1 is an isomorphism. Let h be the morphism such that $m \circ f \circ (m \circ f)^{-1}(n_1) = n_1 \circ h$. Then, by the universal property of pullbacks, there exists a morphism t such that $(m \circ f)^{-1}(n_1) \circ t = f^{-1}(n)$. Since $n_1 =$

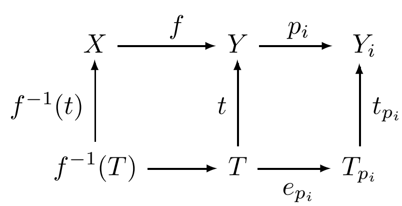
$m \circ n \circ e_1^{-1}$, we obtain that $m \circ f \circ (m \circ f)^{-1}(n_1) = m \circ n \circ e_1^{-1} \circ h$. The fact that m is a monomorphism implies that $f \circ (m \circ f)^{-1}(n_1) = n \circ e_1^{-1} \circ h$. Thus, again from the universal property of pullbacks applied to the left square of the above diagram, we obtain a morphism d such that $(m \circ f)^{-1}(n_1) = f^{-1}(n) \circ d$. This, together with $(m \circ f)^{-1}(n_1) \circ t = f^{-1}(n)$, implies that $f^{-1}(n) \circ d \circ t = f^{-1}(n) = f^{-1}(n) \circ id_{f^{-1}(N)}$. Since $f^{-1}(n)$ is a monomorphism, we obtain that $d \circ t = id_{f^{-1}(N)}$. Therefore, d is a monomorphism and a retraction and so it is also an isomorphism. Hence f must be $\mathcal{N}$-fixed. □

PROPOSITION 2.18

Let $\mathcal{X}$ have a terminal object T and assume that any morphism with T as domain belongs to $\mathcal{M}$. If $\mathbf{E}$ is contained in the class of episinks and $\mathcal{N}$ consists of all morphisms having T as domain, then for every subclass $\mathcal{B} \in S(\mathcal{X})$, we have that $\hat{\Delta}_{\mathcal{N}}(\mathcal{B})$ is closed under monosources.

Proof:

Let $X \in \mathcal{B}$ and let $(Y \xrightarrow{p_i} Y_i)_{i \in I}$ be a monosource with $Y_i \in \hat{\Delta}_{\mathcal{N}}(\mathcal{B})$, for every $i \in I$. Let us consider the following commutative diagram



where (e_{p_i}, t_{p_i}) is the $(\mathbf{E}, \mathcal{M})$-factorization of $p_i \circ t$. Notice that the fact that T is a terminal object implies that e_{p_i} is an epimorphism and a section and consequently it is also an isomorphism. By hypothesis $(p_i \circ f)^{-1}(t_{p_i}) \simeq id_X$. It is just a technicality to show that $f^{-1}(t) \simeq \cap (p_i \circ f)^{-1}(t_{p_i})$, so we omit the lengthy details. This implies that $f^{-1}(t) \simeq id_X$. Thus, $Y \in \hat{\Delta}_{\mathcal{N}}(\mathcal{B})$. □

We observe that in many concrete categories such as for instance topological spaces and groups, under the appropriate assumptions, Propositions 2.17 and 2.18 imply that for every $\mathcal{B} \in S(\mathcal{X})$, $\hat{\Delta}_{\mathcal{N}}(\mathcal{B})$ is an epireflective subcategory (cf. [AHS, Theorem 16.8]).

We have seen in [C_2, Proposition 2.6] that for every $\mathcal{A} \in S(\mathcal{X})^{\mathbf{op}}$, the closure operator $T_{\mathcal{N}}(\mathcal{A})$ is always idempotent. We are going to show that under certain assumptions it is also weakly hereditary. We recall the following corollary from [C_3].

COROLLARY 2.19 ([C_3, Corollary 2.19])

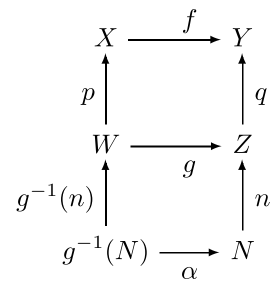
Let $\mathcal{A}$ be a full, reflective subcategory of $\mathcal{X}$ and for $X \in \mathcal{X}$, let $X \xrightarrow{r_X} rX$ denote the reflection morphism. Then, for every $\mathcal{N} \subseteq \mathcal{M}$ and for every $\mathcal{M}$-subobject $M \xrightarrow{m} X$, we have *that*

$$m^{T_{\mathcal{N}}(\mathcal{A})} \simeq \cap\{r_X^{-1}(n) : N \xrightarrow{n} rX \in \mathcal{N} \text{and } m \leq r_X^{-1}(n)\}.$$

□

LEMMA 2.20

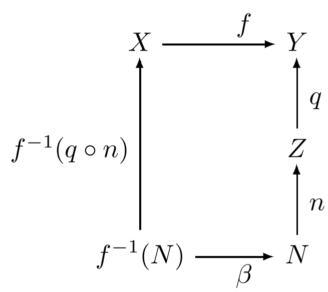
Let us consider the diagram



satisfying the following conditions: the upper square is commutative, p and q are monomorphisms, the lower square is a pullback and n is a monomorphism. Then, we have that $p \circ g^{-1}(n) \simeq f^{-1}(q \circ n) \cap p$.

Proof:

Let us consider the pullback square



By the universal property of pullbacks, there exists a monomorphism $g^{-1}(N) \xrightarrow{t} f^{-1}(N)$ such that $f^{-1}(q \circ n) \circ t = p \circ g^{-1}(n)$ and $\beta \circ t = \alpha$. We need to show that $p \circ g^{-1}(n)$ is isomorphic to the intersection of $f^{-1}(q \circ n)$ and p. To this purpose, let us consider the morphisms $P \xrightarrow{s} X$, $P \xrightarrow{p_1} W$ and $P \xrightarrow{p_2} f^{-1}(N)$ such that $s = p \circ p_1 = f^{-1}(q \circ n) \circ p_2$. Now, $q \circ g \circ p_1 = f \circ p \circ p_1 = f \circ f^{-1}(q \circ n) \circ p_2 = q \circ n \circ \beta \circ p_2$. Since q is a monomorphism, we obtain that $g \circ p_1 = n \circ \beta \circ p_2$. Again by the universal property of pullbacks, we obtain a unique morphism $P \xrightarrow{d} g^{-1}(N)$ such that $g^{-1}(n) \circ d = p_1$ and $\alpha \circ d = \beta \circ p_2$. Consequently we have that $p \circ g^{-1}(n) \circ d = p \circ p_1 = s$. Therefore we obtain that $p \circ g^{-1}(n) \simeq f^{-1}(q \circ n) \cap p$. □

PROPOSITION 2.21

Let $\mathcal{A}$ be a full reflective subcategory of $\mathcal{X}$ with reflection functor r. Assume that the following two conditions are met:

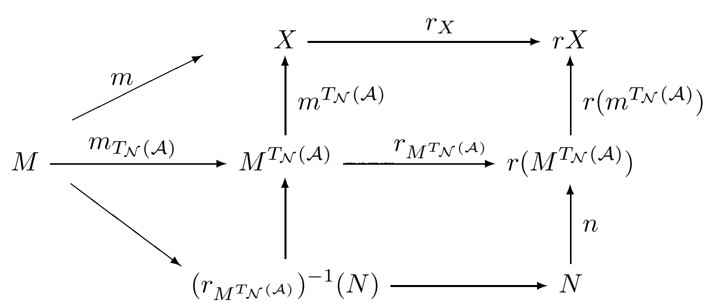
(a) For every $\mathcal{M}$-subobject $M \xrightarrow{m} X$, $r(m) \in \mathcal{M}$;

(b) $\mathcal{N}$ is closed under the formation of direct images along morphisms in $\mathcal{M}$.

Then, $T_{\mathcal{N}}(\mathcal{A})$ is weakly hereditary.

Proof:

Let us consider the $\mathcal{M}$-subobject $M \xrightarrow{m} X$. We are going to show that the morphism $M \xrightarrow{m_{T_{\mathcal{N}}(\mathcal{A})}} M^{T_{\mathcal{N}}(\mathcal{A})}$ is $T_{\mathcal{N}}(\mathcal{A})$-dense. Clearly $(m_{T_{\mathcal{N}}(\mathcal{A})})^{T_{\mathcal{N}}(\mathcal{A})} \leq id_{M^{T_{\mathcal{N}}(\mathcal{A})}}$. To prove that the other inequality holds we will make use of the expression of $T_{\mathcal{N}}(\mathcal{A})$ given in Corollary 2.19. Let us consider the following commutative diagram



with $m \in \mathcal{M}$ and $n \in \mathcal{N}$ such that $m_{T_{\mathcal{N}}(\mathcal{A})} \leq (r_{M^{T_{\mathcal{N}}(\mathcal{A})}})^{-1}(n)$. Let (e_1, m_1) be the $(\mathbf{E}, \mathcal{M})$-factorization of $r(m^{T_{\mathcal{N}}(\mathcal{A})}) \circ n$. Conditions (a) and (b) imply that $m_1 \in \mathcal{N}$ and condition (a) alone yields that e_1 is an isomorphism. To simplify the notation, let $t = r(m^{T_{\mathcal{N}}(\mathcal{A})}) \circ n$. By applying Lemma 2.20, we obtain that $m^{T_{\mathcal{N}}(\mathcal{A})} \circ (r_{M^{T_{\mathcal{N}}(\mathcal{A})}})^{-1}(n) \simeq m^{T_{\mathcal{N}}(\mathcal{A})} \cap r_X^{-1}(t) \simeq m^{T_{\mathcal{N}}(\mathcal{A})} \cap r_X^{-1}(m_1)$. Since $r_X^{-1}(m_1)$ occurs in the construction of $m^{T_{\mathcal{N}}(\mathcal{A})}$ (cf Proposition 2.4), we have that $m^{T_{\mathcal{N}}(\mathcal{A})} \cap r_X^{-1}(m_1) \simeq m^{T_{\mathcal{N}}(\mathcal{A})}$. Therefore, we have that $m^{T_{\mathcal{N}}(\mathcal{A})} \circ (r_{M^{T_{\mathcal{N}}(\mathcal{A})}})^{-1}(n) \simeq m^{T_{\mathcal{N}}(\mathcal{A})}$ for every $\mathcal{N}$-subobject $N \xrightarrow{n} r(M^{T_{\mathcal{N}}(\mathcal{A})})$ satisfying $m_{T_{\mathcal{N}}(\mathcal{A})} \leq (r_{M^{T_{\mathcal{N}}(\mathcal{A})}})^{-1}(n)$. Thus we can conclude that $(m_{T_{\mathcal{N}}(\mathcal{A})})^{T_{\mathcal{N}}(\mathcal{A})} \simeq m^{T_{\mathcal{N}}(\mathcal{A})}$; i.e., $T_{\mathcal{N}}(\mathcal{A})$ is weakly hereditary. $\square$

We have seen in [C_2, Proposition 2.7] that for every $\mathcal{B} \in S(\mathcal{X})$, the closure operator $J_{\mathcal{N}}(\mathcal{B})$ is always weakly hereditary. We can show that under certain assumptions it is also idempotent. We recall the following result from [C_3].

PROPOSITION 2.22 ([C_3, Proposition 2.20])

Let $\mathcal{B}$ be a full, coreflective subcategory of $\mathcal{X}$ and for $Y \in \mathcal{X}$, let $cY \xrightarrow{c_Y} Y$ denote the coreflection morphism. If $\mathcal{N}$ is closed under the formation of direct images then, for every $\mathcal{M}$-

subobject $M \xrightarrow{m} Y$*, we have that*

$$m^{J_{\mathcal{N}}(\mathcal{B})} \simeq sup\left(\{m\} \cup \{(id_{cY})_{cY} : \exists n \in \mathcal{N}_{cY} \text{ with } n_{c_Y} \leq m\}\right).$$

□

PROPOSITION 2.23

Let $\mathcal{B}$ be a full, coreflective subcategory of $\mathcal{X}$ with coreflection functor c and let $\mathcal{N}$ be closed under the formation of direct images. Then the closure operator $J_{\mathcal{N}}(\mathcal{B})$ is idempotent.

Proof:

Let $M \xrightarrow{m} Y$ be an $\mathcal{M}$-subobject. We observe that from Proposition 2.22 we have that either $m^{J_{\mathcal{N}}(\mathcal{B})} \simeq m$ or $m^{J_{\mathcal{N}}(\mathcal{B})} \simeq m \vee (id_{cY})_{cY}$. In the first case it is obvious that $(m^{J_{\mathcal{N}}(\mathcal{B})})^{J_{\mathcal{N}}(\mathcal{B})} \simeq m^{J_{\mathcal{N}}(\mathcal{B})}$. In the second case we obtain that $(m^{J_{\mathcal{N}}(\mathcal{B})})^{J_{\mathcal{N}}(\mathcal{B})} \simeq (m \vee (id_{cY})_{cY})^{J_{\mathcal{N}}(\mathcal{B})}$. Again we either have that $(m \vee (id_{cY})_{cY})^{J_{\mathcal{N}}(\mathcal{B})} \simeq m \vee (id_{cY})_{cY}$ or that $(m \vee (id_{cY})_{cY})^{J_{\mathcal{N}}(\mathcal{B})} \simeq (id_{cY})_{cY} \vee (m \vee (id_{cY})_{cY}) \simeq (m \vee (id_{cY})_{cY})$. So, we conclude that also in the second case we have that $(m^{J_{\mathcal{N}}(\mathcal{B})})^{J_{\mathcal{N}}(\mathcal{B})} \simeq m^{J_{\mathcal{N}}(\mathcal{B})}$. Therefore $J_{\mathcal{N}}(\mathcal{B})$ is idempotent. □

Since a number of examples that illustrate our theory of connectedness and disconnectedness already appeared in [C_3], here we will simply present a significant one that shows the importance of the definition of $\mathcal{N}$-fixed morphism that appears in this paper.

EXAMPLE 2.24

(a). Let $\mathcal{X}$ be the category **Grp** of groups and homomorphisms and let $\mathcal{N}$ consist of all inclusions of normal subgroups. Clearly $\mathcal{N}$ is closed under the formation of pullbacks but not under the formation of direct images. Notice that in this case, $X \xrightarrow{f} Y$ is $\mathcal{N}$-fixed if and only if f is constant in the classical sense.

Let **Sim** denote the subcategory of simple groups, i.e., all those groups that have no nontrivial normal subgroups. Now we show that $\hat{\Delta}_{\mathcal{N}}(\mathbf{Sim}) = \mathbf{Simfree}$, i.e., the subcategory of all groups that have no simple subgroup different from zero. Clearly, since **Sim** is closed under the formation of quotients, if $X \xrightarrow{f} Y$ is a homomorphism with $X \in \mathbf{Sim}$ and $Y \in \mathbf{Simfree}$, then the fact that $f(X) \in \mathbf{Sim}$ and $f(X) \leq Y$ imply that $f(X) = 0$, i.e., f is constant. Therefore we have that $\mathbf{Simfree} \subseteq \hat{\Delta}_{\mathcal{N}}(\mathbf{Sim})$. Conversely, suppose that $Y \in \hat{\Delta}_{\mathcal{N}}(\mathbf{Sim})$ and that $Y \notin \mathbf{Simfree}$. Then, there exists a simple subgroup K of Y different from zero. Consequently, the inclusion $K \xrightarrow{i} Y$ is a non-constant morphism. This is a contradiction with our assumption. Thus, we conclude that $\mathbf{Simfree} = \hat{\Delta}_{\mathcal{N}}(\mathbf{Sim})$.

Using Proposition 2.4 it is easy to see that for every subgroup $M \leq Y$, $M^{J_{\mathcal{N}}(\mathbf{Sim})}$ is the subgroup generated by M and all simple subgroups of Y. Clearly from Theorem 2.13 we have that $D_{\mathcal{N}}(J_{\mathcal{N}}(\mathbf{Sim})) = \mathbf{Simfree}$.

Next we show that $\hat{\nabla}_{\mathcal{N}}(\mathbf{Simfree})$ consists of all groups X such that if K is a normal subgroup of X, then X/K has a simple subgroup different from zero. Let us denote this subcategory by **Simquo**. Let $X \in \mathbf{Simquo}$ and $Y \in \mathbf{Simfree}$. Suppose that there exists a non-constant morphism $X \xrightarrow{f} Y$. Then, $ker f \neq X$ and so $X/ker f$ has a simple subgroup different from zero. Since $X/ker f \simeq f(X) \leq Y$, we obtain a contradiction. Therefore we have that $\mathbf{Simquo} \subseteq \hat{\nabla}_{\mathcal{N}}(\mathbf{Simfree})$. Conversely, suppose that $X \in \hat{\nabla}_{\mathcal{N}}(\mathbf{Simfree})$ and that $X \notin \mathbf{Simquo}$. Then, there exists a normal subgroups K of X such that X/K has no simple subgroup different from zero, that is $X/K \in \mathbf{Simfree}$. Clearly, the quotient morphism $X \xrightarrow{q} X/K$ is not constant. This yields a contradiction. Therefore we can conclude that $\mathbf{Simquo} = \hat{\nabla}_{\mathcal{N}}(\mathbf{Simfree})$.

(b). Part (a) can be generalized as follows. Let $\mathcal{A} \in S(\mathbf{Grp})$. If $\mathcal{A}$ is closed under subgroups and quotients, then it is easy to see that for every subgroup M of a group X, $M^{T_{\mathcal{N}}(\mathcal{A})}$ is the intersection of all normal subgroups H of X such that $X/H \in \mathcal{A}$. That is, $T_{\mathcal{N}}(\mathcal{A})$ agrees with the $\mathcal{A}$-normal closure operator (cf. [FJ], [FW]). As in part (a), using the characterization given in Theorem 2.13 we have that $\mathcal{B} = (I_{\mathcal{N}} \circ T_{\mathcal{N}})(\mathcal{A})$ consists of all those groups X that do not have any proper normal subgroup N such that $X/N \in \mathcal{A}$. Moreover, for every subgroup M of Y, $M^{J_{\mathcal{N}}(\mathcal{B})}$ is the subgroup generated by M and by those subgroups S of Y which do not have any proper normal subgroup N such that $S/N \in \mathcal{A}$.

In the above example we can see the usefulness of being able to describe the connectedness-disconnectedness Galois connection by means of our notion of $\mathcal{N}$-fixed morphism. As a matter of fact it is quite difficult to characterize the Galois closed classes in part (a), using Definition 2.3 directly. The problem lies in the fact that it is not easy to characterize $T_{\mathcal{N}}(\mathbf{Simfree})$. Notice that our previous notion of $\mathcal{N}$-constant morphism cannot be used in this case since $\mathcal{N}$ is not closed under the formation of direct images.

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Department of Mathematics, University of Puerto Rico, Mayagüez campus, P.O. Box 5000, Mayagüez, PR 00681-5000, U.S.A.