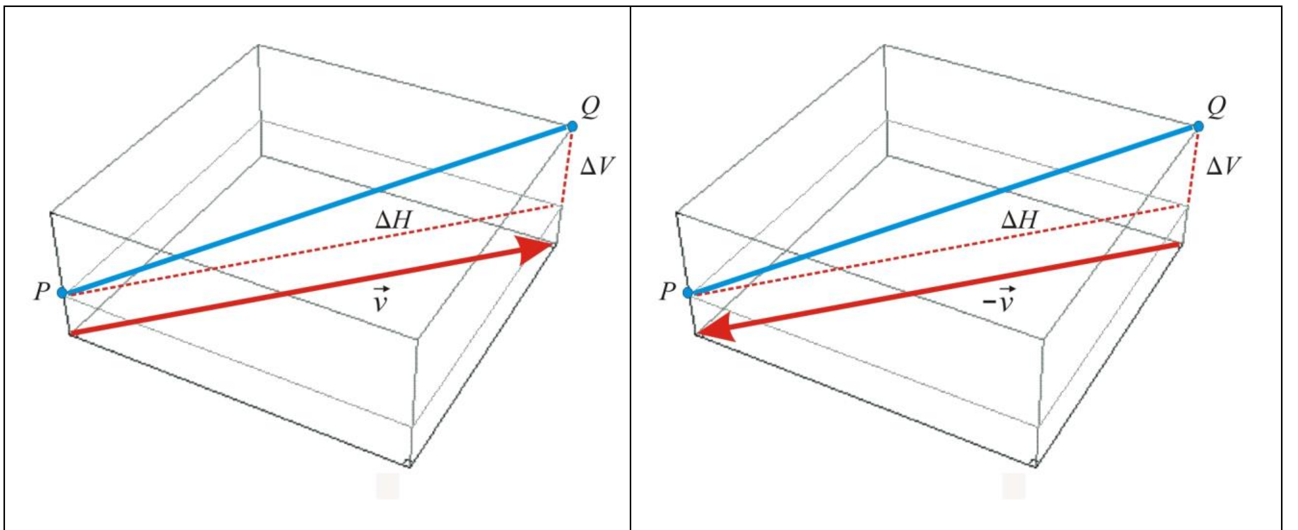
ACTIVITY #10 – DIRECTIONAL DERIVATIVE

Name:______________________________________

A directional derivative is a slope so one starts by defining slopes in three dimensions.

Slope of a line in 3D:

Given a line in three dimensions we wish to obtain its slope as $\frac{\Delta V}{\Delta H} = \frac{\text{vertical change}}{\text{horizontal change}}$. To do that, take two points P and Q on the line and consider the line segment $\overline{PQ}$. In the following figure, observe that if the vertical change ΔV is computed from P to Q it is positive and if it is computed from Q to P it is negative. On the other hand, the horizontal change ΔH must be obtained as a distance or as a vector magnitude, so that it will always be positive (this is what makes slopes in three dimensions different from slopes in two dimensions). Therefore, **slopes in three dimensions are directed slopes**, the sign depends on whether it is computed from Q to P or from P to Q. To give a direction to the line, the segment $\overline{PQ}$ is projected over a horizontal plane and one chooses the vector $\vec{v}$ that corresponds to the direction from P to Q or the vector $-\vec{v}$ that corresponds to the direction from Q to P (see the figure). Then one computes the slope as always:



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The slope of the line in direction of vector $\vec{v}$ is

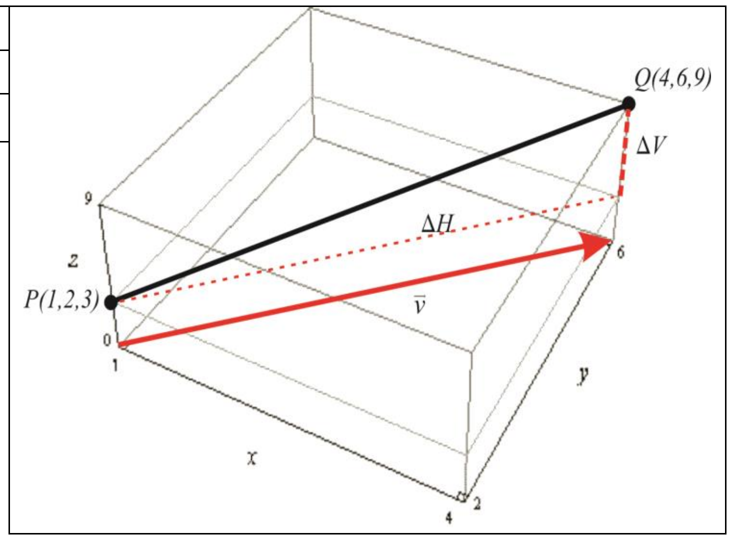
$$\frac{\Delta V}{\Delta H} = \frac{\text{vertical change from } P \text{ to } Q}{|\vec{v}|} \quad \text{where}$$

ΔV, the **vertical change** from P to Q is the z coordinate of the final point Q minus the z coordinate of the initial point P.

ΔH, the **horizontal change** from P to Q is the magnitude of vector $\vec{v}$.

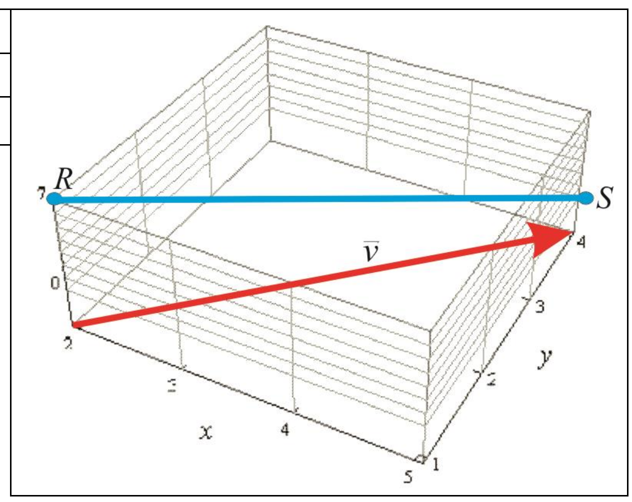
1. In this problem the slope of the line segment $\overline{PQ}$ is computed, first in direction of the vector $\vec{v}$ given in the figure and then in direction of vector $-\vec{v}$. If the problem is done correctly one slope will be the negative of the other. Fill the table below.

Direction	ΔV	ΔH	slope
$\vec{v}$			
$-\vec{v}$			



2. Consider the figure below to fill the table.

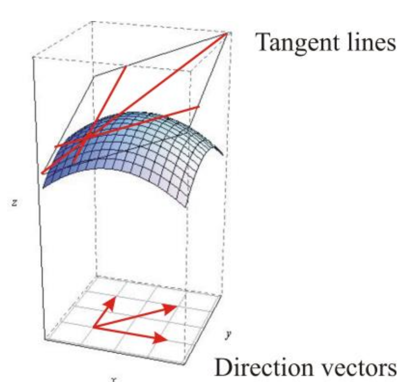
direction	ΔV	ΔH	slope
$\vec{v}$			
$-\vec{v}$			



3. Reflect on the work done in problems 1 and 2.
   a. What is the effect on the slope of a line when the direction $\vec{v}$ of the slope is changed for the opposite direction $-\vec{v}$? Why slopes in three dimensions must be directed slopes?
   b. How does the slope of a line (in 3D) in direction of a vector $\vec{v}$ compare with the slope of the same line in direction of vector $2\vec{v}$ or of vector $\frac{1}{2}\vec{v}$?
   c. A partial derivative of a function of two variables is a slope of a tangent line to the graph of a function in three-dimensional space and hence must be a directed slope. What could be the direction vector (keep in mind that direction vectors only have two coordinates) of a partial derivative with respect to *x*? With respect to *y*?

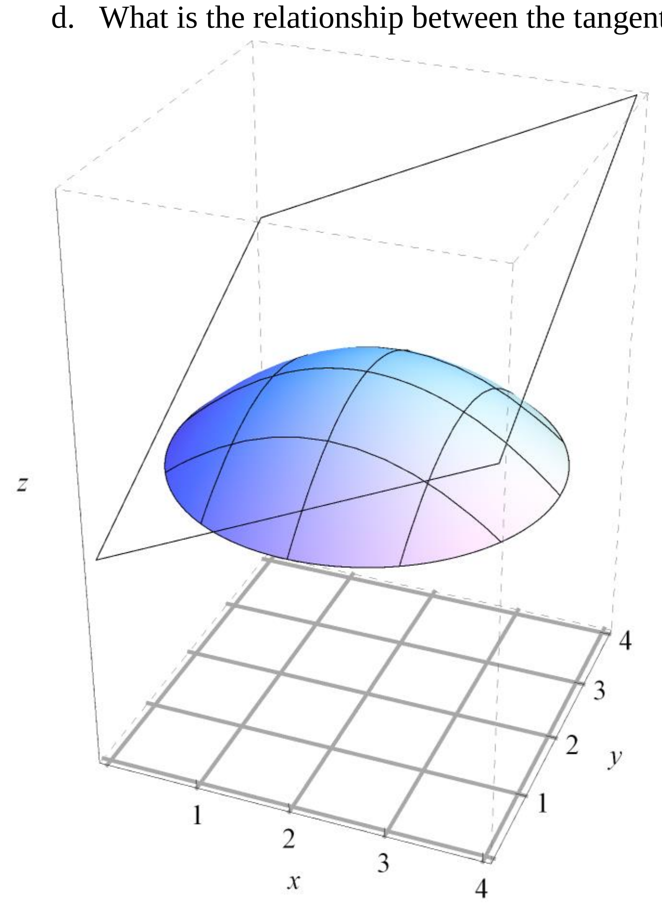
Locating base point, direction vector, and tangent line:

The **tangent plane** at a given point on a surface consists of all the lines that are tangent to the surface at the point at all possible directions.



4. The following is the graph of $z = f(x, y)$ with its tangent plane at the point $(1, 1, f(1, 1))$.

a. Identify the base point $(1, 1, f(1, 1))$ on the figure.
b. Represent the direction vectors $\langle 0,-1\rangle$, $\langle 1,0\rangle$, and $\langle 1,1\rangle$ starting under the base point on the inferior horizontal plane of the figure.
c. For each one of the direction vectors of the previous part, draw as carefully as you can a piece of curve on the surface that is in the given direction, and the tangent line to the surface at the base point in the given direction.
d. What is the relationship between the tangent lines you drew and the tangent plane?



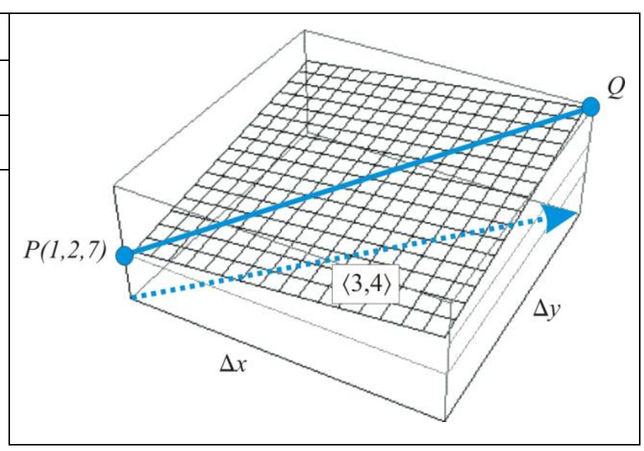
Slope of a line on a plane:

In the following problems we keep on computing the slope of a line in a given direction, the difference is that now the lines are represented as part of a plane. This allows using the plane to find vertical change on the line. Recall that on a plane, vertical change from a point *P* to a point *Q* is given by:

$$\begin{aligned} dz &= dz_x + dz_y \\ &= m_x dx + m_y dy \end{aligned}$$

5. Suppose that in the plane given below $m_x = 5$ and $m_y = 6$. Fill the table.

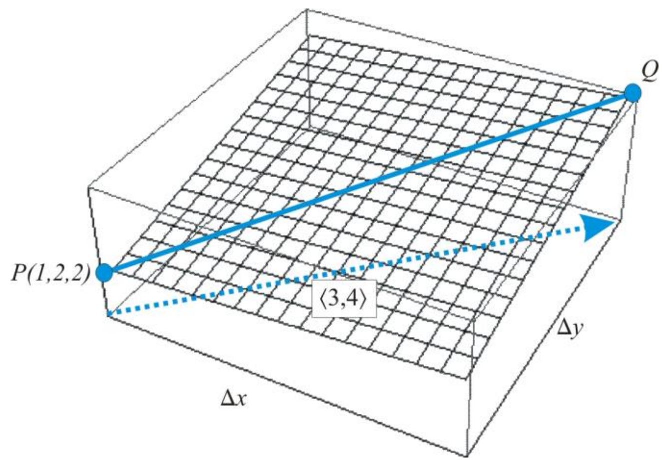
direction	dx	dy	ΔV	ΔH	slope
$\langle 3,4\rangle$					
$\langle -3,-4\rangle$					



The following problems are like the previous one, the only difference is that the planes are tangent planes and hence the slopes may be computed as partial derivatives of the functions.

6. Let $f(x,y) = x^2y$. Observe that the point $P(1,2,2)$ is on the graph of *f*.The figure below represents the tangent plane top the graph of *f* at the point $P(1,2,2)$. The segment $\overline{PQ}$ is tangent to the graph of *f* at the point $P(1,2,2)$ in direction of vector $\langle 3,4 \rangle$. Its slope in direction of vector $\langle 3,4 \rangle$ is the directional derivative of *f* at the point $(1,2)$ and is denoted $D_{\langle 3,4 \rangle} f(1,2)$. In this problem $D_{\langle 3,4 \rangle} f(1,2)$ is computed. Fill the table.

dx	dy	ΔH	m_x	m_y	ΔV	$D_{\langle 3,4 \rangle} f(1,2)$



Observe that **vertical change on a tangent plane** may be obtained using the partial derivatives to compute $m_x dx + m_y dy$.

Directional derivative:

The directional derivative of a function *f*, at a point (a,b), in the direction $\langle dx, dy \rangle$, is the slope of the line tangent to the graph of *f*, at the point $(a, b, f(a,b))$, in the direction of vector $\langle dx, dy \rangle$. It is denoted $D_{\langle dx,dy \rangle} f(a,b)$.

7. Let $f(x,y) = xy + x$. Observe that the point $P(2,4,10)$ is on the graph of *f*. The figure below represents the tangent plane to the graph of *f* at the point $P(2,4,10)$. In this problem $D_{\langle 3,5 \rangle} f(2,4)$ is computed.

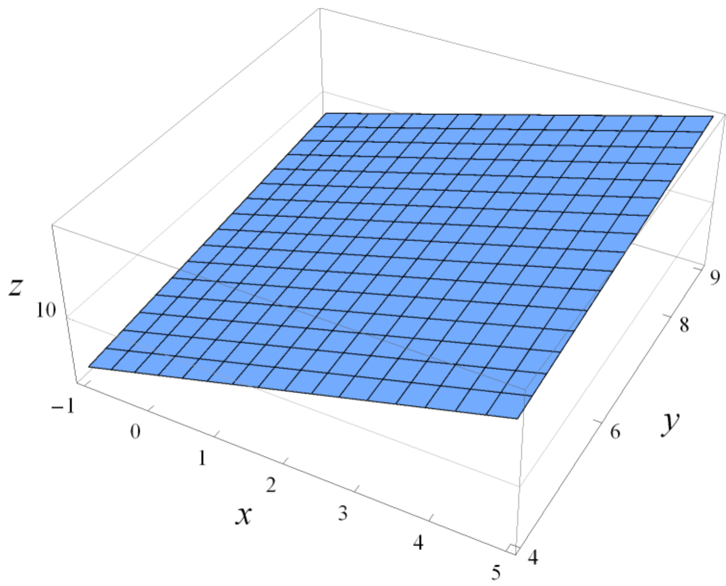
   a. Represent the base point $P(2,4,10)$ in the figure below.

   b. Represent the vector direction $\langle 3,5 \rangle$ in three dimensions as vector $\langle 3,5,0 \rangle$ starting under the base point.

   c. Draw the tangent line to the graph of *f* at the point $P(2,4,10)$ in the given vector direction. That line is part of the tangent plane.

   d. Fill the table to compute of that tangent line in direction of vector $\langle 3,4 \rangle$. That slope is $D_{\langle 3,4 \rangle} f(2,4)$.

dx	dy	ΔH	m_x	m_y	ΔV	$D_{\langle 3,5 \rangle} f(2,4)$



Observe that **a directional derivative** of a function at a given point **is a slope**: the slope of the line that is on the tangent plane of the function at the given point and the given vector direction. Therefore, the vertical change may be obtained as vertical change on a plane.

8. Reflect on what you did on the previous problem and find a formula for the directional derivative of a function *f* at a point (*a*, *b*) in the direction of a vector $\langle dx, dy\rangle$, that is, find a formula for $D_{\langle dx,dy\rangle} f(a,b)$.

9. The following is a table of values of a differentiable two-variable function *f*. Approximate as best you can the value of $D_{\langle 3,4\rangle} f(1,4)$.

y / *x*	2	4	4.02	6	8
0	**5**	**7**	**7.04**	**10**	**11**
1	**6**	**10**	**10.02**	**8**	**9**
1.01	**6.03**	**9.96**	**9.94**	**9.05**	**9.10**
2	**10**	**8**	**7.80**	**7**	**8**
4	**11**	**8.50**	**8.01**	**6**	**5**

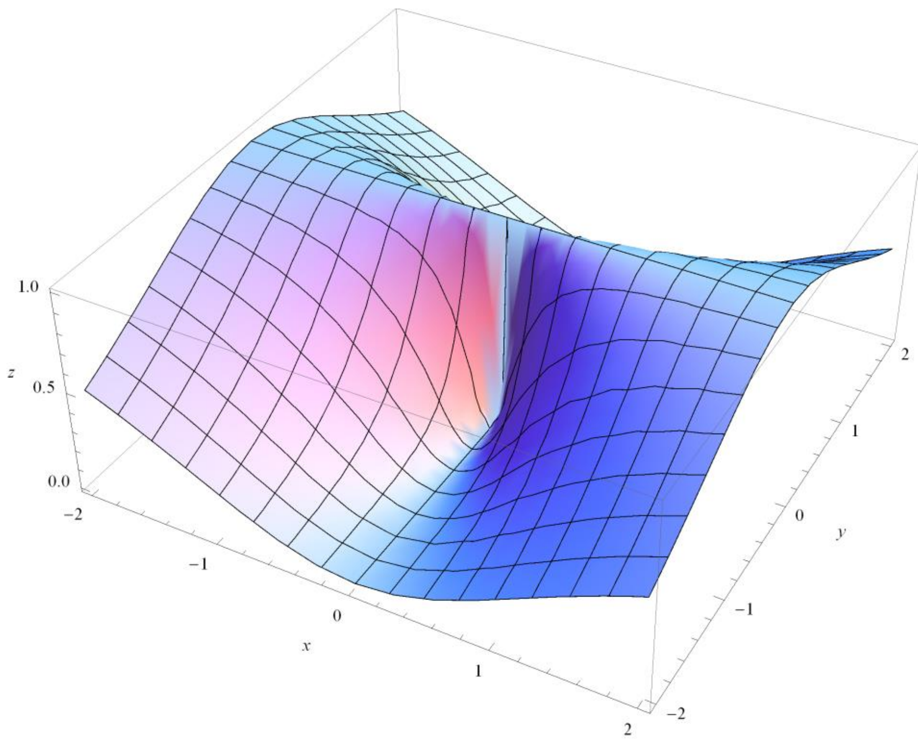
10. The following is the graph of $z = f(x, y)$.

a. First represent on the *xy* plane of the figure below the direction vector $\langle -1,-1\rangle$ starting at the point $(2,-1)$. Then draw in the figure below a piece of the tangent line to the graph of *f* at the point $(2,-1, f(2,-1))$ which is in the given vector direction. The slope of this line in direction of vector $\langle -1,-1\rangle$ is the **directional derivative** of *f* at the point $(2,-1)$ in direction of vector $\langle -1,-1\rangle$ and is denoted $D_{\langle -1,-1\rangle} f(2,-1)$.

b. State the sign of $D_{\langle -1,-1\rangle} f(2,-1)$ briefly justifying your answer.

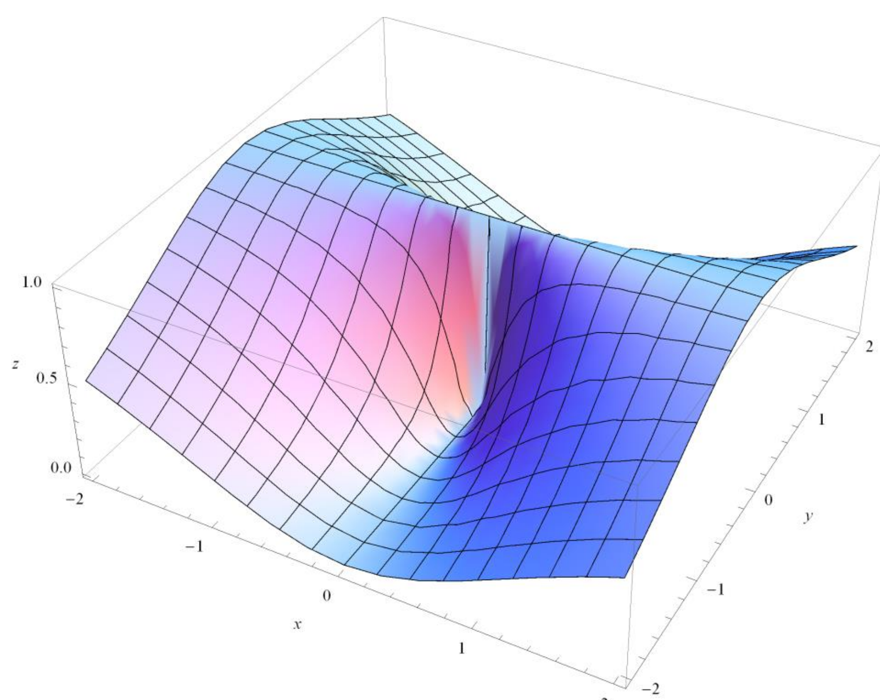
c. Now represent the direction vector $\langle -\frac{1}{2},3\rangle$ on the *xy* plane of the figure below starting at the point $(2,-1)$. Then draw in the figure below a piece of the tangent line to the graph of *f* at the point $(2,-1, f(2,-1))$ which is in direction of the given vector. The slope of this line in direction of vector $\langle -\frac{1}{2},3\rangle$ is $D_{\langle -\frac{1}{2},3\rangle} f(2,-1)$, the directional derivative of *f* at the point $(2,-1)$ in direction of vector $\langle -\frac{1}{2},3\rangle$.

d. State the sign of $D_{\langle -\frac{1}{2},3\rangle} f(2,-1)$ briefly justifying your answer.



11. Let f be the function whose graph appears above. For each value of t the vector $\langle \cos t, \sin t \rangle$ is a direction vector. The value of $D(t) = D_{\langle \cos t, \sin t \rangle} f(1,-1)$ is a number.
    a. Discuss how could the graph of $y = D(t)$ look for values of t in the interval $[0, 2\pi]$.
    b. Discuss how could the graph of $y = G(t) = D_{\langle 1,1 \rangle} f(t,-t)$ look for values of t in $(0,2]$.

12. Suppose the graph of $z = f(x, y)$ is as given below. In this problem use geometric arguments to justify your answers.
    a. Draw the line segment that goes from $(1,-1, f(1,-1))$ to $(2,0, f(2,0))$. How does the slope of this line segment in that direction compare (smaller, equal, larger) with the value of $D_{\langle 1,1 \rangle} f(1,-1)$?
    b. How do $D_{\langle 1,1 \rangle} f(1,-1)$ compare with $D_{\langle 0.01, 0.01 \rangle} f(1,-1)$? Explain.
    c. Which is closer to $D_{\langle 1,1 \rangle} f(1,-1)$, the slope of the line segment that goes from $(1,-1, f(1,-1))$ to $(2,0, f(2,0))$ or the slope of the line segment that goes from $(1,-1, f(1,-1))$ to $(1.01,-0.99, f(1.01,-0.99))$? Explain.



13. The graph of $f(x,y)=\dfrac{x^2}{x^2+y^2}$ appears in the figure above.

a. Compute $D_{\langle -1,-1\rangle} f(2,-1)$. Verify that your answer is consistent with that of problem 4b.

b. Compute $D_{\langle -\frac{1}{2},3\rangle} f(2,-1)$. Verify that your answer is consistent with that of problem 4d.

c. Compute $D_{\langle a,b\rangle} f(2,-1)$.

d. Compute $D_{\langle -\frac{1}{2},3\rangle} f(1,2)$.

e. Compute $D_{\langle -\frac{1}{2},3\rangle} f(a,b)$.