ACTIVITY #9 – THE DIFFERENTIAL

Name:____

The differential: Vertical change on the tangent plane as a function of horizontal change

Let *f* be a function of two independent variables *x* and *y*. The tangent plane to the graph of *f* at the point $(a,b,f(a,b))$ has slope in the *x* direction equal to $f_x(a,b)$ and slope in the *y* direction equal to $f_y(a,b)$. Therefore the vertical change from the point $(a,b,f(a,b))$ an arbitrary point (x,y,z) on the tangent plane is:

$$dz = m_x dx + m_y dy$$

This may be written as:

$$df(a,b) = f_x(a,b)dx + f_y(a,b)dy \quad \text{where}$$

$df(a,b)$ denotes the total vertical change on the tangent plane, that is, $z - f(a,b)$
dx is an independent variable that denotes the horizontal change in the *x* direction
dy is an independent variable that denotes the horizontal change in the *y* direction

***df* (*a*, *b*)** is called **the differential of *f* at the point (*a*, *b*).** It expresses the vertical change $df(a,b)$ on the tangent plane at the point (a,b) as a function of the two independent variables dx, dy that represent the horizontal change.

Note the relationship between the point-slopes equation of the tangent plane at the point $(a,b,f(a,b))$:

$$z - f(a,b) = f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

and the differential at the point (*a*, *b*):

$$df(a,b) = f_x(a,b)dx + f_y(a,b)dy$$

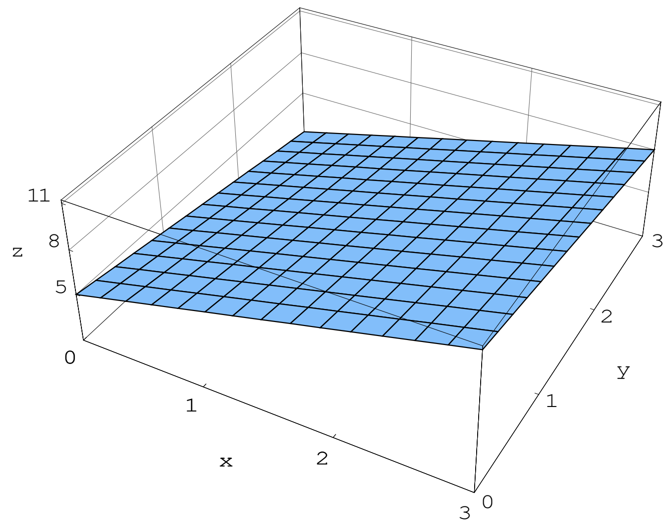
1. Let $f(x,y) = x^3 y$.
   a. Find the equation of the tangent plane to the graph of *f* at the point $(1,2)$ and express it in the form $z - f(a,b) = f_x(a,b)(x-a) + f_y(a,b)(y-b)$.
   b. Use the equation in part a above to find the differential of *f* at the point $(1,2)$.
   c. Evaluate the differential found in part b for $dx = 0.1$ and $dy = 0.2$. Also evaluate the differential for $dx = 3$ and $dy = 4$.

2. The following is a table of the tangent plane to the graph of $z = f(x,y)$ at the point $(1,2,5)$.

x | y	**1**	**2**	**3**
0	4	7	10
1	2	5	8
2	0	3	6

   a. Find the equation of the tangent plane to the graph of *f* at the point $(1,2)$ and express it in the form $z - f(a,b) = f_x(a,b)(x-a) + f_y(a,b)(y-b)$.
   b. Use the equation in part a above to find the differential of *f* at the point $(1,2)$.
   c. Evaluate the differential found in part b for $dx = 0.2$ and $dy = -0.3$. Also evaluate the differential for $dx = -5$ and $dy = 1$.

3. The following is the graph of the tangent plane to the graph of $z = f(x,y)$ at the point (3,0,11).
   a. Find the equation of the tangent plane to the graph of *f* at the point $(3,0)$ and express it in the form $z - f(a,b) = f_x(a,b)(x-a) + f_y(a,b)(y-b)$.
   b. Use the equation in part a above to find the differential of *f* at the point (3,0).
   c. Evaluate the differential found in part b for $dx = -0.2$ and $dy = 0.3$. Also evaluate the differential for $dx = -4$ and $dy = 1$.



4. Let $f(x,y)=x^3y^2+x$.
   a. Find $df(x,y)$. The answer is a function of 4 independent variables: *x*, *y*, *dx*, *dy*.
   b. Find $df(1,2)$. The answer is a function of 2 independent variables: *dx*, *dy*.
   c. Find $df(3,1)$.
   d. Find $df(4,-2)$.

5. Suppose the equation of the tangent plane to the graph of $z=f(x,y)$ at the point (1,2,5) is given by $2x-3y+z=1$.
   a. Find the differential of *f* at the point (1,2).
   b. What is the relationship between the graph of the tangent plane (at the point (1,2,5)) and the graph of the differential (at the point (1,2)? (The graph of the differential is done representing *dx* and *dy* on the horizontal axes and the values of *df*(1,2) on the vertical axis).

The differential and vertical change on the tangent plane as an approximation to change in the values of a function

6. Use a computer program to graph functions[1] of two variables to draw together and compare the graphs of $f(x,y)=xy^2+1$ and its tangent plane at the point $(2,1,f(2,1))$ in each one of the following rectangles. Make sure to show how you found the equation of the tangent plane. In each case print the resulting graph.
   a. $2\le x\le 3$ and $1\le y\le 3$
   b. $2\le x\le 2.1$ and $1\le y\le 1.2$
   c. $2\le x\le 2.01$ and $1\le y\le 1.02$
   d. Describe what you observed in the previous parts.

7. Let $f(x,y)=xy^2+1$.
   a. Find $df(x,y)$.
   b. Find $df(2,1)$
   c. Fill the following table.

Horizontal change	Use the differential to find the vertical change on the tangent plane to the graph of *f* at the point $(2,1,f(2,1))$	Vertical change on the graph of the function	% of error
$dx=1$, $dy=2$			
$dx=0.1$, $dy=0.2$			
$dx=0.01$, $dy=0.02$			

   d. What is the relationship between this problem and the one before?

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[1] For example, the following command in the *Mathematica* software program produces the graph of the function in red and the tangent plane at the point $(2,1,f(2,1))$ in blue. After producing a graph in *Mathematica* it may be rotated using the cursor to obtain a convenient point of view before printing:
Plot3D[{x∗y^2+x,2∗x+4∗y−4},{x,2,3},{y,1,3},PlotStyle→{Red,Blue}]

The differential as vertical change on the tangent plane

8. Each column of the following table corresponds to the tangent plane to the graph of $f(x,y)=xy^2$ at the given point. Different columns correspond to different planes.

   a. Fill the table.

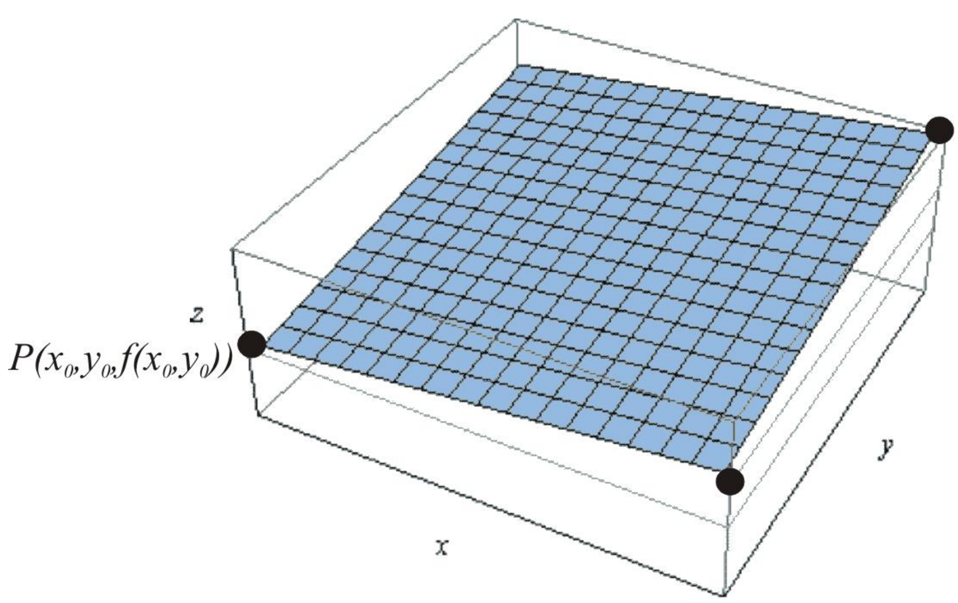
	Tangent plane to the graph of *f* at the point (3,-2,12)	Tangent plane to the graph of *f* at the point (2,1,2)	Tangent plane to the graph of *f* at the point (2,-2,8)
Slope in the *x* direction, m_x			
Slope in the *y* direction, m_y			
Differential of *f* at the given point	$df(3,-2)=$	$df(2,1)=$	$df(2,-2)=$
Vertical change in the x direction, dz_x, if $dx=0.01$			
Vertical change in the *y* direction, dz_y, if $dy=0.02$			
Total vertical change, $dz=dz_x+dz_y$			
Differential of *f* at given point if $dx=0.01$ and $dy=0.02$			

   b. Explain why the last two rows are the same.

9. Let $z=f(x,y)$, (x_0,y_0) an initial point in the domain of *f*, *dx* a change in *x*, *dy* a change in *y*, with $dx>0$ and $dy>0$. The following figure represents a piece of the tangent plane to the graph of *f* at the point $P\left(x_0,y_0,f\left(x_0,y_0\right)\right)$.

   a. Identify horizontal and vertical changes *dx*, *dy*, $f_x(x_0,y_0)dx$, $f_y(x_0,y_0)dy$, $df(x_0,y_0)$, all of them as lengths of horizontal or vertical segments in the figure.

   b. Explain in your own words the relation between the given figure and the differential.



10. A right vertical cone has a base with radius that measures 15 cm and height 45 cm with a maximal error of 0.05 cm in each measure. In this problem the differential will be used to approximate the maximal error that may result when using these measurements to compute the volume.
    a. In order to examine the error in the volume, one needs to start with a formula for the volume of a right circular cone as a function of its radius *r* and its height *a*. Write the formula.
    b. Compute dV at a generic point (r, a). The result should be a function of 4 independent variables: *a*, *r*, dr and da .
    c. Compute dV for the given initial values of radius and height. The result should be a function of two independent variables: dr y da .
    d. Evaluate the differential for the maximal errors dr and da that may occur. The answer should be a number (in units of cm^3).
    e. Compare the value of dV obtained in the previous part with the real change in volume ΔV, and find the percentage of error.
    f. The figure below represents the tangent plane to the graph of $V = f(r, a)$ at the point $P\left(15, 45, f(15, 45)\right)$. Use the figure below to represent the horizontal and vertical changes dr, $V_r(15, 45)dr$, da, $V_a(15, 45)da$, and $dV(15, 45)$ (each one of them is represented as the length of a **horizontal or vertical** line segment).