A Study of Semiotic Registers in the Development of the Definite Integral of Functions of Two and Three Variables

Abstract

Tracing the path from a numerical Riemann sum approximating the area under a curve to a definite integral representing the precise area in various texts and online presentations, we found three semiotic registers that are used: The geometric register, the numerical register and the symbolic register. The symbolic register had three representations: An expanded sum, a sum in sigma notation and the definite integral. Reviewing the same texts, we found that in the presentation of double and triple integrals, not a single textbook continues to present the numerical register and the expanded sum representation of the symbolic register. They are implied and the expectation appears to be that students no longer need them. The omission of these representations is quite ubiquitous and correspondingly affects millions of students. Materials that present the missing numerical register representation and the expanded sum representation of the symbolic register throughout topics associated with double and triple integrals have been created. This paper presents the results of a clinical study on the improvement of student comprehension of multivariable integral topics when these representations are included.

Keywords: “Semiotic Registers”, “Register of Representations”, “Definite Integral”, “Riemann Sum”, “Multivariable Calculus”, “Synergy of Registers”, “Transitional Auxiliary Representation”, “Functions of Two Variables”
Introduction

Duval (2006) defines a register of representation as a representation system that permits a transformation of representations. We will also refer to a register of representation as a “semiotic register”. He proceeds to define treatments as “transformations of representations that happen within the same register” and conversions as “transformations of representation that consist of changing a register without changing the objects being denoted: for example, passing from the algebraic notation for an equation to its graphic representation, passing from the natural language statement of a relationship to its notation using letters, etc.”

Tracing the path from a numerical Riemann sum approximating the area under a curve to a definite integral representing the precise area in some commonly used textbooks (Edwards and Penney, 2008; Rodriguez, 2008; Stewart, 2006; Strauss, Bradley, and Smith, 2002; Swokowski, Olinick and Pence, 1992, Waner and Costenoble, 2007), we found three semiotic registers that are used: the geometric register, the numerical register and the symbolic register. The treatments used within the symbolic register are between an expanded sum, a sum in sigma notation, and the definite integral. An example of these representations commonly used for a simple area is shown in Table 1. With applications to science, engineering and technology such as distance, mass and work, the verbal register is also used. Reviewing the presentation of single integrals in these texts, we found that all of the aforementioned representations are present in every one of them.

Table 1: The registers of representation associated with a simple area.

<table>
<thead>
<tr>
<th>Representation</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Geometric</td>
<td>Approximation of the area under ( y = x^2 ) between ( x = 1 ) and ( x = 5 ) using four rectangles and the midpoint rule for the height of each rectangle.</td>
</tr>
<tr>
<td>Numerical</td>
<td>((1.5)^2 \cdot 1 + (2.5)^2 \cdot 1 + (3.5)^2 \cdot 1 + (4.5)^2 \cdot 1)</td>
</tr>
<tr>
<td>Expanded sum</td>
<td>(x_1^2 \Delta x + x_2^2 \Delta x + x_3^2 \Delta x + x_4^2 \Delta x)</td>
</tr>
<tr>
<td>Sigma</td>
<td>(\sum_{i=1}^{4} x_i^2 \Delta x)</td>
</tr>
<tr>
<td>Sigma</td>
<td>(\lim_{n \to \infty} \left(\sum_{i=1}^{n} x_i^2 \Delta x\right))</td>
</tr>
<tr>
<td>Definite integral</td>
<td>(\int_{1}^{5} x^2 , dx)</td>
</tr>
</tbody>
</table>

Reviewing the same texts that presented all of the representations listed above for integrals of functions of one variable (Edwards and Penney, 2008; Rodriguez, 2008; Stewart, 2006; Strauss...
et al., 2002; Swokowski et al., 1992, Waner and Costenoble, 2007), we found that in the presentation of double and triple integrals, not a single textbook continues to present the numerical register and the expanded sum representation in the symbolic register throughout rectangular, polar, cubic, cylindrical and spherical coordinates. Presentations begin with expressions such as \[ \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i, y_j) \Delta y \Delta x, \sum_{i=1}^{m} \sum_{j=1}^{p} f(x_i, y_j, z_k) \Delta z \Delta y \Delta x \] or a similar expression in polar, cylindrical or spherical coordinates and appear to assume that treatments and conversions that use numerical approximations and expanded sums are now automatically made by students and no longer need to be presented. The omission of these representations is quite ubiquitous and correspondingly affects millions of students.

Hence, a study of the effect of this omission seems appropriate. This paper presents a study of the effect on student comprehension of multivariable integral topics when the aforementioned representations (numerical and expanded sum) and corresponding treatments and conversions are included in presentations associated with double and triple integrals at the multivariable calculus level.

**Theoretical Framework for the Study**

Understanding mathematics involves the possibility of relating different representations of functions. During the last decades, the critical problem of translation between and within representations has been addressed in several studies. For example, Breidenbach, Hawks, Nichols, and Dubinsky (1992) and Sfard (1992) propose that the ability of recognizing and being able to build a bridge between algebraic and graphic representations of functions differentiates between students who have and those who have not encapsulated the notion of function. Gutierrez (1996), Goldin (1998, 2002), and Hitt (2002), among other researchers, insist on the relationship between research on representations, mathematical visualization and understanding of concepts. More recently, Gagatsis, Christou, and Elia (2004) argued that representations constitute different entities and that as such require explicit instruction. We agree with the above mentioned authors on the importance that should be afforded to representations in studying conceptual understanding. Further, for the purpose of this study, we will focus on the role of semiotic representations as a tool for producing knowledge and communicating mental representations (see Duval, 1999, 2006).

From Kant’s (1929) and Piaget’s (1971, 1977) exploration of the epistemological relation between subject and object, Duval (2006) defines a mathematical object as the invariant of some multiplicity of possible representations and for the sake of this article, this is what we will take a “mathematical object” to be. The paradox associated with this definition is that in contrast to other scientific fields, “the only way to have access to them … is using signs and semiotic representations” (Duval 2006). However, as the “mathematical objects must never be confused with the semiotic representations that are used,” considerable care must be taken when assessing conceptual understanding of mathematical objects. Duval puts forth the hypothesis that “comprehension in mathematics assumes the coordination of at least two registers of semiotic representation” and “conceptual comprehension in mathematics involves a two-register synergy, and sometimes a three-register synergy” (Duval 2006). Correspondingly, while we may be unable to directly assess conceptual understanding of a mathematical concept, assessing the coordination or “synergy” of its associated semiotic registers should provide considerable insight into the conceptual understanding of the mathematical object at their core.

---

1 A semiotic representation is a representation of a mathematical object in a specific semiotic register.
In contrast to “institutional representations, the representations found in books or on computer screens,” (Hitt et al, 2008), Duval (2006) accepted that there are spontaneous representations that “a student uses in a mathematical situation” (Hitt, 2008), calling them “transitional auxiliary representations2.” Correspondingly, with integrals of a single variable the numerical representation and the expanded sum representation of the symbolic register would be institutional representations. However, as these representations are generally not seen in multivariable calculus, they may give rise to transitional auxiliary representations that students only use as needed to make sense of a mathematical situation. In the presentation of single integrals of one variable, the numerical representation and the expanded sum representation of the symbolic register are generally used to bridge the cognitive gap between the geometric, symbolic and verbal registers. As this is no longer the case with double and triple integrals, a fundamental question is whether these representations should continue to be institutional in multivariable calculus or whether students are able to either (i) operate without them or (ii) access them as transitional auxiliary representations when they are needed.

In topics associated with integrals, while the order was not always the same, there was always a natural order in the presentation of registers of representation in every textbook we surveyed. Hence, the presentation of semiotic registers is best described as a “semiotic chain” with a natural order (Presmeg, 2006). Only being able to perform the treatments and conversions associated with a semiotic chain is inconsistent with the “synergy” or “simultaneous awareness” of semiotic registers that Duval associates with comprehension of a mathematical concept. An example is shown in Figure 1 where the semiotic chain in Figure 1A has two associated conversions, each of which is represented by an arrow in the diagram: The geometric register to the numerical register and the numerical register to the symbolic register. Simultaneous awareness of these three registers would be associated with the ability to perform more than the two conversions in the semiotic chain and conceivably up to all six possible conversions as shown in Figure 1B.

This paper explores the relationship between a semiotic chain and the achievement of synergy (simultaneous awareness) of semiotic registers for the case of integrals of functions of two variables. It also illustrates how the semiotic chain used can provide students with transitional auxiliary representations that may help them to bridge cognitive gaps between semiotic representations thus allowing them to tackle unfamiliar problems.

2 A “transitional auxiliary representation” is a spontaneous representation that does not depend on a semiotic system and is used in a mathematical activity with a meaning given only by the particular user (Duval, 2006; Hitt et al, 2008).

3 A semiotic chain is a sequence of signified-signifier pairs where the signifier at one stage is contained in the signified of the next stage. The transformation from signified to signifier is either a treatment or a conversion and the final link in the chain is the mathematical concept being studied.
In this article we restrict our attention to the registers of representation considered by Duval (1999, 2006) even though more recent work has suggested the use of other semiotic resources like gestures and artifacts (see for example, Radford, 2002, and Arzarello, 2006). Along this line, Moore-Russo and Viglietti (2012) discussed the evolution of a semiotic chain to a semiotic bundle (see Arzarello, 2006) using what they called a \textit{K5 Connected Cognition Diagram}. This is a graph with 5 vertices and edges connecting each pair of vertices. Four vertices represent observable modes of production: inscriptions, speech, gestures, and artifacts, and the other vertex stands for the mental representation of a particular concept. The strength of the mental representation at the core of the \textit{K5 Diagram} was associated with the strength and fluidity of connections between the 5 vertices. While the nature of what is represented by the vertices is different, this is analogous to the idea that is presented in Figure 1B, where the strength and fluidity in the conversions between the three representation registers (vertices) determine the understanding of the mental object being represented.

Mehanovic (2011) studied the implementation of GeoGebra software to aid in treatments and conversions between registers associated with single integrals of one variable. In contrast to a semiotic chain, such as that presented in Figure 1A, she had success using computer software to simultaneously present an algebraic window and a geometric window to provide “an immediate access to different representation registers, as formulated by Duval.” While Mehanovic was very successful creating the technical tools for simultaneous access to multiple registers, a failure to successfully incorporate these tools into the classroom presentations and environment limited the success of the project. Hence, it is important to not solely view the effectiveness of materials to access different registers of representation as ends in and of themselves but also to consider them in the context of the environment in which they are going to be applied.
In another study of semiotic registers associated with single integrals of one variable, Camacho, Depool, and Santos-Trigo, (2003) used the following model of cognitive development based on semiotic registers:

A mathematical competence model was used to explain students’ level of understanding of basic ideas related to definite integral in which three related phases are identified: the use of a certain language (Semiotic Stage); the use of several registers and their corresponding operations (Structural Stage); and the conversion or transition between different types of representations or registers (Autonomous Stage).

Camacho et al (2003) also used computer software, DERIVE, to simultaneously display the algebraic and geometric registers of representation associated with integrals. The results of the study were obtained from observations and interviews and there was no control group. Their findings categorized students into three distinct groups: One of the three groups showed promise in performing conversions between algebraic registers of representation and geometric registers of representation but was weaker with problem solving skills that involved a verbal register. The other two groups struggled with basic conversions. While an exploration of the ability to perform conversions between semiotic registers was conducted, synergy or simultaneous awareness of registers was not discussed.

This article will first obtain data to address the following specific questions:

1. Most current textbooks assume that the numerical register and the expanded sum representation in the symbolic register do not need to be presented in all topics of multivariable calculus as they are implicitly understood. How well can students that use such textbooks perform treatments and conversions between the explicit registers that are presented and how well can students perform treatments and conversions that include the implied registers and representations that are not presented?

2. As part of this study, experimental materials were written that include these implied registers and representations that are not included in traditional textbooks. How well can students that use these experimental materials perform treatments and conversions between the explicit registers that are presented in current textbooks and how well can these students perform treatments and conversions that include the implied registers and representations that are not presented in traditional textbooks but are presented in the experimental materials?

3. Students using the experimental materials proceed through presentations that use a semiotic chain with the basic structure shown in Figure 2 for each of the following:
   - \( \int_a^b \int_c^d f(x,y)dy \, dx \),
   - \( \int_a^b \int_c^d f(r, \theta)r \, dr \, d\theta \),
   - \( \int_a^b \int_c^d \int_e^f f(x,y,z)dz \, dy \, dx \),
   - \( \int_a^b \int_c^d \int_e^f f(r, \theta,z)r \, dz \, dr \, d\theta \) and
   - \( \int_a^b \int_c^d \int_e^f f(\rho, \phi, \theta)\rho^2 \sin(\phi) \, d\rho \, d\phi \, d\theta \).

While there have been discussions of representations of mathematical objects across coordinate systems (Montiel, Wilhelmi, Vidakovic, and Elstak, 2009), to the author’s knowledge, this is the first time that a semiotic chain has been studied across coordinate systems. As students proceed through the activities associated with these five coordinate systems, we will assess whether there is an evolution of (i) students’ ability to perform
treatments and conversions in the semiotic chain and (ii) students’ ability to perform conversions that are not in the semiotic chain.

4. Does the explicit presentation of the numerical register and the expanded sum representation in the symbolic register throughout the presentation of double and triple integrals affect the multivariable class in terms of (i) the level of difficulty of the class and (ii) the ability to solve standard integral problems of multivariable calculus?

Figure 2: The Semiotic Chain used in presentations for students in the experimental group

These data will then be used to gain insight into the following more general questions:

1. The difficulties that Mehanovic (2011) had with the implementation of new materials limited her ability to obtain positive results regarding student learning. Correspondingly, an assessment of whether the experimental materials were well integrated into the course is relevant when interpreting the theoretical results and this was the first general question we discussed.

2. Whether the numerical register and the expanded sum representation of the symbolic register should remain as institutional representations in multivariable calculus in order to close the cognitive gap between the verbal, geometric and symbolic registers presented in traditional textbooks is the next theoretical question we will address.

3. Finally, we will address whether students, in the course of studying a semiotic chain across coordinate systems, evolve from understanding a specific set of conversions and treatments associated with the semiotic chain towards a synergy of registers that is associated with comprehension of the mathematical object at their core.

Methodology

Nature of the Experimental and Control Groups

In the spring semester of the 2011-2012 academic year at the University of Puerto Rico in Mayaguez (UPRM), a random section of 36 students was selected to use the experimental materials described in the previous sub-section. This section was taught by the author of the experimental materials and is referred to as the experimental group. Two random sections with a total of 68 students were selected as a control group. They were taught by two professors that felt they provided an appropriate balance between a geometric, algebraic and applied approach to multivariable calculus. Based on all available measures shown in Table 2, there was no significant difference between the two populations.

Table 2: A comparison of GPAs and college board exam scores between the experimental and control groups.
As questions related to students’ ability to understand implied registers of representation may be population dependent, we present a brief overview of students in STEM (science, technology, engineering, and mathematics) fields that take multivariable calculus at the UPRM. The National Assessment of Educational Progress (NAEP) is a standardized test with assessments in mathematics and other subject areas. In 2007, the raw scores by subject area for eighth graders in the United States and Puerto Rico are shown in Figure 3. (NAEP, 2009) This data speaks to the extremely worrisome state of mathematics education in Puerto Rico and it is reflected in the preparedness of students in STEM fields at the UPRM. In the fall of 2012, over 55% of freshmen entering STEM fields performed below satisfactory level on junior high school level mathematics material as measured by a diagnostic exam administered by the UPRM. Students in multivariable calculus have completed two semesters of precalculus and two semesters of calculus. Nonetheless, many of the students in multivariable calculus continue to struggle with basic algebra and geometry. Many are also accustomed to a mechanical approach to mathematics which would make it difficult for them to recognize implied representations or concepts that are not explicitly presented. Hence, awareness of the associated population may lend insight into the results of this study.

### Figure 3: Comparison of NAEP Results in 2007 for the U.S. and Puerto Rico

#### Content of the Materials for the Experimental Group

As was mentioned in the previous section, materials and activities were written for this study that incorporate treatments and conversions associated with the numerical register and the expanded sum representation of the symbolic register throughout material associated with double integrals (rectangular and polar coordinates) and triple integrals (cubic, cylindrical and spherical coordinates). The materials consistently followed the semiotic chain presented in Figure 2. The materials used can be seen in chapters 8, 9, 10, 11 and 12 in http://quiz.uprm.edu/calc3 and a sample activity and its solution are shown in Appendix A. It should be noted that the verbal

<table>
<thead>
<tr>
<th>Group</th>
<th>High School GPA (grade point average)</th>
<th>UPRM GPA</th>
<th>Mathematics College Entrance Examination Board score (A version of the SAT given in PR)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Control</td>
<td>3.79</td>
<td>2.89</td>
<td>686</td>
</tr>
<tr>
<td>Experimental</td>
<td>3.76</td>
<td>2.91</td>
<td>670</td>
</tr>
</tbody>
</table>
register sometimes precedes the geometric register when the semiotic chain is associated with a simple practical situation.

Mode of instruction for the experimental and control Groups

The amount of class time dedicated to integrals in these five coordinate systems (rectangular, polar, cubic, cylindrical and spherical) was the same for the experimental and control groups. They used the same text (Stewart, 2006) although, as was mentioned, special materials that incorporated treatments and conversions associated with the numerical register and the expanded sum representation of the symbolic register for the five coordinate systems (rectangular, polar, cubic, cylindrical and spherical) were written and used to supplement Stewart for the introduction to integrals in these coordinate systems. The assignments were the same for the two groups although there were supplemental problems for the experimental group associated with the experimental materials.

Both the experimental and control groups were primarily lecture courses. However, as part of the presentation, five group activities (one for each of the coordinate systems) were provided to the experimental group to guide them through the semiotic chain seen in Figure 2. The activity for rectangular coordinates is shown in Appendix A. The amount of time dedicated to these all activities, except spherical coordinates, was about 20 minutes each. As students struggle more with the geometry associated with spherical coordinates, this activity took about 40 minutes. There was a second instructor in the class to manage basic classroom duties so that during these activities, the professor was free to question and observe students working with the experimental materials.

Data

The results of our study were based on the following four sources of data:

- Classroom observations of the experimental group: As the groups progressed through the various laboratories, their evolution was observed. According to Duval, simultaneous awareness of the various semiotic registers for the mathematical object at their core is strongly associated with students’ comprehension of the mathematical object (Duval, 2006) so of particular interest was students’ simultaneous awareness of different registers and representations and their flexibility in doing treatments and conversions between them. These observations were conducted by the author of the experimental materials.

- Interviews with 5 students from the experimental group and 6 students from the control group to gain insight into their degree of simultaneous awareness of the various registers and representations by assessing their ability to pass between them. As the majority of students in multivariable calculus are from physics, chemistry and engineering programs, particular attention will be paid to the verbal register. All of these interviews were conducted by the author of the experimental materials.

- Exam questions for the experimental group: Exam questions solely for the experimental group provided insight into the degree of difficulty of the added activities for students. Our goal was to see whether the addition of these representations and their associated conversions and treatments significantly affected the level of difficulty for the course.

- Common exam questions for the experimental group and a control group: Common exam questions were given to the experimental group of 36 students and a control group of 68 students on basic integral questions concerning volume that do not require the
numerical and expanded sum representation of the symbolic register. The purpose of this part of the study was to see whether the explicit attention given to the numerical and expanded sum representations and corresponding treatments and conversions improved performance on general integral questions that could appear in any multivariable calculus class. Students in the control group and experimental group both used a standard textbook (Stewart, 2006) and followed a very similar course syllabus. The only differences were the additional activities for the experimental group.

Results

Classroom Observations of the Experimental Group

Students were divided into nine groups of 3-4 students. The groups completed five activities on rectangular, polar, cubic, cylindrical and spherical coordinates over the course of 4 weeks. The following describes, from the point of view of the instructor, the evolution of student perceptions on the representations in the geometric, numerical and symbolic registers as they proceeded through these labs:

- Rectangular Coordinates: The students seemed to understand each representation and treatment quite well and transitioned methodically from the geometric register to the numerical register to the treatments in the symbolic register (passing from the expanded sum representation to the sum with sigma representation and finally to the definite integral notation). However, there was a definite sequential tone to their approach. Duval’s “synergy of registers” would suggest a simultaneous awareness of all of the registers and their relation to the underlying mathematical concept but there was little evidence to suggest that this had occurred. Two of the nine groups of students wanted to know if they needed to include the expanded sum representation of the symbolic register or whether they could go from the numerical approximation right to the sum with sigma representation of the symbolic register. They were told that they could do so as long as they provided an appropriate scheme to assure that the numerical representation and the sum with sigma representation of the symbolic register in fact represented the same quantity. One of the groups worked amongst themselves and indicated that if the chosen points of the four divisions are organized in the form \((x_1, y_1), (x_1, y_2), (x_2, y_1)\) and \((x_2, y_2)\), this would result in the sum with sigma representation of the symbolic register: 
\[
\sum_{i=1}^{2} \sum_{j=1}^{2} f(x_i, y_j) \Delta y \Delta x.
\]
When asked what would happen if there were three divisions in \(x\), they took a couple of minutes to organize themselves and then indicated that \((x_1, y_1), (x_1, y_2), (x_2, y_1), (x_2, y_2), (x_3, y_1), (x_3, y_2)\) would result in the sum with sigma representation of the symbolic register: 
\[
\sum_{i=1}^{3} \sum_{j=1}^{2} f(x_i, y_j) \Delta y \Delta x.
\]
Hence there seemed to be some movement towards a unified view of these registers of representations.

- Polar Coordinates: Once again, all of the groups completed the lab with no real problems. This time three of the groups demonstrated a legitimate scheme to go directly from the numerical register to the sum with sigma representation of the symbolic register. Two of these groups seemed to be simultaneously aware of all the representations and their treatments. When asked to go directly from the geometric register to any of the representations of the symbolic register, these two groups explained how it could be done. So for two groups, there was a strong indication that the semiotic chain was
evolving into the “synergy of registers” associated with true understanding of the mathematical concept.

- **Cubic Coordinates:** Students had no problem completing the lab. As there were now eight divisions and three sigma symbols, there was considerable interest in simplifying the process and going directly from the numerical register to the sum with sigma representation of the symbolic register in triple-sigma notation. There was a lot of interaction between groups and at the end of the lab, seven groups explained that organizing the divisions
  \[(x_1, y_1, z_1), (x_1, y_2, z_1), (x_2, y_1, z_1), (x_2, y_2, z_1), (x_1, y_1, z_2), (x_2, y_1, z_2), (x_1, y_2, z_2), (x_2, y_2, z_2)\] would allow them to go directly to \[\sum_{i=1}^{2} \sum_{j=1}^{3} \sum_{k=1}^{3} f(x_i, y_j, z_k) \Delta z \Delta y \Delta x\]. Two groups organized them differently with
  \[(x_1, y_1, z_1), (x_2, y_1, z_1), (x_1, y_2, z_1), (x_2, y_2, z_1), (x_1, y_1, z_2), (x_2, y_2, z_2), (x_1, y_2, z_2), (x_2, y_2, z_2)\] to go directly to \[\sum_{j=1}^{2} \sum_{i=1}^{3} \sum_{k=1}^{3} f(x_j, y_i, z_k) \Delta z \Delta x \Delta y\]. (This produced a discussion in the following class on order of divisions and how this can change the order of integration.)

  All of the groups could now show how to go directly from the geometric register to any of the representations of the symbolic register so there continued to be strong indications of simultaneous awareness of all registers in the semiotic chain.

- **Cylindrical Coordinates:** Students had no problem completing the lab and were now all going directly from the numerical register to the symbolic register with the sigma representation. For the first time, students demonstrated annoyance at having to explicitly write out the numerical expression for each of the eight divisions associated with the numerical register. They felt that their understanding of the relationship between registers made this step unnecessary. The evolution of the semiotic chain to a “synergy of registers” was now somewhat self-evident. In fact, several groups were no longer filling out the activity in the prescribed order: They first wrote down the most easily expressed registers: The symbolic register with the definite integral representation and the symbolic register with sigma representation. Then, with a slight air of annoyance, they completed the more tedious registers that remained.

- **Spherical Coordinates:** The “simultaneous awareness” of registers that students now had achieved was invaluable in completing this lab. Effectively, students could concentrate entirely on the geometry of spherical coordinates, which they find difficult, without worrying about the associated registers. All groups were able to go from the numerical register to the sum with sigma representation of the symbolic register without passing through the expanded sum representation of the symbolic register. They all completed the lab and arrived at the appropriate definite integral expressed in spherical coordinates.

**Interview Results**

The questions used for the interviews, the treatments and conversions associated with each question and the question by question results are found in Appendix B. The results are summarized in two tables: Table 3 shows how well students in the experimental group and control group could perform treatments and conversions that include registers that are only seen by the experimental group, i.e., treatments and conversions involving the numerical register or any sum treatment of the symbolic register. Table 4 presents the interview results for conversions between registers that students in both the experimental and control group had seen numerous times.
Table 3: Treatments and conversions involving registers only explicitly seen by the experimental group.

<table>
<thead>
<tr>
<th>Conversions and treatments</th>
<th>Control Group</th>
<th>Experimental Group</th>
</tr>
</thead>
<tbody>
<tr>
<td>Geometric register to numerical register</td>
<td>42%</td>
<td>100%</td>
</tr>
<tr>
<td>Numerical register to the expanded sum or sum with sigma representation of the symbolic register</td>
<td>17%</td>
<td>80%</td>
</tr>
<tr>
<td>Sum with sigma representation of the symbolic register to the definite integral representation of the symbolic register</td>
<td>0%</td>
<td>80%</td>
</tr>
<tr>
<td>Verbal register to numerical register</td>
<td>0%</td>
<td>50%</td>
</tr>
</tbody>
</table>

It should be noted that the conversions from

(i) the Geometric register to definite integral representation of the symbolic register and

(ii) the Verbal register to definite integral representation of the symbolic register

that are shown in Table 4 were consistent with problems assigned from the text (Stewart, 2006) for both groups. Only the conversion from the definite integral representation of the symbolic register to verbal register represented a conversion that was inconsistent with problems that they had previously seen.

Table 4: Treatments and conversions involving representations seen by both groups.

<table>
<thead>
<tr>
<th>Conversion</th>
<th>Control Group</th>
<th>Experimental Group</th>
</tr>
</thead>
<tbody>
<tr>
<td>Geometric register to definite integral representation of the symbolic register</td>
<td>67%</td>
<td>80%</td>
</tr>
<tr>
<td>Definite integral representation of the symbolic register to verbal register</td>
<td>17%</td>
<td>80%</td>
</tr>
<tr>
<td>Verbal register to definite integral representation of the symbolic register</td>
<td>33%</td>
<td>80%</td>
</tr>
</tbody>
</table>
One of the C students\(^4\) in the control group, Michelle, liked to doodle as we were discussing the problems and when she was asked to perform the conversions from the geometric register to the sums associated with the numerical and symbolic register, she drew the diagram in Figure 4 indicating that with more divisions, the approximation gets better. This for her is a transitional auxiliary representation which should have led to another representation, but one which she seems not to have consciously related to other representations in her previous work with integrals of functions of two variables. Hence, she struggled with the appropriate representations for this idea: Obtaining the numerical representation only with considerable help and never obtaining sum representations of the symbolic register.

From a semiotic standpoint, Michelle frequently reverted to representations that she clearly understood very well in 2D. When asked for an approximation for a volume in sigma notation, she first wrote down \(\sum_{i=1}^{\infty} f(x_i) \Delta x\) to represent an area in 2D and then wrote down \(\sum_{i=1}^{\infty} f(x,y) \Delta x \Delta y\) as the appropriate extension to a volume in 3D. When asked to find the units associated with a double integral, she wrote down

\[
y = \text{meters} \rightarrow \frac{dy}{dt} = \frac{\text{meters}}{\text{second}} \rightarrow \int \frac{\text{meters}}{\text{second}} dt = \text{meters}.
\]

While Michelle was unsuccessful with both of these problems, she was aware that what she was doing was incorrect which appeared to frustrate her considerably. In conclusion, Michelle appeared far more comfortable in 2D than in 3D at using the available semiotic registers to communicate a concept. She also appeared frustrated by her inability to correctly express concepts in 3D that were similar to concepts she was able to express in 2D.

![Figure 4: A figure drawn by Michelle during the interview](image)

Denisse, a C student from the experimental group, was very successful with all of the treatments and conversions. She was very organized and expressed every detailed in each treatment and conversion that was in the semiotic chain. For example, Denisse expressed every intermediate sigma treatment in order to proceed from the extended sum representation of the symbolic register to the double sigma representation of the symbolic register (See Parts g and h of

---

\(^4\) At the University of Puerto Rico, where the study took place, the grades that a student can receive in a course are: A (excellent), B (good), C (average), D (deficient), F (failed). The control group consisted of 4 students with a grade of A on the course and 2 students with C. The experimental group had 2 students with A, 1 with B and 2 with C.
Appendix A). All the A students in the experimental group leapt directly to the double sigma representation without showing the associated steps. Denisse’s ability to perform conversions that were not in the semiotic chain would suggest that this degree of detail was no longer necessary. However, the ability to perform every single detail seemed to provide her with confidence. Perhaps the most noticeable contrast to Michelle was that with the same problem as in Figure 4, she began with the same formula for the volume \( V = f(x, y) \Delta x \Delta y \) but instead of an abstract diagram, she made a concrete 2-D diagram seen in Figure 5 and was able to quickly proceed to the numerical sum, the sum with sigma representation and the definite integral representation of the symbolic register.

![Figure 5: A figure filled in by Denisse during the interview](image)

Jose, one of the A students of the control group, was one of the few students in the control group to perform the conversion from the geometric register to the definite integral representation of the symbolic register and the conversion from the verbal register to the definite integral representation of the symbolic register. Despite the fact that Riemann sums, normally in their double sigma notation, were the starting point for double and triple integrals, he was completely disinterested and unable to perform any of the treatments or conversions associated with them. When asked if Riemann sums were associated with definite integrals, he responded that “supposedly they are as they’re the basis for the whole theory of integrals however it’s really complicated.” In general, Jose appeared very confident and adept with the limited conversions that he was familiar with. However, while he could represent a situation with a double integral, he could not do this accustomed procedure in reverse. In general, Jose seemed very adept with conversions that he had already seen however his dismissal or inability to deal with any new treatments or conversions would indicate that he was well short of the synergy of registers that would characterize knowing a mathematical object as suggested by Duval (2006).

One of the A students in the experimental group, Alan, was completely successful with every single conversion and treatment that was asked of him during the interview. However, at the conclusion of the interview when Alan was casually discussing the general approach used with the experimental materials, he told the interviewer: “All of this visualization is great but … Mathematics is memorizing stuff. How do I know that this prepares me for subsequent courses?” Eisenberg and Dreyfus (1991) identified three reasons to explain the observed reluctance of some students to visualize: cognitive (visual is more difficult), sociological (visual is harder to teach), and one related to beliefs about the nature of mathematics (visual is not mathematical). We can see in Alan’s response that even though he does not have a cognitive
reason (he managed to perform all conversions involving the geometric register) or a sociological reason (semiotic chains were systematically used in the classroom to facilitate students’ conversions from the geometrical to the symbolic registers), his belief that visualization is not part of mathematics led him to be distrustful of the classroom activities designed to help him acquire mathematical knowledge. It is not clear what effect this has on experimental students’ work with specially designed classroom materials so this suggests that students might need reassurance.

Two members of the experimental group, 1 C student and 1 A student, used a sum representation of the symbolic register to clarify the conversions from the definite integral representation of the symbolic register to the verbal register. It appears that the link between a situation and a definite integral is eased by making an approximation with a finite set of divisions. With this representation institutionalized and readily available, they could go back and forth whereas students in the control group, that did not explicitly see this representation, struggled to do this.

Exam Questions for the Experimental Group

Students in the experimental group obtained an average of 76% on questions associated with the added numerical register and the registers of representations involving sums and the definite integral. They received an average of 69% on all other questions associated with integrals in multivariable calculus. Hence students appeared to find the added topics slightly easier than other topics of integration in multivariable calculus.

Common Exam Questions for the Experimental Group and the Control Group

The goal for the common exam questions was to gauge the effect of the added materials that were given to the experimental group on general multivariable calculus questions. Hence a double integral and a triple integral question that would be considered appropriate for just about any calculus class were given. To minimize the effect of students memorizing standard solids, an effort was made to base the questions on less commonly seen solids. The average scores for students in each group are shown in Table 5. Clearly the experimental group did significantly better than the control group (P<<0.01) so even though the numerical register and the expanded sum representation of the symbolic register are not needed for these questions, the inclusion of the added registers appears to improve students’ ability to solve general integral problems.

<table>
<thead>
<tr>
<th>Question</th>
<th>Control Group (n=68)</th>
<th>Experimental Group (n=36)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Find the volume over the xy-plane and between the surfaces y=0 and ( z = 10 - x^2 - y ).</td>
<td>26%</td>
<td>53%</td>
</tr>
<tr>
<td>2. Find the volume over the plane ( z = 1 ), below the surface ( z = 10 - x^2 ) and bounded by the planes ( y=1 ) and ( y=5 ).</td>
<td>48%</td>
<td>73%</td>
</tr>
</tbody>
</table>

Discussion and Conclusion

Implementation of the Experimental Materials
The first research question we wished to address was whether the incorporation of the experimental materials into the multivariable calculus course was successfully implemented. Students using the experimental materials did moderately better on supplemental questions associated with the added registers that were presented in the experimental materials than they did on standard multivariable calculus questions. Hence they seemed to find the supplemental material slightly easier than the standard multivariable calculus material. They also did 70% better than the control group on general multivariable calculus questions that did not explicitly require the added registers in the experimental materials. These data would suggest that the implementation of the experimental materials was effective and that the forthcoming analyses were done with a population that successfully used the experimental materials.

**Institutional vs. Transitional Auxiliary Representations**

The next research question was whether the missing numerical and expanded sum representations are in fact needed to make the necessary associations between standard registers of representations presented in any multivariable calculus course. To gain insight into this question, we first concentrated on the control group. The results of Table 3 indicate that treatments and conversions relating these missing representations are poorly understood by students in the control group and correspondingly it is incorrect to assume that students can access them as transitional auxiliary representations when a situation calls for it. Hence the control group does, in fact, represent students with little awareness and comprehension of the numerical and expanded sum representations. Table 4 indicates that the students without the benefit of understanding the implied representations have considerable trouble with conversions; especially those that involve verbal registers where the verbal registers reflect simple practical situations. Next, we wished to determine whether the addition of treatments and conversions involving these missing representations will improve the ability to perform the conversions shown in Table 4 that are associated with any multivariable calculus class. Table 3 indicates that students using the experimental materials, in contrast to the control group, are generally able to perform conversions and treatments involving the numerical register and sum representations of the symbolic register and correspondingly, the experimental group does understand the missing representations that the control group does not. Table 4 indicates that they are also far more successful with the treatments and conversions presented in standard textbooks. Hence, while students that cannot perform treatments and conversions associated with the missing representations have a significant cognitive gap between representations in the verbal register and the definite integral representation of the symbolic register, the addition of these missing representations appears to close this cognitive gap significantly.

In some ways, Michelle from the control group and Denisse from the experimental group had similar learning styles. They both seemed uncomfortable with representations or processes that they did not completely understand. Denisse appeared to have the necessary tools to express the concepts in 3D that were asked of her. Michelle appeared comfortable expressing concepts in 2D however she was unable to express very similar concepts using the associated registers in 3D. It seems that classroom activities enabled Denisse to bridge the gap between an object represented in the geometric register with the same object represented as a double integral in the symbolic register by providing her with an intervening structure to ease the conversion.
Of particular interest is the fact that some students in the experimental group explicitly referred to these representations when performing conversions that did not require them which would indicate that these representations can play a role in bridging the cognitive gap between the symbolic, geometric and verbal registers.

These results would support the conclusion that activities where the numerical register and the expanded sum representation of the symbolic register are institutionalized can help students make many of the standard conversions and associations needed in multivariable calculus. If not explicitly presented, students do not appear able to access them as transitional auxiliary representations and correspondingly struggle with standard conversions and associations. More generally, the results suggest that a properly designed semiotic chain can provide students with transitional auxiliary representations that they may use in solving unfamiliar problems.

Synergy of Registers

The final research question was whether repeated study of a semiotic chain across coordinate systems will produce the synergy of registers that is associated with comprehension of the mathematical object at their core. The classroom observation section showed that students’ simultaneous awareness of all of the registers of representations in the semiotic chain definitely increased as students proceeded through more coordinate systems. In the first two labs, students’ attention was very fixed on one representation at a time. However, at the end of the five activities outlined above, observations and conversations with the groups definitely showed a unified perspective of registers, conversions and treatments. While there was the sequential order to the activities that was associated with the semiotic chain, when questioned, students were able to move between registers and do treatments in just about any order that was requested. However, they were able to do this in a very specific environment where they were aware that the entire semiotic chain was forthcoming. With the interview questions, we tried to determine whether this same ability was manifest in less specific environments.

Table 6 shows all of the treatments and conversions in Table 3 and Table 4 that were not in the semiotic chain. With students in the experimental group, the rate of success was 87% for conversions and treatments that were in the semiotic chain and that students had repeated many times. The success rate was 73% for conversions that were not in the semiotic chain. It is unrealistic to expect the same performance between new and familiar treatments and conversions. The fact that there is only moderate drop off from conversions and treatments in the semiotic chain that they have repeated numerous times to new conversions that are not in the chain is consistent with an evolution from a chain to simultaneous awareness of registers as presented in Figure 1B. It is also worth noting that the students in the experimental group, even when committing errors, communicated familiarity and confidence with the registers of representation. There were few extended pauses during the interviews and very few pages were left blank. Hence, both the classroom observations and the interviews support movement from a semiotic chain to simultaneous awareness of the registers contained in the chain as students proceed through five different coordinate systems.

<table>
<thead>
<tr>
<th>Conversion</th>
<th>Success Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Verbal register to numerical register 50%

Geometric register to definite integral representation of the symbolic register 80%

Definite integral representation of the symbolic register to verbal register 80%

Verbal register to definite integral representation of the symbolic register. 80%

**General Conclusion**

The study presented in this article has not addressed the nature of how students understand multivariable integrals. However, it has shown results which indicate that the semiotic tools in the presentations of double and triple integrals in most textbooks may be incomplete and that the numerical register and the expanded sum representation of the symbolic register may be needed by many students to form the associations that are expected of them in multivariable calculus. The study shows how a particular semiotic chain can help students move towards the synergy of registers needed for conceptual understanding of integrals of functions of two variables and how this semiotic chain can give raise to students’ transitional auxiliary representations that may allow them to succeed when faced with unfamiliar problems.

**Acknowledgement**

The material in this article is based upon work supported by the National Science Foundation (NSF) under Grant NSF-DUE 0941877. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the author and do not necessarily reflect the views of the NSF.

**References:**


Radford, L. (2002). The seen, the spoken, and the written: A semiotic approach to the problem of objectification of mathematical knowledge. *For the Learning of Mathematics, 22*(2), 14-23


Appendix A: A Sample Activity of the Experimental Group

Sample Activity with Solution:

Follow the following steps to find the volume below the surface \( z = f(x, y) = 0.5x^2 + y^2 + 4 \) and above the region \( 0 \leq x \leq 4, 0 \leq y \leq 4 \) in the \( xy \) plane.

a. Draw the region on the \( xy \) plane and use a computer, calculator or physical manipulative to visualize the surface over the region.

![Graph of the surface](image)

b. If there are two divisions in \( x \) and two divisions in \( y \) and the volume is to be approximated using the middle value for \( x \) and \( y \) in each division, find \( x_1, x_2, y_1, y_2 \), \( \Delta x \), and \( \Delta y \)

\[ x_1 = 1, x_2 = 3, y_1 = 1, y_2 = 3, \Delta x = 2 \text{ and } \Delta y = 2 \]

c. For each of the four divisions, indicate the point and associated height that will be used to approximate the volume for that division and identify the four cubes with which we will approximate the volume. **Note:** When possible, both a 2D and a 3D representation in the geometric register are encouraged.

![Division Points](image)

d. Use the values obtained in part c to fill in the following table with numerical values. **Note:** The use of the numerical register is introduced here and finalized in 2f.
<table>
<thead>
<tr>
<th>Division</th>
<th>Length</th>
<th>Width</th>
<th>Height</th>
<th>Volume</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>2</td>
<td>5.5</td>
<td>2 x 2 x 5.5</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>2</td>
<td>9.5</td>
<td>2 x 2 x 9.5</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>2</td>
<td>13.5</td>
<td>2 x 2 x 13.5</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>2</td>
<td>17.5</td>
<td>2 x 2 x 17.5</td>
</tr>
</tbody>
</table>

e. Fill in the same table below using $x_1, x_2, y_1, \Delta x$, and $\Delta y$ instead of numerical values. The divisions should not change between this and the previous table. **Note:** The extended sum representation of the symbolic register is introduced here and finalized in 2g

<table>
<thead>
<tr>
<th>Division</th>
<th>Length</th>
<th>Width</th>
<th>Height</th>
<th>Volume</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\Delta y$</td>
<td>$\Delta x$</td>
<td>$0.5x_1^2 + y_1^2 + 4$</td>
<td>$(0.5x_1^2 + y_1^2 + 4)\Delta x\Delta y$</td>
</tr>
<tr>
<td>2</td>
<td>$\Delta y$</td>
<td>$\Delta x$</td>
<td>$0.5x_2^2 + y_1^2 + 4$</td>
<td>$(0.5x_2^2 + y_1^2 + 4)\Delta x\Delta y$</td>
</tr>
<tr>
<td>3</td>
<td>$\Delta y$</td>
<td>$\Delta x$</td>
<td>$0.5x_1^2 + y_2^2 + 4$</td>
<td>$(0.5x_1^2 + y_2^2 + 4)\Delta x\Delta y$</td>
</tr>
<tr>
<td>4</td>
<td>$\Delta y$</td>
<td>$\Delta x$</td>
<td>$0.5x_2^2 + y_2^2 + 4$</td>
<td>$(0.5x_2^2 + y_2^2 + 4)\Delta x\Delta y$</td>
</tr>
</tbody>
</table>

f. Express the approximate volume numerically. **Note:** This is a treatment in the numeric register.

Volume $\approx (2 \times 2 \times 5.5) + (2 \times 2 \times 9.5) + (2 \times 2 \times 13.5) + (2 \times 2 \times 17.5) = 184$

g. Express the same volume obtained in part f using $x_1, x_2, y_1, y_2 \Delta x$, and $\Delta y$. **Note:** This is the expanded sum representation of the symbolic register.

$(0.5x_1^2 + y_1^2 + 4)\Delta x\Delta y + (0.5x_1^2 + y_2^2 + 4)\Delta x\Delta y + (0.5x_2^2 + y_1^2 + 4)\Delta x\Delta y + (0.5x_2^2 + y_2^2 + 4)\Delta x\Delta y$

h. Express the volume obtained in part G in the form $\sum \sum (...)\Delta x\Delta y$. **Note:** The sum with sigma representation of the symbolic register is introduced here.

$\sum_{j=1}^{2} (0.5x_1^2 + y_j^2 + 4) \Delta y\Delta x + \sum_{j=1}^{2} (0.5x_2^2 + y_j^2 + 4) \Delta y\Delta x$

= $\sum_{i=1}^{2} \sum_{j=1}^{2} (0.5x_i^2 + y_j^2 + 4) \Delta y\Delta x$

i. Using part h as a base, find an expression for the volume if there are $n$ divisions in $x$ and $m$ divisions in $y$. 
\[ \text{Volume} \approx \sum_{i=1}^{n} \sum_{j=1}^{m} (0.5x_i^2 + y_j^2 + 4) \Delta y \Delta x \]

j. Use the fundamental theorem and limits to convert the approximation in part i to a precise value. \textbf{Note:} The definite integral representation of the symbolic register is introduced here.

\[ \text{Volume} = \lim_{n \to \infty} \lim_{m \to \infty} \sum_{i=1}^{n} \sum_{j=1}^{m} (0.5x_i^2 + y_j^2 + 4) \Delta y \Delta x = \int_{0}^{4} \int_{0}^{4} (0.5x^2 + y^2 + 4) dy \, dx \]

--End Activity

Some of these activities also include density functions which introduce the verbal register. For example, with cubic coordinates, the units of \(x, y\) and \(z\) are meters, there is a density function \(f\) with units of \(\text{fish/m}^3\), and students are expected to find the number of fish in a cubic tank.
Appendix B: Specific Interview Questions

The following questions were presented to six students in the control group and five students in the experimental group. In the results section of the questions, the final course grade for each student associated with a result is presented in parentheses.

Question 1: Conversion from geometric register to symbolic register

What is an expression for the volume under the surface \( f \) and above the shaded region? (Students were physically shown a surface over the region.)

All efforts were made in question 1 to assure that students visualized the solid associated with the geometric register. The results are presented in Table B1.

Table B1: The results of Interview question 1.

<table>
<thead>
<tr>
<th>Question</th>
<th>Control Group (6 students 4 A’s, 2 Cs)</th>
<th>Experimental Group (2 As, 1B, 2 C’s)</th>
</tr>
</thead>
</table>

Question 2: Conversions from symbolic register to geometric and verbal registers

\[
x = \text{meters} \\
y = \text{meters} \\
f(x,y) = \text{density} = \frac{kg}{m^2}
\]

Given \( \int_{1}^{9} \int_{2}^{6} dy \, dx \)

A. What are the units of this integral?
B. Is it possible to find a result for the integral without evaluating it?
It should be noted that students could interpret the integral as an area or they could recognize that \( \int_1^9 \int_2^6 dy \, dx = \int_1^9 \int_2^6 (1) \, dy \, dx \) and interpret the integral as the volume under the surface \( z = 1 \) over the indicated region. The goal was to see if students would recognize that \( f \) was not a part of this problem. The results are presented in Table B2.

Table B2: The results of interview question 2.

<table>
<thead>
<tr>
<th>Question</th>
<th>Control Group (6 students 4 A’s, 2 Cs)</th>
<th>Experimental Group (2 As, 1 B, 2 C’s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2A</td>
<td>Successful (A).</td>
<td>Successful (AABCC)</td>
</tr>
<tr>
<td></td>
<td>Unsuccessful (AAACC)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Unsuccessful (AAACC).</td>
<td>Unsuccessful (C).</td>
</tr>
</tbody>
</table>

Question 3:

The height of a surface at a given point \((x,y)\) is given by the formula \( f(x,y) = x + y + 1 \) and we wish to find the volume below this surface and over the region shown below.

It should be noted that the students had a large sheet of paper with a very large representation of the region and they were physically shown the plane \( z = x + y + 1 \) over this region to assure that the geometric register was involved.

A. We wish to approximate the volume using 2 divisions in \( x \) and 2 divisions in \( y \), can you draw the associated divisions and indicate the length, width and height of each division if we use the smallest height in each region?

B. Now we wish to approximate the same volume using 3 divisions in \( x \) and 3 divisions
in y, can you draw the associated divisions and indicate the length, width and height of each division if we use the smallest height in each region?

Questions 3A and 3B were to establish the ability to convert from the geometric register to the numerical register.

C. Between (A) and (B), which approximation is better?
D. Is there something we can we do to make these approximations even more precise?

Questions 3C and 3D were to determine whether students understand the underlying limits that are used to transition from the sum with sigma representation of the symbolic register to the definite integral representation of the symbolic register.

E. We’re going to go back to your approximation in A. Can this approximation be expressed using sigma notation? If so, how? If not, why not?

Question 3E was to determine whether students could do the conversion from the numerical register to the symbolic register (sum with sigma representation). Any double sigma expression that could represent the numerical approximation was considered a success.

F. Based on the answer to E
   a. If they have the sigma,
      i. How does this sum in sigma form change if we use the approximation in B instead of the approximation in A?
      ii. How can we make the approximation more and more precise in sigma form?
      iii. How do we obtain the precise value for the volume from this sigma form?
   b. If they don’t have the sigma
      i. If we had been able to put this in sigma notation, would there be any advantage?
      ii. If no, leave the topic
      iii. If yes, what would the advantage be? How would we make the expression more precise in sigma form?
      iv. How do we obtain the precise value of the volume from the sigma form?

The purpose of question 3F was to determine whether the concept which was established in 3C and 3D (that more divisions make the approximation more precise) could be used to do the treatment relating the sum with sigma representation to the definite integral representation in the symbolic register. Any scheme taking the limit of a double sigma to arrive at a double integral was considered a success. The results of question 3 are presented in Table B3.
Table B3: The results of interview question 3.

<table>
<thead>
<tr>
<th>Question</th>
<th>Control Group (6 students 4 A’s, 2 Cs)</th>
<th>Experimental Group (2 As, 1B, 2 C’s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>3A</td>
<td>Successful(AAA). Not Successful(ACC).</td>
<td>Successful(AABCC)</td>
</tr>
<tr>
<td>3B</td>
<td>Successful(AA). Not Successful(ACC).</td>
<td>Successful(AABCC)</td>
</tr>
<tr>
<td>3C</td>
<td>Successful(AAA) Not Successful(ACC)</td>
<td>Successful(AABCC)</td>
</tr>
<tr>
<td>3D</td>
<td>Successful(AAA) Not Successful(ACC)</td>
<td>Successful(AABCC)</td>
</tr>
<tr>
<td>3E</td>
<td>Successful(C) Not Successful(AAAAC)</td>
<td>Successful(ABCC) Not Successful(A)</td>
</tr>
<tr>
<td>3F</td>
<td>Not Successful(AAAACC)</td>
<td>Successful(ABCC) Not Successful(A)</td>
</tr>
</tbody>
</table>

Question 4: Verbal register to numerical register and verbal register to integral representation of the symbolic register

In the region below \( x = \text{km}, \ y = \text{km} \) and the density of people is defined by \( f(y) = 9 - y^3 \) \text{persons/km}^2. \ We are interested in the number of people that live in the region. In the interview students were provided an explanation that perhaps there is a river that runs where \( y < 0 \) and as people prefer to live near the river, the density of people in this region is greater the closer we are to the river.

A. How would you approximate the number of people that live in this region?

Any Riemann sum was considered a success for this part A of the question.

B. If student performs divisions in both \( x \) and \( y \): Do we have to divide in both \( x \) and \( y \)?

Some prompting was provided for part B but only a Riemann sum with divisions in \( y \) and not in \( x \) was considered a success. If they provided such a Riemann sum in part A then part B was automatically considered a success.

C. Can you provide an expression for the precise number of people?

Any correct integral expression was considered a success for part C of the question.
D. If they provide a double integral in part C, ask if the precise number of people can be expressed as a single integral.

As with part B, some prompting was provided for part D but only a single integral expression in terms of y was considered a success. If they provided the correct single integral in part C, it was automatically considered a success in part D. The results are presented in Table B4.

Table B4: The results of interview question 4.

<table>
<thead>
<tr>
<th>Question</th>
<th>Control Group (6 students 4 A’s, 2 Cs)</th>
<th>Experimental Group (2 As, 1 B, 2 C’s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>4A</td>
<td>Unsuccessful (AAAACC)</td>
<td>Successful (AAC)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Unsuccessful (BC)</td>
</tr>
<tr>
<td>4B</td>
<td>Unsuccessful (AAAACC)</td>
<td>Successful (AC)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Unsuccessful (ABC)</td>
</tr>
<tr>
<td>4C</td>
<td>Successful (AC)</td>
<td>Successful (AABC)</td>
</tr>
<tr>
<td></td>
<td>Unsuccessful (AAAC)</td>
<td>Unsuccessful (C)</td>
</tr>
<tr>
<td>4D</td>
<td>Successful (A)</td>
<td>Successful (AC)</td>
</tr>
<tr>
<td></td>
<td>Unsuccessful (AAACC)</td>
<td>Unsuccessful (ABC)</td>
</tr>
</tbody>
</table>