The Number of Digraphs with Cycles of Length \( k \)

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ABSTRACT

In this note, we show that the number of digraphs with \( n \) vertices and with cycles of length \( k \), \( 0 \leq k \leq n \), is equal to the number of \( n \times n \) (0,1)-matrices whose eigenvalues are the collection of copies of the entire \( k \)th unit roots plus, possibly, 0’s. In particular, 1) when \( k = 0 \), since the digraphs reduce to be acyclic, our result reduces to the main theorem obtained recently in [1] stating that, for each \( n = 1, 2, 3, \ldots \), the number of acyclic digraphs is equal to the number of \( n \times n \) (0,1)-matrices whose eigenvalues are positive real numbers; and 2) when \( k = n \), the digraphs are the Hamiltonian directed cycles and it, therefore, generates another well-known (and trivial) result: the eigenvalues of a Hamiltonian directed cycle with \( n \) vertices are the \( n \)th unit roots [2].

KEYWORDS

Acyclic Digraph; Eigenvalue; Power Digraph; (0,1)-Matrix

1. Introduction

A digraph \( G_n \) of \( n \) vertices \( \{1, 2, \ldots, n\} \) is a directed graph whose edges are oriented from vertex \( i \) to vertex \( j \), \( 1 \leq i, j \leq n \). In this note, the digraphs to be considered are with loops or cycles, but parallel edges are forbidden. An acyclic digraph is a digraph that has no cycles of any length. Let \( D_{n,k} \) be a digraph of \( n \) vertices with cycles of length \( k \) plus, possibly, an acyclic digraph with \( n - mk \) vertices, where \( m \) is the number of cycles, and where \( k \) is fixed, \( 0 \leq k \leq n \). Then it is seen that the cycles in \( D_{n,k} \) are disjoint (therefore all the cycles that it has are simple) and if \( k < n \) the digraph is not strongly connected. And, in particular, \( D_{n,n} \) is a Hamiltonian directed cycle of size \( n \) and \( D_{2,0} \) is an acyclic digraph of \( n \) vertices, respectively.

The acyclic directed graphs have been considered by several authors in the past decades. A first related result appeared in the literature seems to be the one described in [2]. It says that a digraph \( G \) contains no cycle if and only if all eigenvalues of its adjacency matrix are 0. Subsequently, to the best of our knowledge, Robinson [3, 4] and Stanley [5] counted the acyclic digraphs independently and showed that if \( R_n \) stands for the number of acyclic digraphs of \( n \) vertices then

\[
R_n = \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} 2^{k(n-k)} R_{n-k} \sim n^2 \frac{2^n}{M^2},
\]

where \( p = 1.488 \ldots \) and \( M = 0.474 \ldots \). Later, in [6, 7], Bender et al. considered the asymptotic number of acyclic digraphs with \( q \) edges, and subsequently, Gessel counted the acyclic digraphs by their sources and sinks in [8]. Most recently, E. Weisstein of Wolfram Research Inc. calculated the number, \( M_{n,q} \), of \( n \times n \) (0,1)-matrices with real positive eigenvalues and showed that for \( n = 1, 2, 3, 4, 5 \) the numbers \( M_{n,q} \) are

1, 3, 25, 543, 29281.
because the numbers were observed to coincide with the first five values of the sequence of the number of acyclic digraphs with n vertices that is obtained by Sloane in [9]. Weisstein conjectured that the two sequences are identical. The conjecture has recently been proven in [1].

Motivated by the above literature, we extend the acyclic digraphs to consider the digraphs \( D_{n,k} \) with cycles of length \( k \) in this note, where \( 0 \leq k \leq n \). Our theorem established in the next section indicates that similar counting theorem holds for more general graphs.

### 2. The Main Results

Let us first prove the following lemma.

**Lemma 1** Given a positive integer \( n \), and a nonnegative integer \( k \) with \( 0 \leq k \leq n \), the eigenvalues of the adjacency matrix \( A = A(D_{n,k}) \) of \( D_{n,k} \) are copies of the entire \( k \)th unit roots plus, possibly, 0’s. Conversely, if \( B \) is an \( n \times n \) \((0,1)\)-matrix whose eigenvalues are copies of the entire \( k \)th unit roots plus, possibly, 0’s then its digraph \( D(B) \) is isomorphic to \( D_{n,k} \), if ignoring the acyclic parts in the two digraphs.

**Proof.** Assume that \( D_{n,k} \) has \( m \) cycles of length \( k \). We show that the eigenvalues of \( A \) are \( m \) copies of the entire \( k \)th unit roots plus \( n - mk \) 0’s. Since relabeling the vertices of a graph does not change the eigenvalues of its adjacency matrix, and since the \( m \) cycles of length \( k \) are disjoined, we may number the vertices consistently with the partial order so that \( A \) has the upper block-triangular as follows:

\[
A = \begin{cases}
    A_1 & \cdots & A_m \\
    0 & A_2 & \cdots & A_m \\
    \vdots & \ddots & \ddots & \ddots \\
    0 & 0 & \cdots & 0
\end{cases}
\]

where each \( A_i, \ i = 1, 2, \cdots, m \), is the adjacency matrix of a (directed simple) cycles of length \( k \), and where ‘*’ is either a block matrix of 0, 1, or 0’s. From linear algebra, it can be easily proven that, for any \( i \leq m \), the characteristic polynomial of \( A_i \) is \( \lambda^k - \lambda + 1 \). So the eigenvalues of \( A_i \) are the \( k \)th unit roots. Since the eigenvalues of \( A \) are collection of the eigenvalues of these \( A_i \) and \( n - mk \) 0’s, its eigenvalues are \( m \) copies of the entire \( k \)th unit roots plus \( n - mk \) zeroes.

Conversely, if \( B \) is an \( n \times n \) \((0,1)\)-matrix whose eigenvalues are \( m \) copies of the entire \( k \)th unit roots plus \( n - mk \) 0’s, then its graph \( D(B) \) is a digraph and, for any \( i \leq n \), the \( i \)th eigenvalues of \( B_i, \lambda_i(B_i), \) is either 1 or 0. We now consider the power digraphs of \( D(B) \) with adjacency matrix \( B \). Since for all \( l = 1, 2, 3, \cdots \)

\[
\text{trace}(B^k) = \sum_{i=1}^{n} \lambda_i^k = mk,
\]

the number of closed walks of length \( l \) in the \( k \)th power graph \( D(B)^k \) of \( D(B) \) is \( mk \). Since the eigenvalues of \( B^k \) are either 1 or 0, the diagonal elements of \( B^k \) must be 1 or 0. In fact, from Perron-Frobenius theory (e.g., [10], p. 28, (1,6) Corollary (a)), we have \( B^l(i,i) \leq \rho(B^l) = 1, \rho(B^l) \) is the largest eigenvalue of \( B^l \), which implies \( B^l(i,i) = 1 \) or 0, and \( B^l \) has exactly \( mk \) 1’s on its diagonal. Thus, counting all the closed walks in the \( k \)th power graph \( D(B)^k \), we conclude that \( D(B) \) is a digraph with \( m \) disjoined cycles of length \( k \) plus an acyclic graph with \( n - mk \) vertices. Putting the thing back to \( B \) implies that \( B \) is the adjacency matrix of a digraph with \( m \) cycles of length \( k \) plus, possibly, an acyclic digraph with \( n - mk \) vertices. The proof is complete.

For \( m = 0, 1, 2, \cdots \), counting the number of the digraphs and the \((0,1)\)-matrices in the above lemma, we immediately have the following

**Theorem 1** For each \( n = 1, 2, 3, \cdots \), and nonnegative integer \( k, 0 \leq k \leq n \), the number of digraphs \( D_{n,k} \) with cycles of length \( k \) is equal to the number of \( n \times n \) \((0,1)\)-matrices whose eigenvalues are the collection of copies of the entire \( k \)th unit roots plus, possibly, 0’s.

Note that when \( k = 0 \), because of the one to one corresponding, this leads to an alternative proof of the above conjecture (the main theorem of [1]). That is, for each \( n = 1, 2, 3, \cdots \), the number of acyclic digraphs \( D_{n,0} \)
is equal to the number of \( n \times n \) \((0,1)\)-matrices whose eigenvalues are 0’s --- from linear algebra, which is equivalent to saying that, for each \( n = 1, 2, 3, \cdots \), the number of acyclic digraphs \( D_{n,0} \) is equal to the number of \( n \times n \) \((0,1)\)-matrices whose eigenvalues are equal to \( 1 \) --- and from [1], which is also equivalent to saying that, for each \( n = 1, 2, 3, \cdots \), the number of acyclic digraphs \( D_{n,0} \) is equal to the number of \( n \times n \) \((0,1)\)-matrices whose eigenvalues are positive real numbers.

**Corollary 1** If a digraph \( D \) has cycles of lengths \( k_i \), \( i = 1, 2, \cdots, t \), and the cycles are piecewise disjoined then the eigenvalues of its adjacency matrix \( A \) are collection of the entire \( k_i \)th unit roots, \( i = 1, 2, \cdots, t \), plus 0’s. And vice versa.

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