The Asymptotic Number of Spanning Trees in Circulant Graphs

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Definition:
The circulant graph

\[ C_{n}^{s_1,s_2,\ldots,s_k} \]

is the graph with \( n \) vertices such that every vertex is connected to its left and right \( s_i^{th} \) neighbors, for each \( i = 1, 2, \ldots , k \).
Definition:
The circulant graph \( C^{s_1, s_2, \ldots, s_k}_n \) is the graph with \( n \) vertices such that every vertex is connected to its left and right \( s_i \)th neighbors, for each \( i = 1, 2, \ldots, k \).

\[
V = \{0, 1, \ldots, n - 1\}
\]

\[
E = \{(i, j) : |i - j| \mod n \in \{s_1, s_2, \ldots, s_k\}\}
\]
Our Problem:
Find the number of spanning trees in a given circulant graph.
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Finding the number of spanning trees in a general given graph is polynomial in the graph size (Kirchoff’s Matrix Tree theorem).

We are interested in finding the number of S.T.s in a parametrized class of circulant graphs.

Given $s_i, \ldots, s_k$, and $m$ as functions of $n$ derive a formula for the number of S.Ts in $C_{m}^{s_1, s_2, \ldots, s_k} = (V, E)$ as a function of $n$. 
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**Example:** $C_{4n+1}^{1, n+1}$
Background

- \( T(C_{n}^{1,2}) = nF_{n}^{2} \)
  Proven by Keitman and Golden (1975)
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- \( T(C_{n}^{s_1, s_2, \ldots, s_k}) \) for different fixed \( s_i \).
  Baron, Prodinger, Tichy, Boesch & Wang (1985),
  Boesch & Prodinger (1986), Sjogren (1991),
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- \( T(C_{n}^{s_{1},s_{2},\ldots,s_{k}}) = na_{n}^{2} \) where \( a_{n} \) satisfies linear r.r. of order \( 2^{s_{k}-2} - 1 \) (and has unique root of maxima modulus)
  when \( s_{i} \) are fixed.
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- \( T(C_{n}^{s_1,s_2,\ldots,s_k}) = na_{n}^{2} \) where \( a_{n} \) satisfies linear r.r.
  of order \( 2^{s_k-2} - 1 \) (and has unique root of maxima modulus)
  when \( s_i \) are fixed.

- \( T\left(C_{m(n)}^{s_1(n),s_2(n),\ldots,s_k(n)}\right) \) satisfies a linear r.r.
  when \( s_i(n) \) and \( m(n) \) are linear in \( n \), e.g., \( C_{4n+1}^{1,n+1} \)
In general, given fixed $s_1, s_2, \ldots, s_k$ or linear $s_1(n), s_2(n), \ldots, s_k(n), m(n)$ there is no “easy” way to find formula for $T(C_{s_1,s_2,\ldots,s_k})$ or $T(C_{s_1(n),s_2(n),\ldots,s_k(n)})$.
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This talk: A formula for asymptotic value

$$\lim_{n \to \infty} \left( T(C_{n}) \right)^{\frac{1}{n}}$$

as a function of the $s_i$
In general, given fixed $s_1, s_2, \ldots, s_k$ or linear $s_1(n), s_2(n), \ldots, s_k(n), m(n)$ there is no “easy” way to find formula for $T(C_{s_1, s_2, \ldots, s_k}^n)$ or $T \left( C_{s_1(n), s_2(n), \ldots, s_k(n)}^{m(n)} \right)$.

This talk: A formula for asymptotic value

$$\lim_{n \to \infty} \left( T(C_n) \right)^{\frac{1}{n}}$$

as a function of the $s_i$.

Caveat: limits are taken over $n$ such that $\gcd(n, s_1, s_2, \ldots, s_k) = 1$ since, if $\gcd \neq 1$, there are no spanning trees!
In general, given fixed $s_1, s_2, \ldots, s_k$ or linear $s_1(n), s_2(n), \ldots, s_k(n), m(n)$ there is no “easy” way to find formula for $T(C_{n}^{s_1,s_2,\ldots,s_k})$ or $T \left( C_{m(n)}^{s_1(n),s_2(n),\ldots,s_k(n)} \right)$.

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Caveat: limits are taken over $n$ such that $\gcd(n, s_1, s_2, \ldots, s_k) = 1$ since, if $\gcd \neq 1$, there are no spanning trees!

Note: When $s_i$ are fixed, $T(C_n) = na_n^2$ where $a_n \sim c\alpha^n$ for some constants $c$ and $\alpha$. So we derive value of $\lim \left( na_n^2 \right)^{1/n} = \alpha^2$. 
Major Results

• Theorem 1: \[
\lim_{n \to \infty} T\left( C_{n}^{s_1, s_2, \ldots, s_k} \right)^{\frac{1}{n}}
\]

exists and can be calculated as a function of the \( s_i \)
Major Results

• Theorem 1: \( \lim_{n \to \infty} \frac{T(C_{n}^{s_1, s_2, \ldots, s_k})}{n} \)

exists and can be calculated as a function of the \( s_i \)

• Theorem 2

\( \lim_{s_1, s_2, \ldots, s_k \to \infty} \lim_{n \to \infty} \frac{T(C_{n}^{s_1, s_2, \ldots, s_k})}{n} \)

exists and can be calculated. Limit is same as for \( k \)-dimensional grid graphs and tori
Major Results

- **Theorem 1:**
  \[
  \lim_{n \to \infty} T(C_n^{s_1, s_2, \ldots, s_k}) \frac{1}{n}
  \]
  exists and can be calculated as a function of the \( s_i \)

- **Theorem 2**
  \[
  \lim_{s_1, s_2, \ldots, s_k \to \infty} \lim_{n \to \infty} T(C_n^{s_1, s_2, \ldots, s_k}) \frac{1}{n}
  \]
  exists and can be calculated. Limit is same as for \( k \)-dimensional grid graphs and tori

- **Theorem 3**
  \[
  \lim_{n \to \infty} T\left(C_{p n + q}^{s_1, \ldots, s_k, a_1 n + b_1, \ldots, a_l n + b_l}\right) \frac{1}{n}
  \]
  exists and can be calculated as a function of the \( s_i \), \( p \) and \( a_i \)
Theorem 1:
For any fixed integers $1 \leq s_1 < s_2 < \cdots < s_k$,

$$\lim_{n \to \infty} T\left(C^n_{s_1, s_2, \ldots, s_k}\right) \frac{1}{n} = 4^k \exp \left( \int_0^1 \ln \left( \sum_{i=1}^{k} \sin^2 \pi s_i x \right) \, dx \right).$$
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$$= 4^k \exp \left( \int_0^1 \ln \left( \sum_{i=1}^{k} \sin^2 \pi s_i x \right) \, dx \right).$$

*Note: This could also be derived in a different way using a recent result of Lyons (2005)*
Theorem 2:

\[
\lim_{s_1, s_2, \ldots, s_k \to \infty} \lim_{n \to \infty} T(C_{n}^{s_1, s_2, \ldots, s_k}) \frac{1}{n} = 4^k \exp \left( \int_0^1 \cdots \int_0^1 \ln \left( \sum_{i=1}^{k} \sin^2 \pi x_i \right) dx_1 \cdots dx_k \right)
\]
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$$\lim_{s_1, s_2, \ldots, s_k \to \infty} \lim_{n \to \infty} T(C_{n}^{s_1, s_2, \ldots, s_k})^\frac{1}{n}$$

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Note: In the special case of $k = 2$ we calculate the limit

$$4 \exp(\int_0^1 \int_0^1 \ln(\sin^2 \pi x + \sin^2 \pi y) \, dx \, dy) = 3.20991230 \ldots$$
**Theorem 2:**

\[
\lim_{s_1, s_2, \ldots, s_k \to \infty} \lim_{n \to \infty} T(C_{n}^{s_1, s_2, \ldots, s_k})^{\frac{1}{n}}
\]

\[
= 4^k \exp \left( \int_0^1 \cdots \int_0^1 \ln \left( \sum_{i=1}^{k} \sin^2 \pi x_i \right) \, dx_1 \cdots \, dx_k \right)
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\]

\[
= \lim_{m,n \to \infty} (T(Grid(m,n)))^{\frac{1}{mn}} = \lim_{m,n \to \infty} (T(Torus(m,n)))^{\frac{1}{mn}}
\]

Theorem 3:

everything integral) Given $1 \leq s_1 < \cdots < s_k$, $1 \leq a_1 \leq \cdots \leq a_l < p$, $b_1, b_2, \ldots, b_l$ and $0 \leq |q| < p$. Then

$$
\lim_{n \to \infty} T\left(C_{pn+q}^{s_1, \ldots, s_k, a_1 n+b_1, \ldots, a_l n+b_l}\right)^{\frac{1}{n}}
$$

$$
= 4^k \exp \left[ \sum_{t=1}^{p} \int_{0}^{1} \ln \left( \sum_{i=1}^{k} \sin^2 \pi s_i x + \frac{1}{2} \left[ l - \sum_{i=1}^{l} \cos \frac{2t \pi a_i}{p} \right] \right) \, dx \right]
$$
\[
\lim_{n \to \infty} T \left( C_{pn+q}^{s_1, \ldots, s_k, a_1 n+b_1, \ldots, a_l n+b_l} \right) \frac{1}{n} = 4^k \exp \left[ \sum_{t=1}^{p} \int_0^1 \ln \left( \sum_{i=1}^{k} \sin^2 \pi s_i x + \frac{1}{2} \left[ l - \sum_{i=1}^{l} \cos \frac{2t \pi a_i}{p} \right] \right) \, dx \right]
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\]

Note this does **NOT** depend upon \( q \) and \( b_i \) so we can calculate

\[
\lim_{n \to \infty} T \left( C_{n}^{s_1, s_2, \ldots, s_k, \lfloor n/d_1 \rfloor+e_1, \lfloor n/d_2 \rfloor+e_2, \ldots, \lfloor n/d_l \rfloor+e_l} \right) \frac{1}{n}
\]

and show it only depends upon the \( s_i \) and \( d_i \) and not the \( e_i \).
\[
\lim_{n \to \infty} T \left( C_n^{s_1, s_2, \ldots, s_k, \lfloor \frac{n}{d_1} \rfloor + e_1, \lfloor \frac{n}{d_2} \rfloor + e_2, \ldots, \lfloor \frac{n}{d_l} \rfloor + e_l \right) \frac{1}{n}
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exists and only depends upon the \( s_i \) and \( d_i \).
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exists and only depends upon the \( s_i \) and \( d_i \)

Example: In limit

\[
T \left( C_n^{1, \lfloor \frac{n}{3} \rfloor} \right) = 4 \exp \left[ \sum_{t=1}^{3} \int_0^1 \ln \left( \sin^2 \pi x + \frac{1}{2} \left[ 1 - \cos \frac{2t\pi}{3} \right] \right) dx \right]
\]
$$\lim_{n \to \infty} T \left( C_n^{s_1, s_2, \ldots, s_k, \lfloor \frac{n}{d_1} \rfloor + e_1, \lfloor \frac{n}{d_2} \rfloor + e_2, \ldots, \lfloor \frac{n}{d_l} \rfloor + e_l \right) \frac{1}{n}$$

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A-priori unexpected since $n \mod 3$ implies different graph structures
\[ \lim_{n \to \infty} T \left( C_n^{s_1, s_2, \ldots, s_k, \lfloor \frac{n}{d_1} \rfloor + e_1, \lfloor \frac{n}{d_2} \rfloor + e_2, \ldots, \lfloor \frac{n}{d_l} \rfloor + e_l \right) \frac{1}{n} \]

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A-priori unexpected since \( n \mod 3 \) implies different graph structures

\( n \mod 3 = 0 \):
\[
\lim_{n \to \infty} T \left( C_n^{s_1, s_2, \ldots, s_k, \lfloor \frac{n}{d_1} \rfloor + e_1, \lfloor \frac{n}{d_2} \rfloor + e_2, \ldots, \lfloor \frac{n}{d_l} \rfloor + e_l \right) \frac{1}{n}
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\]

A-priori unexpected since \( n \mod 3 \) implies different graph structures

\( n \mod 3 = 0 : \quad \text{1 cycle of length } n \quad + \quad \text{n/3 cycles of length 3} \)
\[
\lim_{n \to \infty} T \left( C_n^{s_1,s_2,\ldots,s_k,\lfloor \frac{n}{d_1} \rfloor + e_1,\lfloor \frac{n}{d_2} \rfloor + e_2,\ldots,\lfloor \frac{n}{d_l} \rfloor + e_l \right) \frac{1}{n}
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exists and only depends upon the \( s_i \) and \( d_i \)

Example: In limit

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T \left( C_n^{1,\lfloor \frac{n}{3} \rfloor} \right) = 4 \exp \left[ \sum_{t=1}^{3} \int_0^1 \ln \left( \sin^2 \pi x + \frac{1}{2} \left[ 1 - \cos \frac{2t\pi}{3} \right] \right) \, dx \right]
\]

A-priori unexpected since \( n \text{ mod } 3 \) implies different graph structures

\( n \text{ mod } 3 = 0 : \quad 1 \text{ cycle of length } n \quad + \quad n/3 \text{ cycles of length } 3 \)

\( n \text{ mod } 3 = 1 : \)
$$\lim_{n \to \infty} T \left( C_n^{s_1, s_2, \ldots, s_k, \lfloor \frac{n}{d_1} \rfloor + e_1, \lfloor \frac{n}{d_2} \rfloor + e_2, \ldots, \lfloor \frac{n}{d_l} \rfloor + e_l \right) \frac{1}{n}$$

exists and only depends upon the $s_i$ and $d_i$

Example: In limit

$$T \left( C_n^{1, \lfloor \frac{n}{3} \rfloor} \right) = 4 \exp \left[ \sum_{t=1}^{3} \int_{0}^{1} \ln \left( \sin^2 \pi x + \frac{1}{2} \left[ 1 - \cos \frac{2t\pi}{3} \right] \right) \, dx \right]$$

A-priori unexpected since $n \mod 3$ implies different graph structures

$n \mod 3 = 0 : \quad 1 \text{ cycle of length } n \quad + \quad n/3 \text{ cycles of length } 3$

$n \mod 3 = 1 : \quad 2 \text{ cycles of length } n$
\[
\lim_{n \to \infty} T \left( C_n^{s_1, s_2, \ldots, s_k, \lfloor \frac{n}{d_1} \rfloor + e_1, \lfloor \frac{n}{d_2} \rfloor + e_2, \ldots, \lfloor \frac{n}{d_l} \rfloor + e_l \right) \frac{1}{n}
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Example: In limit

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A-priori unexpected since \( n \mod 3 \) implies different graph structures

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\( n \mod 3 = 1 : \quad 2 \text{ cycles of length } n \)

\( n \mod 3 = 2 : \)
\[
\lim_{n \to \infty} T \left( C_n^{s_1, s_2, \ldots, s_k, \lfloor \frac{n}{d_1} \rfloor + e_1, \lfloor \frac{n}{d_2} \rfloor + e_2, \ldots, \lfloor \frac{n}{d_l} \rfloor + e_l \right) \frac{1}{n}
\]

exists and only depends upon the \( s_i \) and \( d_i \)

Example: In limit

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T \left( C_n^{1, \lfloor \frac{n}{3} \rfloor} \right) = 4 \exp \left[ \sum_{t=1}^{3} \int_{0}^{1} \ln \left( \sin^2 \pi x + \frac{1}{2} \left[ 1 - \cos \frac{2t\pi}{3} \right] \right) \, dx \right]
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A-priori unexpected since \( n \mod 3 \) implies different graph structures

- \( n \mod 3 = 0 \): 1 cycle of length \( n \) + \( n/3 \) cycles of length 3
- \( n \mod 3 = 1 \): 2 cycles of length \( n \)
- \( n \mod 3 = 2 \): 1 cycle of length \( n \) + a 2nd cycle of length \( n \) or 2 cycles of length \( n/2 \)
**Theorem 1:**

For any fixed integers \(1 \leq s_1 < s_2 < \cdots < s_k\),

\[
\lim_{n \to \infty} T(C_n^{s_1, s_2, \cdots, s_k}) \frac{1}{n} = 4^k \exp \left( \int_0^1 \ln \left( \sum_{i=1}^{k} \sin^2 \pi s_i x \right) \, dx \right).
\]
**Theorem 1:**
For any fixed integers \(1 \leq s_1 < s_2 < \cdots < s_k\),

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\lim_{n \to \infty} T(C_{n}^{s_1,s_2,\ldots,s_k})^{\frac{1}{n}} = 4^k \exp \left( \int_0^1 \ln \left( \sum_{i=1}^{k} \sin^2 \pi s_ix \right) \, dx \right).
\]

We will now prove this theorem.
A matrix is a *circulant matrix* if each row is a copy of the previous row (circularly) shifted by one column to the right.
Background

• A matrix is a *circulant matrix* if each row is a copy of the previous row (circularly) shifted by one column to the right

• The adjacency matrices of circulant graphs are circulant matrices

\[
\begin{pmatrix}
0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \\
1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \\
1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\
0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \\
1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \\
1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\
\end{pmatrix}
\]
Background

The eigenvalues of a $n \times n$ 0/1 circulant matrix with 1’s in position $i_1, i_2, \ldots, i_k$ on the first row are

$$\lambda_j = \varepsilon^{i_1 j} + \varepsilon^{i_2 j} + \cdots + \varepsilon^{i_k j},$$

where $\varepsilon = e^{\frac{2\pi \sqrt{-1}}{n}}$ and $j = 0, 1, \cdots, n - 1$. 
The eigenvalues of a $n \times n$ 0/1 circulant matrix with 1’s in position $i_1, i_2, \ldots, i_k$ on the first row are

$$\lambda_j = \varepsilon^{i_1 j} + \varepsilon^{i_2 j} + \cdots + \varepsilon^{i_k j},$$

where $\varepsilon = e^{\frac{2\pi \sqrt{-1}}{n}}$ and $j = 0, 1, \cdots, n - 1$.

$\Rightarrow$ The eigenvalues of adj matrix of $C_{n}^{s_1, \ldots, s_k}$ will be

$$\lambda_j = \sum_{i=1}^{k} \varepsilon^{s_i j} + \sum_{i=1}^{k} \varepsilon^{-s_i j}$$

$$= 2 \sum_{i=1}^{k} \cos \left( \frac{2\pi s_{i,j}}{n} \right)$$
If $G$ is a $d$-regular graph then

$$T(G) = \frac{1}{n} \prod_{j=1}^{n-1} (d - \lambda_j)$$

where $\lambda_0 = d$, $\lambda_1$, $\lambda_2$, $\cdots$, $\lambda_{n-1}$ are the eigenvalues of the corresponding adjacency matrix of the graph.
Background

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  \[ T(G) = \frac{1}{n} \prod_{j=1}^{n-1} (d - \lambda_j) \]
  where $\lambda_0 = d$, $\lambda_1, \lambda_2, \ldots, \lambda_{n-1}$ are the eigenvalues of the corresponding adjacency matrix of the graph.

- This and previous page imply (well known fact)
  \[ T(C_n^{s_1, s_2, \ldots, s_k}) = \frac{1}{n} \prod_{j=1}^{n-1} \left( 2k - 2 \sum_{i=1}^{k} \cos \frac{2\pi s_i j}{n} \right) \]
Background

- If $G$ is a $d$-regular graph then

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$$T(C_{n}^{s_1,s_2,\cdots,s_k}) = \frac{1}{n} \prod_{j=1}^{n-1} \left( 2k - 2 \sum_{i=1}^{k} \cos \frac{2\pi s_{ij}}{n} \right)$$

Proof is about massaging RHS into nice form
\[
\lim_{n \to \infty} T(\frac{C_n^{s_1, s_2, \ldots, s_k}}{n})^{1/n}
\]
\[
\lim_{{n \to \infty}} T(C_n^{s_1, s_2, \ldots, s_k})^{1/n} = \lim_{{n \to \infty}} \left[ \frac{1}{n} \prod_{j=1}^{n-1} \left( 2k - 2 \sum_{i=1}^{k} \cos \frac{2\pi s_{ij}}{n} \right) \right]^{1/n}
\]
\[
\lim_{n \to \infty} T\left(C_n^{s_1, s_2, \ldots, s_k}\right)^{1/n}
\]

\[
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\]
\[
\lim_{n \to \infty} T(C_n^{s_1, s_2, \ldots, s_k})^{1/n}
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if \( f(x) \) is a continuous non-negative real function defined on \((0, 1]\),

s.t. \( \int_0^1 \ln(f(x)) \, dx \) exists, then

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\lim_{n \to \infty} \left( \sum_{j=1}^{n-1} \ln \left( f \left( \frac{j}{n} \right) \right) \times \frac{1}{n} \right) = \int_0^1 \ln(f(x)) \, dx.
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QED
Our approach for the constant $s_i$ case was to

1. Start with the limit of the $\frac{1}{n}$th root of a product of $n$ items,

2. Change this into the limit of $\frac{1}{n}$ times the sum of some function evaluated at the values $\frac{j}{n}$, $j = 1, 2 \ldots n$ and

3. Show that this converges to an integral
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- Show that this converges to an integral

The same approach will work for the linear jump case, but requires many more manipulations.
Open Question
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We saw how to calculate value of

\[ \lim_{n \to \infty} \left( T(C_n) \right)^{\frac{1}{n}} \]

In constant jump case we already knew that \( T(C_n) = na_n^2 \)
where \( a_n \) is described by linear recurrence relation with unique max-modulus root, so this gives us first order asymptotics
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$$\lim_{n \to \infty} \left( T(C_n) \right)^{\frac{1}{n}}$$

In constant jump case we already knew that $T(C_n) = n a_n^2$ where $a_n$ is described by linear recurrence relation with unique max-modulus root, so this gives us first order asymptotics.

In linear-jump case all we know is that $T(C_n)$ satisfies some linear r.r but know nothing about form of solution. How can we derive first order asymptotics?