Algebraic and Combinatorial Properties of the Transfer Matrix of the 2-Dimensional $(1, \infty)$-Runlength Limited constraint

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1 Introduction

The codes here are the ones used in magnetic, digital or optical recordings.

- A \((d,k)\)-Runlength Limited \((RLL)\) constrained code is the set of codewords over binary alphabet \(\{0,1\}\) all satisfying the constraints that the number \(d\) (\(k\)) is the minimum (maximum) permitted number of 0’s separating consecutive 1’s in a legal binary sequence. For example,

\[
1001000100001000000000001000
\]

is a word that satisfies the \((2,10)\)-\(RLL\) constraint used in compact audio discs.

- A 2-D code has constraints both horizontally and vertically. The two constraints may be different.

The read/write isolated constraint is an example. (constraints in reading no two consecutive 1’s and in each rewriting cycle, no two consecutive positions are allowed to change.)
2 2-D \((d,k)-RLL\) constraint

The 2-D \((d, k)-RLL\) code \(S^{(2)}_{d,k}\) satisfies the \textit{RLL} constraint both horizontally and vertically.

For example (if we read in \(y\) direction and write in \(x\) direction):

\[
\begin{pmatrix}
1 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}, \quad
\begin{pmatrix}
1 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}.
\]

The first matrix is fine, but the second is not permitted.
3 The Capacity

- The capacity $\text{cap}(S)$ measures the growth rate of the number $N(m, n; S)$ of $m \times n$ arrays in $S$, i.e., the number of $m \times n$ $(0, 1)$-matrices that satisfy the constraints in two directions,

$$\text{cap}(S) = \lim_{n,m \to \infty} \frac{\log_2 N(m, n; S)}{nm},$$

intuitively, it represents the maximum amount of information that is transmitted or stored per bit in communication.
4 The history

The 2-D \textit{RLL Constrained code} $S_{1,\infty}^{(2)}$ also arises in statistical physics, graph theory of coding theory, until recently it was attached separately by different fields.

- also called \textit{hard-square}, or \textit{hard-core lattice gas system} (Burton and Steif, 1994)
- Engel (1982) called it \textit{Fibonacci number of a lattice},
- Calkin and Wilf (1998) called it \textit{independent sets in grid graph}. Denote it simply by $S_{hs}$.
- Combinatorically, it contains all matrices of size $m \times n$ that do not have adjacent horizontal or vertical ‘1’s.
5 Previous results

As far as we know in estimating the Shannon capacity, there has been no one who has paid much attention to the algebraic and combinatorial properties of the transfer matrix for a constrained code.

For the Hard Square code, Weber seems to be the first to consider its capacity. Weber (1988, [6]) obtained

\[ 0.53602 \leq \text{cap}(S_{hs}) \leq 0.63598, \]

and then Engel (1990, [2])

\[ 0.58789 \leq \text{cap}(S_{hs}) \leq 0.59756, \]

and then Calkin and Wilf (1998,[5]) proved that

\[ 0.587891 \leq \text{cap}(S_{hs}) \leq 0.588339. \]

Now they have been further improved by Nagy and Zeger [14] to be

\[ 0.587891161775 \leq \text{cap}(S_{hs}) \leq 0.587891161868. \]
6 The objective of this talk

Focuses on deriving the properties of transfer matrix of the *Hard Square* system $S_{hs}$. Similar properties of the other constrained codes can be obtained using the techniques and ideas.

- We derive some algebraic and combinatorial properties of the transfer matrix of the *Hard Square* system $S_{hs}$. 
7 Transfer matrix

Let $C_m (m \geq 0)$ be the set of all column $(m + 1)$-vectors $v_i$ of 0’s and 1’s, such that $v$ contains no two consecutive 1’s.

Let $T_{F_{m+3}} = (t_{ij})$ where $t_{ij} \overset{\text{def}}{=} v_i v_j$ is 1 if the concatenation $v_i v_j$ satisfies the constraints and is 0 otherwise. Then $T_{F_{m+3}}$ is called the transfer matrix of the problem.

The number of $m \times n$ matrices $f(m, n)$ that satisfy the constraints is given by

$$f(m, n) = 1^t \cdot T_{F_{m+3}}^n \cdot 1,$$

where 1 is the vector of all 1’s.
8 The capacity and the transfer matrix

\[ \text{cap}(S_{hs}) = \lim_{n,m \to \infty} \frac{\log_2 f(m,n)}{nm} \]
\[ = \lim_{n,m \to \infty} \frac{\log_2 \left( \frac{1}{T_{F_{m+3}}} \cdot 1 \right)}{nm} \]
\[ = \lim_{m \to \infty} \log_2 \frac{\lambda_m^{1/m}}{\lambda_m} \]
\[ = \lim_{m \to \infty} \log_2 \frac{\lambda_{m+1}}{\lambda_m} , \]

and

\[ \text{cap}(S_{hs}) \geq \log_2 \frac{\lambda_{2m+1}}{\lambda_{2m}} , \]

where \( \lambda_m \) is the largest eigenvalue of \( T_{F_{m+3}} \).
9 Characterizations of transfer matrix

We can easily find its transfer matrices.

Lemma 1. If we arrange the \((m + 1)(0,1)\)-sequences in lexicographical order, then the transfer matrix is given by

\[
T_{F_{m+3}} = \begin{pmatrix}
T_{F_{m+2}} & T_{F_{m+2}}^{(F_{m+1})} \\
* & 0_{F_{m+1}}
\end{pmatrix},
\]

where \(T_{F_{m+2}}^{(F_{m+1})}\) is the first \(F_{m+1}\) columns of \(T_{F_{m+2}}\), i.e., \(T_{F_{m+2}} = (T_{F_{m+2}}^{(F_{m+1})}, \cdot)\), \(0_{F_{m+1}}\) is the \(F_{m+1} \times F_{m+1}\) null matrix, the “*” signifies the transpose part.

The first three matrices:

\[
T_{F_3} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix},
T_{F_4} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix},
T_{F_5} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{pmatrix}.
\]
10 The $L^tDL$ decomposition

Lemma 3. The transfer matrix $T_{F_{m+3}}$ has the following $L^tDL$ decomposition:

$$T_{F_{m+3}} = L_{F_{m+3}}^t D_{F_{m+3}} L_{F_{m+3}},$$

where

$$L_{F_{m+3}} = \begin{pmatrix} L_{F_{m+2}} & \begin{pmatrix} L_{F_{m+1}} \\ 0 \end{pmatrix} \\ 0 & L_{F_{m+1}} \end{pmatrix}, \quad L_3 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad L_4 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$D_{F_{m+3}} = \begin{pmatrix} D_{F_{m+2}} & 0 \\ 0 & -D_{F_{m+1}} \end{pmatrix}, \quad D_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad D_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$
11 The distributions of its eigenvalues

**Theorem 1.** Let $P_m$ and $N_m$ be the numbers of positive and negative eigenvalues of $T_{F_{m+3}}$. Then

$$P_m - N_m = -\frac{2}{\sqrt{3}} \sin \frac{m\pi}{3},$$

where $N_m = N_{m-1} - N_{m-2} + F_{m-2}$ with $N_0 = 1$ and $N_1 = 2$. 
12 The inverse of the matrix

**Corollary 1.** The inverse of $T_{F_{m+3}}$ is an $(-1, 0, 1)$-matrix (the elements are $-1, 0, 1$), and is given recursively by

$$T_{F_{m+3}}^{-1} = \begin{pmatrix}
    -T_{F_{m}}^{-1} & 0 & T_{F_{m}}^{-1} & T_{F_{m+1}}^{-1} \\
    0 & 0 & 0 & 0 \\
    (T_{F_{m}}^{-1}, 0) & -T_{F_{m}}^{-1} & 0 & 0 \\
    T_{F_{m+1}}^{-1} & 0 & -T_{F_{m+1}}^{-1} & 0
\end{pmatrix}.$$
13 The power of the matrix

Lemma 4. The transfer matrix $T_{F_{m+3}}$ satisfies the following relations: $T_{F_{m+3}}^k P_{F_{m+3}} = P_{F_{m+3}} T_{F_{m+3}}^k, k = \pm 1, \pm 2, \cdots$, where $P_{F_{m+3}}$ is a symmetric permutation matrix of size $F_{m+3}$.

This gives $T_{F_{m+3}}^2 = A_{F_{m+3}}^2$, where $A_{F_{m+3}} = T_{F_{m+3}} P_{F_{m+3}} = P_{F_{m+3}} T_{F_{m+3}}$.

Corollary 3. If $\lambda_i$ and $\mu_i$ are the eigenvalues of transfer matrix $T_{F_{m+3}}$ and matrix $A_{F_{m+3}}$, respectively, then we have

$|\lambda_i| = |\mu_i| (i = 1, 2, \cdots, F_{m+3})$. So $T_{F_{m+3}}$ and $A_{F_{m+3}}$ share the same largest eigenvalue. Furthermore,

$f(m, n) = 1^t T_{F_{m+3}}^n 1 = 1^t A_{F_{m+3}}^n 1$. 
14 The general properties

Similar relations hold for other constraints. Even if the corresponding transfer matrix, say $T$, is not symmetric, then we have the per-symmetric property:

$$T^k P = P(T^l)^k, \quad k = 0, 1, 2, \cdots$$

where the $P$ is a symmetric permutation matrix. Note that the graph of $P$ is of loops plus cycles of order 2.
15 The order of the recursive relation that $f(m, n)$ satisfies

$f(m, n)$ satisfies a recursive relation with order

$$
\begin{cases}
\frac{(F_{m+3} + F_{m+3})}{2}, & \text{if } m \text{ is odd}; \\
\frac{(F_{m+3} + F_{m+6})}{2}, & \text{if } m \text{ is even}.
\end{cases}
$$

**Example 3.** Following are the first three recursive relations for $f(n, m) = 1^T T_{F_{n+3}}^m 1$ (for $m = 1, 2, 3$). Their recurrence relations have orders 2, 4, 5, respectively.

$$
\begin{align*}
f(n, 1) &= 2f(n - 1, 1) + f(n - 2, 1); \\
f(n, 2) &= 2f(n - 1, 2) + 6f(n - 2, 2) - f(n - 4, 2); \\
f(n, 3) &= 4f(n - 1, 3) + 9f(n - 2, 3) - 5f(n - 3, 3) \\
&\quad - 4f(n - 4, 3) + f(n - 5, 3).
\end{align*}
$$

Their initial conditions are given by $f(m, n) = f(n, m) = 1^T T_{F_{m+3}}^n 1$, for example, $f(1, 1) = 7$, $f(2, 1) = 17$, and so on.
16 The analytic expression of $f(m, n)$

**Theorem 3.** Let $\varphi(\lambda) = det(\lambda I - T)$ be the characteristic polynomial of $T$. Then

$$f(m, n) = \frac{|\lambda_m I - T_{11}|}{\varphi'(\lambda_m)} \lambda_m^{n+2} + \frac{|\lambda_{j_2} I - T_{11}|}{\varphi'(\lambda_{j_2})} \lambda_{j_2}^{n+2} + \cdots$$

$$+ \frac{|\lambda_{j_{rm}} I - T_{11}|}{\varphi'(\lambda_{j_{rm}})} \lambda_{j_{rm}}^{n+2},$$

where $T_{11}$ is the bottom-right $(F_{m+3} - 1) \times (F_{m+3} - 1)$ principal submatrix of $T$, i.e., $T = \begin{pmatrix} 1 & T_{11}^t \\ 1 & 1 \end{pmatrix}$, and all coefficients $\frac{|\lambda_j I - T_{11}|}{\varphi'(\lambda_j)} > 0$. 
17 Numerical computations

<table>
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<tr>
<th>$m$</th>
<th>$\frac{\lambda_{m+1}}{\lambda_m}$</th>
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<td>1</td>
<td>1.4920660376475357643782160628584</td>
</tr>
<tr>
<td>2</td>
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</tr>
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<td>3</td>
<td>1.5029282260930080313112631785226</td>
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</tr>
<tr>
<td>13</td>
<td>1.5030480824752615741231348797605</td>
</tr>
</tbody>
</table>

From the table,

\[
cap(S_{hs}) > \log_2 1.5030480824752323465499639032462 = 0.587891161775232....
\]

We computed them using the recursive relations in Lemma 1 and the *Power Method*. The corresponding two matrices are of sizes 377, 610, respectively.
18 Conclusion and Conjecture

We considered some algebraic and combinatorial properties of the transfer matrix of the *Hard Square* system. Some similar results can be obtained by making use of the approaches.

We would like to pose the following Conjecture. It seems true for the transfer matrices of the *Read/Write Isolated* memory ([23], [24]) and the *Hard Square* system.

**Conjecture.** If the transfer matrix $T$ of a constrained system is symmetric, and if $\lambda$ is an eigenvalue of $T$, and $\lambda$ is not 0 or $-1$, then the number $-\frac{1}{\lambda}$ is also an eigenvalue of $T$. 

19 The case of Hard Square system

In the case of the \textit{Hard Square} system, $T$ is invertible. If the conjecture is true, then we can prove easily that

$$\frac{1^tT^{-2n}1}{1^tT^{-(2n-2)}1} \leq \frac{1^tT^{-(2n+2)}1}{1^tT^{-2n}1} \leq -\lambda_m.$$ 

where $\lambda_m$ is the largest eigenvalue of $T$. This would give a very good upper bound of its capacity.
References


[21] X. YONG, M. GOLIN, The number of displacements of kings in an $2n \times 2m$ chessboard, Preprint.
