# COMP 6839 Problems

1. Let $Y = \mathbf{R}^d$ with the Euclidean norm

$$\|y\| = \left(\sum_{j=1}^{d} y_j^2\right)^{1/2}$$

   (a) Cauchy-Schwarz inequality:

$$\left|\sum_{j=1}^{d} y_j z_j\right| \le \|y\| \, . \, \|z\|$$

   Hint: consider the product $(\lambda y + z)\cdot(\lambda y + z)$ as a function of $\lambda$; does it take negative values?

   (b) Using (a), establish the triangle inequality for the Euclidean norm.

2. Show that $\|y\| = \max_{x\in[a,b]}|y'(x)|$ defines a norm for the linear space

$$Y_0 = \left\{ y\in C^1[a,b] : \int_a^b y(x)\,dx = 0 \right\}$$

   but does not define a norm for $Y = C^1[a,b]$.

3.

   (a) Verify that $\|y\| = |\,y(a)\,| + \max_{x\in[a,b]}|y'(x)|$ defines a norm for $Y = C^1[a,b]$.

   (b) Show that $\max_{x\in[a,b]}|y(x)| \le (1 + b - a)\|y\|,\ \forall y\in Y$. Hint: $y(x) = y(a) + \int_a^x y'(t)\,dt$.

4. With $Y = C[0,1]$ and $(y_n) = ((x/2)^n)$,

   (a) Show that $y_n \to 0$ as $n \to \infty$, using $\|y\|_1 = \int_0^1 |y(x)|\,dx$.

   (b) Show that $y_n \to 0$ as $n \to \infty$, using $\|y\|_\infty = \max_{x\in[0,1]}|y(x)|$.

5. Let $Y = C[0,1]$ and $(y_n) = (x^n)$.

(a) Show that $y_n \to 0$ as $n \to \infty$, for the one-norm of the previous problem.

(b) But $y_n$ does not tend to 0 for the infinity-norm. This shows that sequential convergence to a value depends on the norm.

6. Let $Y = C[a, b]$ and use the definition of continuity to establish that $J(y) = \int_a^b (\sin x) y(x)\, dx$ is continuous on $Y$ using:

(a) $||y||_\infty = \max_{x \in [a,b]} |y(x)|$.

(b) $||y||_1 = \int_a^b |y(x)|\, dx$.

Make a similar analysis for $F(y) = \int_a^b \sin\big(y(x)\big)\, dx$. Hint: use a mean value inequality.

7. Verify that $J(y) = \int_0^1 |y(x)|\, dx$ does not achieve a minimum value on

$$D = \left\{ y \in C[0, 1] : y(0) = 0, y(1) = 1 \right\}$$

although $J$ is bounded below by zero on $D$.