COMP 6839 Problems

We will use witout proof the

> **Lemma:** let Ω be an open subset of the real line. (v_n) converges weakly to v in $L^2(\Omega)$ if:
>
> (a) $|v_n|_2$ is uniformly bounded.
>
> (b) For any closed interval $[a, b]$ contained in Ω, $\int_a^b v_n \, dx \to \int_a^b v \, dx$

8.

(a) Using this lemma, show that the sequence $\sin nx$ converges weakly to zero in $L^2(0, 2\pi)$, but not strongly.

(b) The example on p. 25 of Aubert-Kornprobst strives to illustrate this fact, but appears to contain a mistake. Can you find it?

9. (*) Let X be a Banach space and $x \in X$. Show that the following properties of a function $F : X \to \overline{\mathbf{R}}$ are equivalent:

(a) For all $t \in R$ with $t < F(x)$, there exists a ball U centered at x such that $t < F(y)$ for every $y \in U$. (Definition of lower semi-continuity at x).

(b)

$$F(x) \leq \sup_{U \in N(x)} \inf_{y \in U} F(y)$$

where $N(x)$ is the set of all balls centered at x.

(c) $F(x) \leq \liminf_{n \to \infty} F(x_n)$ for every sequence (x_n) converging to x in X.

10. (*) Same setting as the previous problem. Show that the following properties are equivalent:

(a) F is lower semicontinuous on all of X

(b) For every $t \in R$, the set $\{F > t\}$ is open in X

(c) For every $t \in R$, the set $\{F \leq t\}$ is closed in X

11. Show that the function $f(u, \xi) = u^2 + (1 - \xi^2)^2$ of the Bolza problem is not convex.

12. Compute the polar in R^n of:

(a) $f(\xi) = \frac{1}{p}|\xi|^p, \ 1 < p < \infty$

(b) $f(\xi) = e^{\xi}, \ n = 1$

13. (Matlab Image processing toolbox). Load an image in the matlab session, and add different kinds of noise to it, saving each sample.

14. Is $\log x$ in $L^2(0, 1)$? In $L^p(0, 1)$ for which $p > 1$?