MATE 3021-050-100 Sample examination Due December 2, 2013

1. Part (b) and (c) are independent of (a).

   (a) Find $\lim_{x\to 0}\dfrac{1-\cos(5x)}{2x}$.

   (b) Sketch the graph of $y = \log x$. Noting that the logarithm function is monotone increasing, show that for $x \geq e$, $\log x \geq 1$. Conclude that for $x \geq e$, $\dfrac{1}{x} \leq \dfrac{\log x}{x}$.

   (c) One can also show that for the same range of x-values, $\log x < \sqrt{x}$. Show then that for $x \geq e$, $\dfrac{\log x}{x} < \dfrac{1}{\sqrt{x}}$, and finally use this and the result of (b), plus a certain theorem, to investigate the limit of $\dfrac{\log x}{x}$ as $x \to \infty$.

2.

   (a) Find the rate of change $\dfrac{dx}{dy}$ on the circle $(x-1)^2 + (y+0.5)^2 = 4$ as a function of x, y. (Plot circle).

   (b) Find a function $y = p(x) = ax + b$ which satisfies $p'(0) = 2$, $p(0) = 3$.

3.

   (a) Study the variation and regions of convexity-concavity of the function $y = \log(x^2 + 1)$. Sketch the graph, without tabulating, but marking exactly the extrema and inflection points. What symmetry does it have?

   (b) Plot the graph of $y = 2\log x$. Using inequalities, show that for $x \geq 1$, the graph of $\log(x^2 + 1)$ is intermediate between the graph of $2\log x$ and that of $2\log x + \log 2$. What do you conclude about the existence of straight-line asymptotes? ($\log x$ does not have straight-line asymptotes).

4.

   (a) Discuss the limit as $x \to \infty$ of:

   i) $\exp(-\log(x))$ ii) $\dfrac{3e^{2x}}{2e^{2x} - e^x}$

   (b) Find the straight-line asymptote, if there is one:

   i) $y = \dfrac{3 - x^2}{1 - 2x^2}$ ii) $y = \dfrac{x^2 - 2}{2x + 1}$

5.

   (a) Use the formula for $\cos(a + b)$ and the definition of the derivative to show that

   $$\frac{d}{dx}\cos x = -\sin x.$$

   (b) Find the derivative of $x \longmapsto \dfrac{\sin(2x)}{1 + x^2}$.

6.

(a) Draw three instances of the function $x \longmapsto x^a$: one which decreases, one which increases, and one which is constant. State which range of a corresponds to each case.

(b) Draw three instances of the function $x \longmapsto a^x$: one which decreases, one which increases, and one which is constant. State which range of a corresponds to each case.

(c) In both (a) and (b), find the regions of convexity-concavity.

7. Choose one of the exercises which were assigned on 11/11 (finding direct and inverse image of sets of values).