MATE 3021 problems

1. For $0 \le n \le 5$, graph the function $P_n = 10 \cdot 2^n$.

2. A population quadruples every 20 minutes. Give a formula for P_n , size of the population after n time units elapsed, if $P_0 = 10$ and the unit of time is 10 minutes.

3. Suppose that $P(t) = 100 \cdot 2^t$, $t = 0, 1, 2, \ldots$ and one unit of time corresponds to 2 hours. Determine the amount of time it takes the population to triple in size.

4. A strain of bacteria reproduces asexually every 23 minutes: every 23 minutes, each cell splits into two. If initially there are 2 bacteria, how long will it take until there are 256 bacteria?

5. Find the exponential growth equation for a population that quadruples in size every unit of time and that has five individuals at time 0.

6. Find the exponential growth for a population that doubles in size every unit of time and has 46 individuals at time 0.

7. Find P_n as a function of n in each case:

   (a) $P_0 = 5, \quad P_{n+1} = 2P_n.$

   (b) $P_0 = 6000, \quad P_{n+1} = \frac{1}{3}P_n.$

8. Write the first five terms of the sequence and find $\lim_{n\to\infty} a_n$:

   (a) $a_n = \dfrac{(-1)^n}{n^3 + 2}, \quad n = 0, 1, 2, \ldots$

   (b) $a_n = \left(\dfrac{1}{2}\right)^n, \quad n = 0, 1, 2\ldots$

9. The sequence a_n is recursively defined. Find all fixed points:

   (a) $a_{n+1} = \dfrac{4}{a_n}$

   (b) $a_{n+1} = \sqrt{6a_n}.$

10. For the sequence of part (b) of the previous problem, show by induction that if $a_0 = 2$, then $a_n \ge 2$ for all n. In this case, what are the posible limiting values of the sequence?

11. Using numerical experiments, can you find values of a_0 such that the sequence defined by $a_{n+1} = 3/a_n$ has a limit? (Do not choose a_0 to be the fixed point).

12. Assume that $\lim_{n\to\infty} a_n$ exists. Find all fixed points of the sequence, and use a table, or other reasoning to guess which fixed point is the limiting value for the given initial condition:

(a) $a_{n+1} = \sqrt{2a_n}, \quad a_0 = 0.$

(b) $a_{n+1} = \frac{1}{3}(a_n + 1), \quad a_0 = 1.$

(c) $a_{n+1} = \frac{1}{2}\left(a_n + \frac{4}{a_n}\right), \quad a_0 = 1.$

13. Use the limit laws to evaluate each limit:

a) $\lim_{x\to -3} \frac{x^3 - 20}{x + 1}$ b) $\lim_{x\to -4} \frac{x + 4}{16 - x^2}$

14. Use the limit laws to evaluate each limit:

a) $\lim_{x\to 3} \frac{x^2 - 2x - 3}{x^2 - 4}$ b) $\lim_{x\to -2} \left(\frac{x^2}{2} - \frac{2}{x^2}\right)$

15. Use the limit laws to evaluate each limit:

a) $\lim_{x\to -1} (x^3 - 4x + 2)$ b) $\lim_{x\to 1} \frac{(x-1)^2}{x^2 - 1}$

16. Let $f(x) = \frac{x^2 - 9}{x - 3}$ if $x \neq 3$. What value should one assign to $f(3)$ so that f is continuous everywhere?

17. Let $f(x) = \frac{x^2 + x - 2}{x - 1}$ if $x \neq 1$. Which value to asssign to $f(1)$ so that f is continuous everywhere?

18. Find the values of $x \in R$ where the function, in each case, is continuous:

a) $f(x) = x^4 - x^2$ b) $f(x) = \frac{x^2 + 1}{x - 1}$

c) $f(x) = \cos(2x)$ d) $f(x) = \exp(-\sqrt{x-1}).$

19. Let

$$f(x) = \begin{cases} 1/x, & x \geq 1 \\ 2x + c, & x < 1 \end{cases}$$

Graph f when $c = 0$. Is f continuous for this choice of c? How must you choose c so that f is everywhere continuous?

20. Find the limit in each case:

a) $\lim_{x \to \frac{\pi}{3}} \sin\left(\frac{x}{2}\right)$ b) $\lim_{x \to \frac{\pi}{2}} \frac{\cos^2 x}{1 - \sin^2 x}$

c) $\lim_{x \to 1} \frac{1}{\sqrt{3 - 2x^2}}$ d) $\lim_{x \to 1} \log[e^x \cos(1 - x)]$.

21. a)–d): exercises 10, 13, 15, 18 of § 3.3 of the text.

22. a)–d): exercises 4, 12, 20, 24 of § 3.3.

23. Exercise 1 (b), (c) of § 3.4.

24. Exercise 4 (b), (c) of § 3.4.

25. a)–b): exercises 7, 14 of § 3.4.

26. a)–b): exercises 13, 18 of § 3.4.

27. Exercise 4 of § 3.5.

28. Exercise 6 of § 3.5.

29. Exercise 8 of § 3.5.

30. Exercise 10 of § 3.5.

31. a)–b): exercises 12, 15 of § 4.1. In each case, first find the derivative at x by using the definition as limit of the difference quotient.

32. Exercise 23 of § 4.1.

33. Exercise 31 of § 4.1.

34. Exercise 34 of § 4.1.

35. Exercise 35 of § 4.1.

36. a)–d): exercises 9, 11, 20, 21 of § 4.2.

37. a)–d): exercises 26, 27, 35, 37 of § 4.2.

38. a)–d): exercises 42, 46, 50, 54 of § 4.2.

39. a)–c): exercises 55, 58, 61 of § 4.2.

40. a)–c): exercises 70, 74, 80 of § 4.2.

41. a)–c): exercises 13, 17, 24 of § 4.3.

42. a)–c): exercises 26, 34, 36 of § 4.3.

43. a)–c): exercises 39, 43, 46 of § 4.3.

44. a)–c): exercises 51, 69, 71 of § 4.3.

45. a)–c): exercises 75, 81, 82 of § 4.3.

46. a)–c): exercises 84, 88, 94 of § 4.3. Extra credit for (c): how many hands are found in this problem?

47. a)–c): exercises 5, 11, 22 of § 4.4.

48. a)–c): exercises 29, 35, 39 of § 4.4.

49. a)–b): exercises 44, 46 of § 4.4.

50. a)–c): exercises 49, 54, 55 of § 4.4.

51. a)–b): exercises 64, 67 of § 4.4.

52. a)–b): exercises 70, 72 of § 4.4.

53. a)–b): exercises 78, 85 of § 4.4.

54. a)–c): exercises 13, 16, 44 of § 4.5.

55. a)–c): exercises 15, 17, 48 of § 4.5.

56. a)–c): exercises 20, 29, 35 of § 4.5.

57. a)–b): exercises 60, 62 of § 4.5.

58. a)–b): exercises 67, 70 of § 4.5.

59. a)–c): exercises 4, 9, 35 of § 4.6.

60. a)–c): exercises 20, 23, 36 of § 4.6.

61. a)–c): exercises 24, 44, 48 of § 4.6.

62. a)–b): exercises 52, 54 of § 4.6.

63. a)–b): exercises 59, 60 of § 4.6.

64. a)–b): exercises 61, 62 of § 4.6.

65. Exercise 65 of § 4.6.

66. a)–b): exercises 68, 71 of § 4.6.

67. a)–b): exercises 69, 70 of § 4.6.

68. Find the inverse function of f, then find its derivative in two ways: i) By differentiating it directly ii) Using (4.12) of § 4.7:

a) $f(x) = \sqrt{x+1},\ x \geq -1$ b) $f(x) = \frac{x^2-1}{x^2+1},\ x < 0.$

69. In parts (a)–(c), find the derivative of the inverse function of the given $f(x)$, at the given input value $y = y_0$:
a) $f(x) = e^x + 2x,\ y_0 = 1.$
b) $f(x) = x + \sin x,\ y_0 = \pi.$
c) $f(x) = \sqrt{2+x^2}, x \geq 0,\ y_0 = \sqrt{6}.$

70. Exercise 22 of § 4.7. Draw the graph of $\arcsin x$.

71. Differentiate the functions:

a) $f(x) = (\ln(1-x^2))^2$ b) $g(x) = \ln(\cos(1-x^2))$ c) $h(x) = \ln|x^2-3|.$

For (c), draw the graph of $|x^2-3|$ and sketch the graph of $\ln|x^2-3|$.

72. Find the derivative, using logarithmic differentiation: a) $f(x) = x^{2\ln(x)}$ b) $g(x) = (\ln(2x))^x$ c) $h(x) = x^{\frac{1}{2x}}.$

73. Exercise 76 of § 4.7.

74. Exercises 9–12 of § 5.1.

75. a)–b): exercises 40, 41 of § 5.1.

76. Exercise 44 of § 5.1.

77. Exercise 50 of § 5.1.

78. a)–b) exercises 9, 14 of § 5.2.

79. a)–b): exercises 17, 20 of § 5.2.

80. Exercise 29 of § 5.2.

81. Exercise 30 of § 5.2.

82. Exercise 40 of § 5.2.

83. a)–b): exercises 2, 14 § 5.3.

84. a)–b): exercises 20, 22 § 5.3.

85. Exercise 26 § 5.3.

86. Exercise 29 § 5.3.

87. Exercise 31 § 5.3.

88. Exercise 36 § 5.3.

89. Exercise 42 § 5.3.

90. Exercise 43 § 5.3.

91. a)–b): exercises 43, 47 § 6.1.

92. a)–b): exercises 56, 59 § 6.1.

93. a)–b): exercises 62, 63 § 6.1.

94. Exercise 68 of § 6.1.

95. a)–b): exercises 72, 73 § 6.1. (Draw the graph).

96. a)–b): exercises 76, 77 § 6.1. Draw the graph, and explain which property of the integral comes into play.

97. a)–b): exercises 80, 82 § 6.1. Here too, the reasoning is graphical.

98. a)–c); exercises 2, 5, 10 § 6.2.

For exercises 99–104, use table 6.1, and the occasional given hint.

99. a)–b): exercises 41, 45 of § 6.2.

100. a)–b): exercises 50, 56 of § 6.2.

101. a)–b): exercises 62, 64 of § 6.2. (Guess a function of the form $a\cos((1-x)/3)$ and apply the chain rule).

102. a)–c): exercises 71, 72, 78 of § 6.2.

103. a)–b): exercises 84, 88 of § 6.2. See example 10 p. 301.

104. a)–b): exercises 93, 95 of § 6.2. Note that $\sqrt{e^x} = e^{x/2}$.

105. a)–b): exercises 99, 102 of § 6.2.

106. a)–b): exercises 108, 114 of § 6.2.

107. a)–b): exercises 116, 122 of § 6.2. (Find $d(e^{t^2})/dt$, $d(\log|t-1|)/dt$).

108. a)–b): exercises 125, 126 of § 6.2.