Some common corrections

1. The following symbols have different meanings:

   (a) The symbol “ = ” means *equals*, and nothing else. It connects expressions, sets, quantities, or any objects of the same nature, but not statements. It is not an all-purpose connector, nor does it open a paragraph.

   (b) The symbol “ $\Rightarrow$ ” means *implies*. It connects sentences or logical statements; it does not connect values or expressions. For instance:

   $$\sin^2 x + \cos^2 x = 1 \;\Rightarrow\; 2\cos^2 x + 3\sin^2 x = 2 + \sin^2 x$$

   is correct, an equality being a particular kind of statement.

   $$\sin^2 x + \cos^2 x \Rightarrow 1$$

   does not make sense: “$\sin^2 x + \cos^2 x$” and “1” are not statements.

   (c) The symbol “ $\to$ ” means *tends to*, as in “has limit”. It cannot be substituted either to “ $\Rightarrow$ ” or “ = ”. I recommend its use at the exclusion of the lim symbol when computing limits; see [3-6] below.

2. Axes are marked with a single arrow, not two. The arrow is to show the orientation of the axis, not simply to show that it extends to infinity.

3. Do not confuse a sequence (or a function) with its limit. “$a_n = L$” is an absurdity, unless a_n is the constant sequence $(L, L, L, \ldots)$. If you mean that a_n has limit L, write

   $$a_n \to L\,.$$

   For example,

   $$4 - \frac{1}{n} = 4$$

   is incorrect, and does not replace the correct form

   $$4 - \frac{1}{n} \to 4\,.$$

Most of the abuse of notation occurs using the lim symbol. Simple principle: functions and single values (numbers) are different objects (see also [9] below). A function is a whole graph, whereas a number is a lonely point on the line. Therefore, a function may equal a function, a value may equal a value, and a function may approach a value as its input (a variable) itself approaches a value. This is the only interaction between function and value. Keep this in mind reading [4], [5] and [6] below.

4.

(a) The lim symbol is never used by itself. So:

$$\text{“}\lim_{t\to 0} = -1\text{”}$$

is not valid; you must specify limit of what.

$$\text{“}\lim_{t\to 0}(t-1) \Rightarrow -1\text{”}$$

is also wrong: “ $\Rightarrow$ ” means “implies”, which connects logical statements, not quantities. See [1] above. What about

$$\lim_{t\to 0}(t-1) \to -1\,?$$

It is still wrong. A limit, which is a fixed number, does not approach another number, and “ $\to$ ” means “approaches”. The only correct form is

$$\lim_{t\to 0}(t-1) = -1\,.$$

(b) One common mistake in using the “lim” symbol is to omit it. In the following sequence, both “ = ” signs are wrong:

$$\lim_{x\to\infty}\frac{3x+5}{x-2} = \frac{3+5/x}{1-2/x}$$

$$= \frac{3+0}{1-0}$$

(fixed number = function of x = fixed number). The correct form is instead:

$$\lim_{x\to\infty}\frac{3x+5}{x-2} = \lim_{x\to\infty}\frac{3+5/x}{1-2/x}$$

$$= \frac{3+0}{1-0}$$

(number = number = number). See following note about how to avoid all this trouble.

5. For problems on limits, follow this style: the first line is only to cast the problem statement, ending with the colon symbol “:”

(a) $\displaystyle\lim_{x\to\infty}\frac{3x+5}{x-2}:$

As $x \to \infty$, $$\frac{3x+5}{x-2} = \frac{3+5/x}{1-2/x}$$

$$\to \frac{3+0}{1-0}$$

(since $5/x \to 0$ and $2/x \to 0$)

$$= 3$$

Note the use of $=$ and $\to$: function equals function, number equals number, function approaches number. We never write function equals number, or number approaches function. A limit is a number, not a function.

(b) $\lim\limits_{t\to\infty} e^{-1/t^2}$:

As $t \to \infty$, $$-\frac{1}{t^2} \to 0$$

so $$e^{-1/t^2} \to e^0$$

$$= 1 .$$

(c) $\lim\limits_{x\to\infty} \dfrac{1-e^x}{1+2e^x}$:

As $x \to \infty$, $$\frac{1-e^x}{1+2e^x} = \frac{e^{-x}-1}{e^{-x}+2}$$

$$\to \frac{0-1}{0+2}$$

(since $e^{-x} \to 0$)

$$= -\frac{1}{2} .$$

(d) $\lim\limits_{t\to\infty} \sin(2t)$:

As $t \to \infty$, $\sin(2t)$ oscillates by taking infinitely often all values between -1 and 1 , and therefore has no limit.

Here, a plot would be useful. Provide plots whenever possible.

(e) The clause “as $x \to a$” (a can be a finite value or infinity) does not have to come first, it can come last. Illustrate on exercise 43 of hw6:

$\lim_{\theta \to 0} \frac{\sin \theta}{\theta + \tan \theta}$:

$$\frac{\sin \theta}{\theta + \tan \theta} = \frac{\sin \theta / \sin \theta}{\theta / \sin \theta + 1/\cos \theta}$$

$$\to \frac{1}{1 + 1}$$

as $\theta \to 0$

where we used $\theta/\sin \theta \to 1$ and $\cos \theta \to 1$ as $\theta \to 0$.

6. Finding a derivative without using the lim symbol. Find the derivative of $f(x) = x^3$ using the definition in the form (4) p.160: limit as $h \to 0$ of $[f(x+h) - f(x)]/h$:

$$\frac{f(x+h) - f(x)}{h} = \frac{(x+h)^3 - x^3}{h}$$

$$= \frac{x^3 + 3x^2h + xh^2 + h^3 - x^3}{h}$$

$$= \frac{3x^2h + 3xh^2 + h^3}{h}$$

$$= 3x^2 + h(3x + h^2)$$

$$\to 3x^2 \text{ as } h \to 0.$$

So that the derivative of x^3 is $3x^2$.

7. Never write $0/0$, $a/0$, a/∞, ∞/∞, $0 \times \infty$, $\infty - \infty$, $\ln(0)$ or any other variant of this style. In particular, ∞ is not a number.

8. As with any language, a solution in mathematics is a sequence of sentences, separated by periods. The sentences are often, but not always, themselves sequences of equalities (or inequalities). They may sometime contain words ("since the function is continuous, the hypothesis of such theorem holds…"). A "solution" which consists of a heap of disconnected expressions will be considered nonsensical, and may not get any credit.

9. Distinguish between the expression of a function and its value at a particular point. If I found that $g(x) = 1 + x$ and I need its value at $x = 1$, I don't write

$$g(x) = 1 + x = 2$$

since "$g(x) = 1 + x$" is an identity (true for all x) but "$1 + x = 2$" certainly is not, but I write two separate statements:

$$g(x) = 1 + x$$

$$g(1) = 2\,.$$

Keep this in mind in problems where you must find, say, the equation of the line tangent at a point.

There are of course cases when it is appropriate to write “$1 + x = 2$”: for example, as part of the sentence “Solve for x the equation $1 + x = 2$”. But this is practically the only exception.