Some common corrections

1. As with any language, a solution in mathematics is a sequence of sentences, separated by periods. The sentences are often, but not always, themselves sequences of equalities (or inequalities). They may sometime contain words ("since the function is continuous, the hypothesis of such theorem holds…"). A "solution" which consists of a heap of disconnected expressions will be considered nonsensical, and may not get any credit.

2. Some are using the following symbols as if they meant the same thing:

   (a) The symbol " $=$ " means *equals*, and nothing else. It connects expressions, sets, quantities, or any objects of the same nature, but not statements. It is not an all-purpose connector, nor does it open a paragraph.

   (b) The symbol " $\Rightarrow$ " means *implies*. It connects sentences or logical statements; it does not connect values or expressions. For instance:

   $$\sin^2 x + \cos^2 x = 1 \;\Rightarrow\; 2\cos^2 x + 3\sin^2 x = 2 + \sin^2 x$$

   is correct.

   $$\sin^2 x + \cos^2 x \Rightarrow 1$$

   does not make sense: "$\sin^2 x + \cos^2 x$" and "1" are not statements.

   (c) The symbol " $\to$ " means *tends to*, as in "has limit". It cannot be substituted either to " $\Rightarrow$ " or " $=$ ".

3. (Adventures with square root). To solve, say, $x^2 = 5$, some like to use as a step:

   $$\sqrt{x^2} = \sqrt{5} \qquad (1)$$

   Omit this step: in real numbers it is redundant, and in complex numbers it is defective. The correct steps are:

   $$x^2 - 5 = 0$$

   $$(x + \sqrt{5})(x - \sqrt{5}) = 0$$

   yielding the solutions $x = -\sqrt{5}, x = \sqrt{5}$. This way, in complex numbers, $x^2 = -5$ gives:

   $$x^2 + 5 = 0$$

   $$(x - \sqrt{5}i)(x + \sqrt{5}i) = 0\ldots$$

   As for (1), which I am asking you to omit, keep in mind that in real numbers, $\sqrt{x^2} \equiv |x|$, not x.

4. We never use the integral symbol without the differential symbol. $\int f(x)$ is incorrect. If you are integrating with respect to x, you must write $\int f(x)\,dx$.

5. We never equate differential expressions with scalars:

$$d(\cos\theta) = -\sin(\theta)d\theta$$

is correct (on the right, scalar times differential = differential).

$$d(\cos\theta) = -\sin(\theta)$$

is not.

6. Never write $0/0$, $a/0$, a/∞, ∞/∞, $0 \times \infty$, $\infty - \infty$, $\ln(0)$ or any other variant of this style. In particular, ∞ is not a number.

7. Use of parentheses. Algebraic symbols $+$, $-$, $\cdot$, $\times$ etc… are not allowed to collide and must be separated by parentheses. To multiply x by $-y$, we write $x(-y)$. "$x \cdot\ -y$" or "$x \times -y$" is incorrect, even if you reduce the size of the minus sign, raise it and stick it very close to y. Also, $x - y$ and $x + (-y)$ are correct. "$x + -y$" is not, even if you reduce the size of the minus sign, raise it and stick it very close to y.

8. A definite integral is a single number:

$$\int_0^{\pi/4} (1 + \tan^2\theta)\, d\theta = 1$$

An indefinite integral is a collection of functions:

$$\int (1 + \tan^2\theta)\, d\theta = \tan\theta + C$$

This integral is a single function of x:

$$\int_0^{x} (1 + \tan^2\theta)\, d\theta = \tan x$$

The first might be denoted I, the second $I(\theta)$, the third, $I(x)$ or $g(x)$. We do not have separate notation for a single function or a collection of functions; I have used capital letters for collections of functions. It is a mistake to use the wrong notation for any of these three objects, and an even worse mistake to try to equate any two of different kind.
Note than when given a definite integral, there is no advantage whatsoever to replace it with an indefinite integral. Any transformation of the integrand is done as a separate step.

9. In problems where you have to find the volume of a solid of rotation, you are to provide *two* figures: one of the plane region, including a plane section of a typical element (annulus or shell), the other of the generated solid, including the generated whole element (annulus or shell). Make the figures to be at least 10cm wide. For examples in the text (in case you were

not paying attention in class), see figure 8 (a) and (b) p.442, and figure 10 p.453. Along with the element, show which of dx or dy plays the role of thickness. Before writing the formula $dV = \ldots$, I have used the step “at position x (or y), inner radius = …, outer radius = …” or “at position x (or y), radius = …, height = …”. Follow these steps.

10. Trigonometric substitutions: you must show not only which substitution you perform, but also the θ interval. This involves drawing a certain figure (one or two arcs of the unit circle). You must also explain why a certain quantity is non-negative. Refer to the preamble of homework 4, and to the examples that I repeated in class on Feb 19. Here is a model solution for this kind of problem: exercise 3 p.491 of Stewart asks to evaluate

$$\int \frac{\sqrt{x^2-4}}{x}\,dx\,.$$

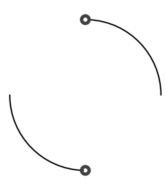
The solution has three steps:

Step 1: name things and indicate the substitution, and what it implies.
Step 2: transform $f(x)\,dx$ in terms of θ, starting with the square root, and then the rest.
Step 3: integrate, then transform back to $I(x)$.

The solution will be, verbatim:

Step 1

Let $f(x)$ be the integrand and $I(x) = \int f(x)\,dx$. Let $x = 2\sec\theta$, with θ in one of these quadrants:



$$\text{Then } dx = 2\sec\theta\tan\theta\,d\theta \quad \text{and} \quad \tan\theta \geq 0\,. \tag{1}$$

Step 2

$$\begin{aligned}
\sqrt{x^2-4} &= \sqrt{4\sec^2\theta - 4} \\
&= 2\sqrt{\sec^2\theta - 1} \\
&= 2\left|\tan\theta\right| && (2) \\
&= 2\tan\theta, \quad \text{since } \tan\theta \geq 0 && (3)
\end{aligned}$$

$$f(x)\,dx = \frac{2\tan\theta}{2\sec\theta}\,2\sec\theta\tan\theta\,d\theta$$

$$= 2\tan^2\theta\,d\theta\,.$$

Step 3

$$\int 2\tan^2\theta\,d\theta = \int 2(1+\tan^2\theta - 1)\,d\theta$$

$$= 2(\tan\theta - \theta) + C\,.$$

$$I(x) = 2\sqrt{x^2-4} - 2\,\mathrm{arcsec}(x/2) + C\,.$$

In summary, your solution must contain the figure showing the quadrants for θ and the corresponding caption (both labelled (1)) and the sub-steps labelled (2) and (3) including " since $\tan\theta \geq 0$". You will not regret later this attention to detail.

11. Break down your solution into sentences. In the above example, there are seven sentences (an uninterrupted sequence of equalities makes one sentence): three in step 1, two in step 2, two in step 3. Try writing the same solution without giving names to the integrand $f(x)$ and the integral $I(x)$. You will have either to write the same expression in full more than once, or write a long sequence of equalities, separated by interruptions ("let $x = \ldots$") and yielding a number of dangling " = " signs. So it is good practice to name things in problems that involve intermediate steps: substitution, integration by parts, partial fractions. This means most or all of the sections in chapter 7.

12. The comparison theorem for integrals implies that a positive integrand over an interval cannot have a negative integral. The same theorem also implies that a negative integrand cannot have a positive integral.

13. We do not mix variables inside integrals. If in

$$\int_0^1 \frac{e^{3x}}{1+e^x}\,dx$$

you perform the substitution $u = e^x$, the result will be

$$\int_1^e \frac{u^2}{1+u}\,du$$

(only u appears). $\int_1^e \frac{e^{2x}}{1+u}\,du$ is incorrect.