Tips and common corrections

[1] Some comments about notation.

(a) The symbol " $=$ " means *equals*, and nothing else. It connects expressions, sets, quantities, or any objects of the same nature, but not statements. It is not an all-purpose connector, nor does it open a paragraph.

(b) The symbol " $\Rightarrow$ " means *implies*. It connects sentences or logical statements; it does not connect values or expressions. For instance:

$$\sin^2 x + \cos^2 x = 1 \;\Rightarrow\; 2\cos^2 x + 3\sin^2 x = 2 + \sin^2 x\,.$$

The symbol " $\to$ " means *tends to*, as in "has limit". It cannot be substituted to " $\Rightarrow$ ".

(c) The synthetic set notation describes a subset of a known set by the generic property of its members. For instance, the region of the plane where $x < y$ can be written

$$D = \{(x, y) : x < y\}$$

Alternative notations such as "$\{R^2 : x < y\}$" and variants thereof are nonsensical, and better discarded.

(d) Use of parentheses.

To multiply x by $-y$, we write $x(-y)$. "$x \cdot -y$" or "$x \times -y$" is incorrect, even if you reduce the size of the minus sign, raise it and stick it very close to y.
Also, $x - y$ and $x + (-y)$ are correct. "$x + -y$" is not, even if you reduce the size of the minus sign, raise it and stick it very close to y.

(e) The notations $\dfrac{dz}{dx}, \dfrac{\partial z}{\partial x}$ mean different things. The first is a total derivative (z is a function of x only, possibly through intermediate variables), whereas the second is a partial derivative.

(f) Some write "$\lim\limits_{(x,y)\to(x,0)} f(x, y)$" to mean: "limit along the x-axis". This is incorrect: $(x, 0)$ is a variable point, not a fixed point. You may write:

$$\lim_{x\to 0} f(x, 0)$$

since, along a curve (the x-axis), the function becomes essentially a function of one variable only. A better way (which works not only for the axes, but for any curve in the plane) is to use parametric equations for the path along which you take the limit, as I showed in class. In the case, say, of the parabola $x = y^2$, those equations might be

$$(x, y) = (t^2, t), \qquad -\infty < t < \infty,$$

and then your limit becomes $\lim\limits_{t\to 0} f(t^2, t)$.

(g) The symbol f, in itself, does not mean any generic function, the particular instance of which has to be guessed from context by the puzzled reader. There is a tendency to use f (as in f_x, f_y) in problems where no f is given. In general, you must not use any symbol not given by the problem, unless you define it first. If you mean "$z = f(x, y)$", then you must state so.

(h) We never use the integral symbol without the differential symbol. $\int f(x)$ is incorrect. If you are integrating with respect to x, you must write $\int f(x)\,dx$. If you are integrating on a plane region with respect to area, you must use dA (or $dx\,dy$, or $r\,dr\,d\theta$, as the case may be).

[2] In the xy plane, an equation

$$F(x, y) = C \qquad (1)$$

correspond to a curve, and the inequality

$$F(x, y) \le C \qquad (2)$$

to a region bounded by the above curve. This is how you find (and draw) the region defined by (2). For instance, the region $y^2 \le x$ is the inside of a parabola (draw it). On the other hand, avoid nonsensical algebra such as this:

$$\text{"} y^2 \le x \Rightarrow y \le \pm\sqrt{x} \text{"}$$

(What does $y \le \pm\sqrt{x}$ mean? that "$y \le \sqrt{x}$ *or* $y \le -\sqrt{x}$" or "$y \le \sqrt{x}$ *and* $y \le \sqrt{x}$"? You may check that neither of the regions defined by these two statements is the inside of the parabola.)

[3] Finding the range of a function.

We illustrate with this model example: find the range of the function

$$f(x, y) = \log(g(x, y)) = \log(x^2 - y^2).$$

The domain of f is the region $D = \{(x, y) : x^2 - y^2 > 0\}$, which is the union of two open quadrants, of boundary the two lines $y = \pm x$.

The range of f is the image under log of the image under g of the domain D, where

$$g(x, y) = x^2 - y^2.$$

The crucial step is to determine $g(D)$. By property of D, this image is contained in $I = (0, \infty)$. If we can show that $g(D)$ fills all of I, then $f(D) = \log(g(D))$ will be the whole real line: the log function being monotone increasing, it sends intervals to intervals, and it sends $I = (0, \infty)$ to all of R, as is easily seen from its graph.

The difficulty of determining $g(D)$ is that g is a function from two variables to one, and its graph is a surface. However, if we can find a *curve* C contained in D such that $g(C) = I$, then we are done, since in the chain of inclusions

$$g(C) \subset g(D) \subset I,$$

the sets between which $g(D)$ is squeezed are equal. C intersects then all the level sets of g (why?) The reason we look for a curve is that, upon parametrising the curve, the restriction of g to C is a function of a single variable (namely, the parameter), and to find the image of C, we use the graph of that function.

Illustrate on this example: the level sets of g are all the hyperbolae of equation $x^2 - y^2 = k$, where $k > 0$. Knowing this, or simply by guessing, we are lead to a suitable choice of the curve C: a curve which intersects all the level sets is the open half of the x axis,

$$C = \{(t, 0) : 0 < t < \infty\}.$$

Then $g(C) = \{t^2 - 0 : 0 < t < \infty\}$ This is the image under the square function $h(t) = t^2$ of the interval $(0, \infty)$, and, from the graph of the square function, this image is indeed all of $I = (0, \infty)$. Which ends the proof.

[4] "$\sqrt{x^2} = x$" is wrong, since x may be negative. The correct formula is $\sqrt{x^2} = |x|$, which is more precise than "$\sqrt{x^2} = \pm x$".

[5] The preferred way to find a tangent plane to a surface is to represent the surface as a level set, even if given in functional form. For example, $x = y^2 - z^2$ is equivalent to $x - y^2 + z^2 = 0$, and not equivalent at all to $z = \sqrt{y^2 - x}$, as pointed out in note [4].

[6] Distinguish between the expression of a function and its value at a particular point. If I found that the gradient of f is $\langle x^2, z, -y\rangle$ and I will need its value at $(-1,1,0)$, I don't write

$$\nabla f(x, y, z) = \langle x^2, z, -y\rangle = \langle 1, 0, -1\rangle$$

but I write two separate statements:

$$\nabla f(x, y, z) = \langle x^2, z, -y\rangle$$

$$\nabla f(-1,1,0) = \langle 1, 0, -1\rangle.$$

On the same subject, reserve, in general, symbols like $x, y, z\ldots$ for generic coordinates, and refer to particular values by using subscripted variables. In the example above, it is better to state "let $(x_0, y_0, z_0) = (-1,1,0)$" than "let $(x, y, z) = (-1,1,0)$".

[7] "$a = \cos b$" and "$b = \cos^{-1}(a)$" are not equivalent statements at all (one implies the other, but unfortunately, not in the direction you would like). Refer to your precalculus text as to why. The only use of the so-called "inverse trigonometric functions" (which are not true inverse functions) is along with tables or calculators. Do not use them in this course (at the risk of penalty), and use them wisely and sparingly outside this course.

[8] When computing iterated integrals, do not use the style of the text:

$$\int_0^1 \int_y^{\sqrt{y}} (x + y)\, dx\, dy = \int_0^1 \left[\frac{x^2}{2} + yx \right]_{x=y}^{x=\sqrt{y}} dy = \ldots$$

but rather, break into steps:

$$I = \int_0^1 \int_y^{\sqrt{y}} (x + y)\, dx\, dy$$

$$\int_y^{\sqrt{y}} (x + y)\, dx = \frac{1}{2}(y - y^2) + y(\sqrt{y} - y)$$

$$I = \int_0^1 \frac{1}{2}(y - y^2) + y(\sqrt{y} - y)\, dy = \ldots$$