Some common corrections

1. The following symbols have different meanings:

   (a) The symbol " $=$ " means *equals*, and nothing else. It connects expressions, sets, quantities, or any objects of the same nature, but not statements. It is not an all-purpose connector, nor does it open a paragraph.

   (b) The symbol " $\Rightarrow$ " means *implies*. It connects sentences or logical statements; it does not connect values or expressions. For instance:

   $$\sin^2 x + \cos^2 x = 1 \;\Rightarrow\; 2\cos^2 x + 3\sin^2 x = 2 + \sin^2 x$$

   is correct, an equality being a particular kind of statement.

   $$\sin^2 x + \cos^2 x \Rightarrow 1$$

   does not make sense: "$\sin^2 x + \cos^2 x$" and "1" are not statements.

   (c) The symbol " $\rightarrow$ " means *tends to*, as in "has limit". It cannot be substituted either to " $\Rightarrow$ " or " $=$ ".

2. Axes are marked with a single arrow, not two. The arrow is to show the orientation of the axis, not simply to show that it extends to infinity.

3. On problems where you have to match equations with graphs, you only have to provide enough detail to force the conclusion. This may involve using mostly words. For instance, problem 3 of hw1:

   > 21 and 22 are the only bounded surfaces among those given. 21 corresponds to an ellipsoid of diameter along the x-axis (the diameter is the largest distance between points of the surface), 22 corresponds to an ellipsoid of diameter along the z-axis. Therefore, 21 matches VII and 22 matches IV.
   >
   > 27 is the only cylindre among the given surfaces. So 27 matches VIII.

   Etc…

4. "Inverse trigonometric functions" is a misnomer, as trigonometric functions have no inverse. $\sin(x-y) = k$ does not become $x - y = \sin^{-1}(k)$ unless, by coincidence, $-\pi/2 \le x - y \le \pi/2$. $\sin^{-1}(k)$ is a single value, whereas the solutions t of $\sin t = k$ form an infinite periodic set, of which $\sin^{-1}(k)$ is one representative, useful for finding the numerical value(s), given other information on t. So $\sin^{-1}$ is only useful in conjunction with table or calculator. Refer to your precalculus text for the definition of $\sin^{-1}$ and $\cos^{-1}$. You will never need these functions, except when solving certain numerical problems.

5. The inequality $x^2 \le 4$ corresponds to the region in the xy plane bounded by the lines of equations $x = \pm 2$. "$x \le \pm 2$" is meaningless: what is the choice? each of $x \le -2$ and $x \le 2$

represents a vertical half-plane, one containing the other. Neither their intersection nor their union is the appropriate solution set, since both contain the region $x \le -2$.

6. Never write $0/0$, $a/0$, a/∞, ∞/∞, $0 \times \infty$, $\infty - \infty$, $\ln(0)$ or any other variant of this style. In particular, ∞ is not a number.

7. Distinguish between the expression of a function and its value at a particular point. If I found that $g(x) = 1 + x$ and I need its value at $x = 1$, I don't write

$$g(x) = 1 + x = 2$$

since "$g(x) = 1 + x$" is an identity (true for all x) but "$1 + x = 2$" certainly is not, but I write two separate statements:

$$g(x) = 1 + x$$

$$g(1) = 2 .$$

There are of course cases when it is appropriate to write "$1 + x = 2$": for example, as part of the sentence "Solve for x the equation $1 + x = 2$". But this is practically the only exception.

8. Much confusion mars the use of the lim symbol. One way to misuse it is to equate a limit (which is a fixed value) with a function, as in

$$\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^2 + yx^2} = \frac{y}{1+y}$$

(nonsense: fixed value = function of y). Other incorrect forms are:

$$\lim_{(x,0)\to(0,0)} f(x,y)$$

(do you mean $\lim_{x\to 0} f(x,0)$?), or

$$\lim_{(x,y)\to(x,0)}$$

($(x,0)$ is not a fixed value). You will not find the symbol used this way in the text.

To keep one variable fixed, or, more generally, to replace a two-variable limit by a single-variable limit, I have shown how to do it using parametrisation (for instance, $y = 0$, $x = t$; more generally, $x = x(t)$, $y = y(t)$). One can solve any problem about limits without using (or using minimally) the lim symbol. For such an example, read [9] below.

9. Limits, continued. Problem 41 of §14.2 asks to find

$$\lim_{(x,y)\to(0,0)} \frac{e^{-(x^2+y^2)} - 1}{x^2 + y^2} .$$

I copy the expression as shown, and thereafter I only use "lim" for functions of a single variable. Solution:

Let $f(x, y) = \dfrac{e^{-(x^2+y^2)} - 1}{x^2 + y^2}$. Using polar coordinates, $f(x, y) = \dfrac{e^{-r^2} - 1}{r^2} = g(r)$. By l'Hospital's rule,

$$\begin{aligned} \lim_{r\to 0+} g(r) &= \lim_{r\to 0} g(r) \\ &= \lim_{r\to 0} \frac{-2re^{-r^2}}{2r} \\ &= -1\,. \end{aligned}$$

Note how introducing function names (here, f, g) alleviates the notation. Note also that writing each of

$$\lim_{r\to 0} g(r) = \frac{-2re^{-r^2}}{2r} \qquad \text{(value = function)}$$

or

$$g(r) = \frac{-2re^{-r^2}}{2r} \qquad \text{(function = a different function)}$$

amounts to an absurdity.

Finally, observe that the solution shown (as an example of correct style) is not the best. It is better in general to avoid using l'Hospital's rule, on account of all the conditions to be verified. Here, observe that $(e^{-r^2} - 1)/r^2$ is of the form $(e^{-t} - 1)/t$, which is a difference quotient of the function e^{-t} at $t = 0$, and since e^{-t} has known derivative $-e^{-t}$, the difference quotient has limit $\dfrac{d}{dt} e^{-t}\Big|_{t=0} = -1$ as $t \to 0$.

10. Existence of limit. We gave an example (ex 2) of a function which has the same limit along any line going through the origin (there are infinitely many), yet has no limit as $(x, y) \to (0, 0)$. The moral is: finding the same limit along certain paths, or as many paths as you want, does not mean the limit exists. Finding limits along paths only serves to show that there is no limit.

    To show that there is a limit, there are two ways we know: direct use of the definition, or squeeze theorem (the use of polar coordinates, when possible, makes itself use of the squeeze theorem). A convenient criterion, equivalent to the squeeze theorem and which I explained in class, (and which you can substitute to the squeeze theorem from now on) is:

$$f(x,y) \to 0 \text{ as } (x,y) \to 0 \text{ if } f(x,y) = g(x,y)\,h(x,y) \qquad (1)$$

where $g(x, y) \to 0$ as $(x, y) \to 0$ and $\left|h(x, y)\right|$ is bounded by some M.

This criterion can also be used when the limit of f is some $L \neq 0$, since $f(x,y) \to L$ amounts to $f(x,y) - L \to 0$, where (1) now reads instead:

$$f(x,y) - L = g(x,y)h(x,y)$$

11. Distinguish between total and partial derivative. If $f(x,y)$ depends on t via $x(t), y(t)$, the rate of change of f with respect to t is denoted df/dt. The rate of change of f wrt x is $\partial f/\partial x$.

12. Use of parentheses. Algebraic symbols $+$, $-$, $\cdot$, $\times$ etc… are not allowed to collide and must be separated by parentheses. To multiply x by $-y$, we write $x(-y)$. "$x \cdot\ -y$" or "$x \times -y$" is incorrect, even if you reduce the size of the minus sign, raise it and stick it very close to y. Also, $x - y$ and $x + (-y)$ are correct. "$x + -y$" is not, even if you reduce the size of the minus sign, raise it and stick it very close to y.

13. Negative numbers are not the square root of their square.

$$z^2 = 8$$

has two solutions z, not only one, as you see by factoring:

$$z^2 - 8 = (z - \sqrt{8})(z + \sqrt{8}).$$

To put it another way:

$$\text{“} z = \sqrt{z^2} \text{”}$$

is a fallacy, which will cost you dearly when solving problems. The correct identity is

$$\sqrt{z^2} \equiv |z|.$$

14. We never use the integral symbol without the differential symbol. $\int f(x)$ is incorrect. If you are integrating with respect to x, you must write $\int f(x)\,dx$.

15. We never equate differential expressions with scalars:

$$d(\cos\theta) = -\sin(\theta)d\theta$$

is correct (on the right, scalar times differential = differential).

$$d(\cos\theta) = -\sin(\theta)$$

is not.

16. (This was emphasized in every example we did in class): do not follow the style of the text when computing nested integrals. Compute them one at a time, *starting from the inside.*

Bad style:

$$\begin{aligned}\int_0^1\int_x^1 \sin(y^2)\,dy\,dx &= \iint_D \sin(y^2)\,dA \\ &= \int_0^1\int_0^y \sin(y^2)\,dx\,dy \\ &= \int_0^1 \Big[x\sin(y^2)\Big]_{x=0}^{x=y}\,dy \\ &= \dots\end{aligned}$$

Stop right here: red flag, siren wails. The use of the difference expression inside the integral means you are on the wrong track, and all set to annoy your instructor.

Correct style:

$$\begin{aligned}\int_0^1\int_x^1 \sin(y^2)\,dy\,dx &= \iint_D \sin(y^2)\,dA \\ &= \int_0^1\int_0^y \sin(y^2)\,dx\,dy \\ &= I\,.\end{aligned}$$

Inner integral:

$$\int_0^y \sin(y^2)\,dx = y\,\sin(y^2)\,.$$

Outer integral:

$$\begin{aligned}I &= \int_0^1 y\,\sin(y^2)\,dy \\ &= \dots\end{aligned}$$

The correct style, in addition to being preferred by your instructor, makes it easier for you to detect and correct your mistakes.

17. As with any language, a solution in mathematics is a sequence of sentences, separated by periods. The sentences are often, but not always, themselves sequences of equalities (or inequalities). They may sometime contain words ("since the function is continuous, the hypothesis of such theorem holds…"). A "solution" which consists of a heap of disconnected expressions will be considered nonsensical, and may not get any credit.