Notes and comments on section 12.6

This innocent-looking section actually contains a lot of fundamental ideas. Take enough time to study it.

1. Geometric sets and cartesian equations. A *cartesian* equation is an equation involving x, y, z and no other variables, unlike *parametric* equations, which involve auxiliary variables. Ex: in the plane, the equation $x = y$ and the parametric equations

$$x = t,\ y = t,\ -\infty < t < \infty$$

describe the same set, the first bisector.

In space, surfaces correspond to a single equation (so $x = 0$ is the yz-plane) and curves, being the intersection of two surfaces, require two equations. The equations (there are two) of the x-axis are $y = 0, z = 0$. Indeed, the x-axis is the intersection of the xz plane (y = 0) and the xy plane (z = 0). As you see, the more the equations, the smaller the set.

2. Symmetries. In the plane xy, symmetries are about lines or points. The curve $x - y^2 = 0$ is symmetric about the x-axis, since replacing y by -y in the equation does not change its validity. Eg., the points $(4, 2)$ and $(4, -2)$ are symmetric (mirror image) about the x-axis.

In space, symmetries can be about planes, lines or points. The cylindre $z = x^2$ (see ex 1 p.834) is symmetric about the yz-plane, since replacing x by -x does not change the validity of the equation. Is it symmetric about the z-axis? Yes, since replacing each (x, y) by $(-x, -y)$ in the equation also leaves it undisturbed. Is it symmetric about the origin? For this, replacing (x, y, z) by $(-x, -y, -z)$ should have no effect. Since this is not the case ($z \not\equiv -z$), the answer is no.

Quiz question: is it symmetric about the y-axis?

Another important kind of symmetry is radial symmetry. It is the symmetry of a surface of revolution about its axis. In example 2(b), the cylindre $y^2 + z^2 = 1$ has radial symmetry about the x-axis, since each trace $x = k$ is a circle (in this case, the circle, which is a curve, has the two equations $x = k,\ y^2 + z^2 = 1$).

The surface obtained by rotating the curve $x = \sqrt{y}$ about the y-axis is such that each trace with the plane $y = k,\ k \geq 0$ is a circle, and the circle has radius $\sqrt{k}$, since this is the distance from $(\sqrt{k}, k, 0)$ to the y-axis. This circle has equations (remember, two):

$$y = k,\ x^2 + z^2 = \sqrt{k}^{\,2} = k\,.$$

So the surface, which is the union of these circles as k varies from $-\infty$ to $+\infty$, has equation $x^2 + z^2 = y$.

Note that we used the identity $\sqrt{k}^{\,2} \equiv k,\ k \geq 0$. Is also $\sqrt{k^2} = k$ for any real number k?

3. Horizontal vs vertical. These words are in no way symmetric. A vertical geometric set (say, a plane) is one which is *parallel* to a certain *line* (the z-axis). A horizontal plane is one which is *perpendicular* to the z-axis, or, equivalently, parallel to the xy-plane. We do not reserve any

words for those planes perpendicular to the x-axis, or parallel to the yz-plane (such a plane would of course be vertical, but not all vertical planes have this property).

4. Standard forms (p.835). Note that these equations are in a new set of variables, which depend on the original ones by some given equations. See example 8 p.838: the equation

$$x^2 + 2z^2 - 6x - y + 10 = 0\,,$$

upon completing the square: $x^2 - 6x = (x-3)^2 - 9$, becomes:

$$(x-3)^2 - 9 + 2z^2 + 10 - y = 0$$

which, setting $X = x - 3$, $Y = y - 1$ and $Z = z$, reads:

$$Y = X^2 + 2Z^2\,.$$

5. Use of traces (ex 3–7) A general trace is of the form $z = k$ (or $x, y = k$), where k is not necessarily 0. In ex 7, the trace is empty for $-2 < k < 2$, showing that the surface is not connected, and has two sheets.

   In ex 5, the value 0 of k ($z = 0$) corresponds to a singularity: $y^2 - x^2 = k$ will be a hyperbola with 2 branches for $k \neq 0$, but degenerates to two crossing lines $y = \pm x$ when $k = 0$. Note that these two lines are the fixed asymptotes of the whole family of hyperbolas: the same for all values of k.

   Moral: to use traces, you must consider all possible values of the level k, not only $k = 0$.

6. Table 1. Parsing the terminology. An *elliptic paraboloid* is a surface for which traces perpendicular to *one direction* (in this case, the z axis: such traces are in a plane $z = k$) are ellipses, and those traces perpendicular to *each of two directions* (resp. $y = k$, $x = k$) are parabolas. So even if one switches the roles of x, y, z, an elliptic paraboloid is never a “parabolic ellipsoid”. By this convention, hyperboloids of both kinds are elliptic hyperboloids. In this table, the use of “vertical” and “horizontal” is simply due to the fact that in standard form, except for the ellipsoid, z plays a different role than x or y.