Notes and comments on section 14.1

Keywords of this section: function, domain, range, graph, level set.

1. Domain vs range: if f is a function (synonyms: map, mapping) from part of the plane to the real line, its domain, set of *points (x,y)* where it is defined, is a region in the plane R^2 (possibly all the plane, as in ex 5,8,10,12). The range is the set of all possible values taken by the function. Finding the domain of a function is always required; finding the range is not always necessary, unless the problem so specifies.

   To summarise: for a numerical function of two variables, the domain is a region of the plane, the range, being a set of values, a subset of the real line.

2. Finding the domain: this is best done graphically. Refer to ex1(a) p.888. The domain is the *intersection* of 2 regions: where $x + y + 1 \geq 0$, and where $x \neq 1$. In part (b), the domain is the single region of the plane where $y^2 - x > 0$. Plotting a single inequality in the plane is done graphically, as we show next.

3. Take the inequality $y^2 - x > 0$. This is a region of the plane which has *boundary* the curve $y^2 - x = 0$. See figure 3 p.889. On which part of the boundary does the region lie? This is found by testing the inequality $y^2 - x > 0$ at *two points, on each side of the curve* $y^2 - x = 0$. $(0,0)$ is a bad choice, since it falls on the curve. $(1,0)$ and $(-1,0)$ is a suitable pair, since they fall on each side of the parabola. Now $0^2 - 1 \leq 0$ (the exclusion of $>$ is $\leq$), and $0^2 - (-1) > 0$, which tells us that the region in question is the one containing $(-1,0)$, which is the "outside" of the parabola. It does not include the curve itself, since $>$ is a strict inequality. Ex 4 is solved similarly.

   Note that we needed to test the inequality on *both sides* of the curve. An example to illustrate why a single side is not enough: assume the inequality is $x^2 > 0$. Then the region has boundary $x^2 = 0$, which is the y-axis. But here, testing on the points $(-1,0)$ and $(1,0)$ shows that the region contains both the left half-plane $x < 0$ and the right half-plane $x > 0$.

   Never try to resolve inequalities using Marvel Comics algebra:

   $$\text{“}x < y^2 \Leftrightarrow \sqrt{x} < \pm y\text{”}$$

   which, if you think about it for a moment, makes no sense at all (what does "$< \pm 2$" mean? and why assume that x has a square root?)

4. Graphs. $z = f(x,y)$, being a single cartesian equation, is in general the equation of a surface, see notes on §12.6. In order to be the graph of a function $z = f(x,y)$, a surface has to pass the vertical line test: each vertical line intersects it at one point at most. Similarly, in order to represent a function $x = f(y,z)$, the surface must pass the "lines parallel to the x-axis" test.

   Quiz question: is the sphere $x^2 + y^2 + z^2 = 9$ the graph of any function $z = f(x,y)$ (or $x = f(y,z)$, or $y = f(x,z)$)?

5. Level curves. Level curves of a function $z = f(x,y)$ are curves in the *xy*-plane. Level curves of a function $y = f(x,z)$ are curves in the *xz*-plane, etc… Level curves, if you can find them, are

the best way to find the range of a function (which, as noted above, is in general more difficult to find than the domain). Indeed:

> *The range of a function $z = f(x, y)$ is the set of all level values k such that the corresponding level curve $f(x, y) = k$ is not empty.*

Go back to ex4 to illustrate: the level curves of $g(x, y) = \sqrt{9 - x^2 - y^2}$ are curves

$$\sqrt{9 - x^2 - y^2} = k \qquad (1).$$

If $k < 0$, this set is empty (why?). If $k \geq 0$, then (1) is equivalent to

$$9 - x^2 - y^2 = k^2, \;\; k \geq 0$$

which itself reads

$$x^2 + y^2 = 9 - k^2, \;\; k \geq 0$$

and, when not empty, this is a family of circles. The largest one has radius 3 (for $k = 0$) and as k increases, the smallest one has radius 0 (for $k = 3$). Since the only non-empty level sets correspond to $0 \leq k \leq 3$, this tells us that the range is the interval $[0, 3]$.