## Sections 14.3, 14.4

**Section 14.3.** There are not many difficulties with this section. Once the definition is understood, it is a matter of practicing the mechanical computing skills.

Refer to the definition 4 on p.914, and you see that partial derivatives are simply derivatives along paths as we have seen in §14.2, where the paths are lines, parallel either to the $x$-axis or the $y$-axis. So $f_x(a, b)$ is the derivative of $f$ along the path parametrised by

$$x = t, \; y = b$$

which, if we set $g(t) = f(t, b)$, is the same as $g'(a)$. This is what Equation 1 p.913 states. Similarly,

$$f_y(a, b) = h'(b),$$

where $h(t) = f(a, t)$. Graphical illustrations are on p.915.

Note that the notation is complete without being redundant: $f_x(a, b)$ means the derivative of $f$ with respect to $x$ at $x = a$, holding $y$ fixed to the value $b$.

Another notation for partial derivative with respect to $x$ at $(a, b)$ is $\frac{\partial f}{\partial x}(a, b)$. This is not the same as $\frac{df}{dx}(a, b)$. The latter notation is only used when $f$ is a function of a single variable. We will elaborate on this later, when we study the chain rule, and composition of functions. See also note 6 of §14.4 below.

**Section 14.4.** This section is at the heart of Calculus. Whereas the goal of Algebra is to find exact solutions to equations or systems of equations, the goal of Calculus is to appoximate, and find lower or upper bounds. When the function to approximate depends on two variables, the approximating object is the tangent plane.

1. In Calculus I, you have seen the role of the tangent line to the graph of $f$ at $x = a$: this line exists only if $f$ has a derivative at $x = a$, and then its slope is $f'(a)$. The line is the graph of a certain function $L(x)$, which is called the "linear approximation of $f$ at $a$". Change $f$ or $a$, and you get a different linear approximation. The usefulness of $L$ is the simplicity of its equation:

$$L(x) = f(a) + f'(a)(x - a)$$

   which is easy to compute, once you know the fixed values $f(a)$ and $f'(a)$. (Newton, Hooke and Leibniz did not have pocket calculators). The closer $x$ is to $a$, the better the approximation of $f(x)$ by $L(x)$.

Approximating a function of two variables uses the same idea. The difference is that now the graph of $z = f(x, y)$ is a surface, so that the tangent object is a plane instead of a line. For the rest of these comments, let us list the following points:

2. Replacing $(a, b)$ by $(x_0, y_0)$, Equation (2) p.128 reads:

$$z = L(x, y) = L(x_0, y_0) + c(x - x_0) + d(y - y_0)$$

where $c = f_x(x_0, y_0)$ and $d = f_y(x_0, y_0)$ are *fixed values.* A common mistake is to replace, say, $f_x(x_0, y_0)$ in the formula by $f_x(x, y)$, which now makes $c$ a function of $x, y$, so that (2) is no longer the equation of a plane.

3. When the equation of the tangent plane is asked, you must provide an *equation.* One side of the equation won't do. A common mistake would be, following example 1 p.928, to write the "equation" as

$$4x + 2y - 3$$

which is not the same as

$$z \text{ (or } L(x, y)\text{ )} = 4x + 2y - 2\,.$$

"$4x + 2y - 3$" by itself is not an equation, but an expression.

4. Before deriving equation 1 on p.928, the text refers to the general equation of a plane passing through $(x_0, y_0, z_0)$ and perpendicular to $\langle A, B, C\rangle$:

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0 \qquad (0)$$

This form is more general than

$$z - z_0 = a(x - x_0) + b(y - y_0) \qquad (1)$$

since it allows $C$ to be zero (vertical plane). While (1) works when the surface arises as the graph of a function $z = z(x, y)$, the symmetric form (0) works even if the surface is not the graph of a function (think of the sphere). We will come back to this point in §14.6.

5. Differentials. How to read equation 9 p.932:

$$dy = f'(x)\, dx$$

This says that if $dx$ is some arbitrary increment away from $a$ (so $dx = x - a$), then *on the tangent line to the graph of* $f$ *at* $(a, f(a))$, $dy = f'(a)\, dx$, where $dy = y - f(a)$. This is indeed the case, since the equation of the tangent line is

$$y - f(a) = f'(a)(x - a)\,.$$

In the language of signal processing, for the electrical engineering majors among you (or statics, for the mechanical engineers), $dy$ is the response to the stimulus $dx$, *if* $f$ *were replaced by its linearisation* $L$. The actual response $\Delta y = f(a + dx) - f(a)$ will in general be different from $dy$, unless $f = L$. But for small $dx$, $\Delta y$ is very close to $dy$. See fig 6.

In the same way then, if $f(x, y)$ is a function of two variables, and $L(x, y)$ its linear approximation (also called linearisation, or tangent map) at $(a, b)$, then $dz = L(a + dx, b + dy) - L(a, b)$ is expressed in terms of the independent increments $dx$, $dy$ by eq 10 p.932. See fig 7 on the same page. $dz$ will be a good approximation to $\Delta z$ if $dx$, $dy$ are small.

6. Use of $d$ and $\partial$ and consistent notation. Consider again the equations 9 and 10 p.932:

$$dz = f'(a)\, dx \qquad (9)$$

$$dz = f_x(a, b)\, dx + f_y(a, b)\, dy \qquad (10)$$

where in (9), we call $z$ the dependent variable for better analogy with (10) (so in (9), $z = f(x)$).

(a) In the one-variable case, along the tangent line, the ratio $dz/dx$ is constant, equal to $f'(a)$. Along the graph of $f$, $\Delta z/\Delta x = \Delta z/dx$ has no reason to be constant, but if $f'(a)$ exists, then the limit of $\Delta z/dx$ as $dx \to 0$ is $f'(a)$. This justifies the notation

$$f'(a) = \frac{df}{dx}(a)\,.$$

(b) In the two-variable case, equation (10) holds exactly on the tangent plane, which is the graph of $L$. In what sense does the ratio $dz/dx$ or $dz/dy$ have a limit as $dx$ (resp $dy$) $\to 0$? Unfortunately, the answer is beyond our scope, but note that if we try to divide eq (10) by $dx$:

$$\frac{dz}{dx} = f_x(a, b)\,\frac{dx}{dx} + f_y(a, b)\,\frac{dy}{dx}$$

the second term on the right is ambiguous as both $dx, dy \to 0$. The best we can do is to keep $dy = 0$ as $dx \to 0$ (or vice-versa), but then this restriction is reflected in denoting the limit not by $\frac{dz}{dx}(a, b)$, but: $\frac{\partial z}{\partial x}(a, b)$.

7. Scalar vs differential relations. Is $f$ is a function (also called scalar expression), $df$ is called a differential expression. If you think of "d" as an operation, this operation obeys the rules of differentiation:

$$d(u + v) = du + dv\,.$$

$$d(uv) = v\, du + u\, dv\,.$$

$$d(u/v) = (v\, du - u\, dv)/v^2\,.$$

Differentials represent increments, so that scalars and differentials are distinct objects. Don't mix the two:

$$V = \pi r^2 h/3$$

is a scalar relation,

$$dV = \frac{2\pi r h}{3} dr + \frac{\pi r^2}{3} dh$$

is a differential relation.

$$PV = 8.31T$$

is a scalar relation,

$$VdP + PdV = 8.31dT$$

is a differential relation.

$$VdP + PdV = 8.31T$$

is a crude mistake: “differential = scalar”.

However, both scalars and differentials take, upon substitution, real values. So $PV$ is a product, say, of kilopascals and cubic meters, and $d(PV)$ is an increment, also expressed in kilopascals times cubic meters.