Section 14.5

1. Formulæ. The chain rule allows to express the rate of change of a function u of several variables, say $x_1, \ldots x_n$ in terms of original variables $t_1, \ldots t_m$ if the x variables are known functions of the t variables. We are assuming here that all partial derivatives exist. As in the text, we consider a special case and a general case, but our special case is slightly different from the one in the text: instead of two intermediate variables, assume there are n.

   Case 1: one initial (independent) variable t, n intermediate variables $x_1, x_2, \ldots x_n$. Then if u is function of $x_1, \ldots x_n$, we have the equivalent of equation (2) p.938:

$$\frac{du}{dt} = \frac{\partial u}{\partial x_1}\frac{dx_1}{dt} + \ldots + \frac{\partial u}{\partial x_n}\frac{dx_n}{dt}$$

   (as many terms as there are intermediate variables). Note the use of symbols d and ∂: the rates of change $\partial u/\partial x_i$ are partial derivatives, but the rates of change dx_i/dt and du/dt are total derivatives, since each x_i is a function of t only, and via the intermediate variables, u is considered function of t only.

   Case 2: this corresponds to the general formula, equation (4) p.940: there are now m independent variables $t_1, \ldots t_m$. All rates of change are partial derivatives:

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \ldots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_i}.$$

   There are still n terms, but there are now m equations, one for each independent variable t_i.

2. Implicit differentiation. Formulæ 6 and 7 are superfluous to memorize; you will end up using them "implicitly" by writing the equation of the curve (or surface) and differentiating with respect to the desired independent variable. The principle at play here is that *on the curve, one variable becomes (locally) a function of the other.* Or: *a curve has one degree of freedom*. We illustrate on the same examples as in the text:

   (a) Example 8. Find $y' = dy/dx$ on the curve

$$x^3 + y^3 = 6xy.$$

   Note that without the proviso "on the curve", y' would have no meaning, since without this constraint, x, y are independent variables. Differentiate the equation (which you can also write $x^3 + y^3 - 6xy = 0$) with respect to x. The power rule and the chain rule tell us that $d(y^3)/dx = 3y^2 dy/dx$. We obtain:

$$3x^2 + 3y^2y' - 6y - 6xy' = 0$$

   or:

$$(3y^2 - 6x)y' = 6y - 3x^2$$

and the last step is to divide both sides by the factor multiplying y' . As you see, I did not need to memorize the formula.

(b) Example 9 is solved similarly. Let us illustrate how to find $\partial z/\partial y$ on the surface

$$x^3 + y^3 + z^3 + 6xyz = 1 .$$

The principle at play is that *on the surface*, z becomes “locally” a function of x, y, which itself means:

> given a point x_0, y_0, z_0 on the surface, if x, y are allowed to vary near x_0, y_0, and if z is constrained to stay near z_0, then to each (x, y) there corresponds only one value of z .

So *a surface has two degrees of freedom* . The constraint “z near z_0” is important, as the surface may fail the vertical line test. Think of the sphere.

Now differentiate both sides of the equation with respect to y, and use the fact that x does not depend on y, but z depends on both x and y:

$$0 + 3y^2 + 3z^2 z_y + 6xz + 6xyz_y = 0$$

then grouping terms:

$$(3z^2 + 6xy)z_y + 3y^2 + 6xz = 0$$

and here again, the last step is simple algebra, and I did not have to memorize formula (7).

3. The following example will illustrate a little further the process of implicit differentiation.

A point is moving on the curve of intersection of the surface

$$x^2 + xy + y^2 - z^2 = 0 \qquad (1)$$

and the plane

$$x - y + 2 = 0 \qquad (2) .$$

When x is 3 and is increasing 2 units per second, and z is negative, find (a) the rate at which y is changing, (b) the rate at which z is changing and (c) the speed with which the point is moving.

(a) Let t be time, in seconds. Denoting dy/dt by y', and differentiating (2) with respect to time, we have

$$y' = x' = 2 \text{ units per second.}$$

(b) First we find the y,z coordinates of the point. Equation (2) tells us that $y = 5$, and upon substitution in eq (1), $z^2 = 3^2 + 3(5) + 5^2 = 49$. Since z is negative, $z = -7$. Now differentiate equation (1) with respect to time:

$$2xx' + x'y + xy' + 2yy' - 2zz' = 0$$

and replacing x, x', y, y', z by their values:

$$2(3)2 + 2(5) + 3(2) + 2(5)(2) - 2(-7)z' = 0\,.$$

Solving for z' gives $z' = -24/7$ units per second.

(c) Speed is the modulus (length) of velocity:

$$s = |\langle x', y', z' \rangle| = \sqrt{x'^2 + y'^2 + z'^2}$$

and at that moment, $s = \sqrt{4 + 4 + (-24/7)^2} \simeq 4.44$ units per second.