Section 14.6

This section is not as heavy in terms of new ideas as the previous ones. The concept of directional derivative generalizes that of partial derivative, where the direction can now be other than along the directions of the axes.

1. Directional derivative. See definition 2 p. 979: the rate of change of the function f at the point (x_0, y_0) in the direction of the unit vector $\langle a, b\rangle$ is also the derivative at $h = 0$ of

   $$g(h) = f((x_0, y_0) + h\langle a, b\rangle)\,.$$

   This is simply the definition of derivative; see Notes on §14.2 and the comments on "hidden derivative". This can be thought of as rate of change along a straight path. Replacing the unit vector by $\langle 1, 0\rangle$ or $\langle 0, 1\rangle$ recovers the partial derivatives of §14.3.

2. Gradient. Since there are infinitely many directions in the plane, we expect the directional derivative to depend on many different things. The formula 9 on p. 950 brings some reassurance:

   $$D_u f(x, y) = \nabla f(x, y) \cdot u$$

   tells us that the effect can be split in two parts: the direction vector u (if we write $u = \langle \cos\theta, \sin\theta\rangle$, this is a one-parameter family, the parameter being the angle θ), and the gradient vector ∇f . But the gradient vector itself depends on only two things, namely, the partial derivatives of f at (x, y) . So in total, the directional derivative has 3 degrees of freedom (meaning independent parameters): 2 from the partial derivatives, and one from the direction in which the point varies.

   This can be extended to functions of more than 2 variables: see formula 14 p. 951.

3. Gradient vector and tangent plane. The gradient vector allows us to write the equation of the tangent plane of a surface which is not necessarily a graph (i.e., fails the vertical line test: think of the sphere). The formula 19 p. 954 is symmetric in all the variables, whereas formula 2 p. 928 distinguishes the dependent variable z from the independent variables x, y . Note also that what plays the role of the function in the two cases is not the same: in the case of the graph,

   $$z = f(x, y) \qquad (1)$$

   is a representation where f is a function of two variables. In the case of the arbitrary surface (p. 954), F such that the equation of the surface is

   $$F(x, y, z) = k \qquad (2)$$

   is a function of 3 variables. Equation (1) is said to be *explicit*, equation (2) is an *implicit* representation of the surface. Note that while form (2) cannot be reduced to form (1) (think of the sphere), form (1) is always a particular case of form (2):

$$z = f(x, y) \Leftrightarrow z - f(x, y) = 0$$

and this is of the form $F(x, y, z) = 0$ where $F(x, y, z) = z - f(x, y)$. Another way to state this: any surface which is the graph of a function of two variables, is also the level set of a function of three variables.

4. Normal line. The normal line at a point of the surface, being perpendicular to the tangent plane, is in the same direction as the gradient. The text shows how to obtain the cartesian equations of the line, also called the symmetric equations (there are two: curves in R^3 have 2 equations). But for most purposes, the parametric equation of the line is more useful: if the surface has implicit equation $F(x, y, z) = c$, then at the point $P = (x_0, y_0, z_0)$, the vector equation of the normal line is:

$$(x, y, z) = (x_0, y_0, z_0) + t\nabla F(x_0, y_0, z_0)$$

where (x,y,z) is the point on the line corresponding to the parameter t. Illustrate on the example 8 p. 955, where $P = (-2, 1, -3)$ and $\nabla F(-2, 1, -3) = \langle -1, 2, -2/3 \rangle$:

$$(x, y, z) = (-2, 1, -3) + t\,\langle -1, 2, -2/3 \rangle$$

Note that the symmetric equations

$$\frac{x+2}{-1} = \frac{y-1}{2} = \frac{z+3}{-2/3}$$

amount to eliminating t in the parametric equations: the three equal quantities are all equal to t.

Let us illustrate the usefulness of the parametric form of the normal line on two exercises from this section. In one exercise, we must find where the normal line intersects a surface. In the other, we must check that a certain point belongs to the normal line.

59. Where does the normal line to the paraboloid $z = x^2 + y^2$ at the point $P = (1, 1, 2)$ intersect the paraboloid a second time? (Plot the figure).

Step 1: find the gradient of F. The paraboloid has implicit equation

$$z - x^2 - y^2 = 0 \qquad (3).$$

Then $\nabla F(x, y, z) = \langle -2x, -2y, 1 \rangle$.

Step 2: find the equation of the normal at (1, 1, 2):

$$(x, y, z) = (1, 1, 2) + t\,\langle -2, -2, 1 \rangle \qquad (4)$$

since $\nabla F(1, 1, 2)$ is the vector $\langle -2, -2, 1 \rangle$.

Step 3: in how many points does the normal intersect the surface? Each such point corresponds to a value of t such that $(1 - 2t, 1 - 2t, 2 + t)$ (obtained from equation (4)) belongs to the surface, or, using equation (3):

$$2 + t - (1 - 2t)^2 - (1 - 2t)^2 = 0$$

and this is a quadratic equation in t. Upon simplification:

$$8t^2 - 9t = 0$$

which has two roots. One is $t = 0$ which gives the point P, and the other is $t = 9/8$. Plug in this value in (3) to get the coordinates of the other point of intersection.

As an exercise, solve pb 60 on the same page.

58. Show that every normal line to the sphere

$$x^2 + y^2 + z^2 = r^2 \qquad (5)$$

passes through the center of the sphere.

The centre of the sphere is the origin. How many variables do we need in the solution? Each point has 3 coordinates. We will need to describe an arbitrary point on the sphere where to take the normal, and an arbitrary point on the normal itself, to write the equation of the normal. Let us call the point on the sphere $P = (x_0, y_0, z_0)$, and (x, y, z) will be an arbitrary point on the normal line. Now we can proceed:

Step 1: find the gradient at the point P: here $F(x, y, z) = x^2 + y^2 + z^2$, and

$$\nabla F(x_0, y_0, z_0) = \langle 2x_0, 2y_0, 2z_0 \rangle$$

Step 2: so the parametric equations of the normal line at P are

$$(x, y, z) = (x_0, y_0, z_0) + t \langle 2x_0, 2y_0, 2z_0 \rangle .$$

Step 3: the line will contain the origin if for some value of t, there holds $(x, y, z) = (0, 0, 0)$. So we get the system of 3 equations in the single variable t:

$$x_0 + 2tx_0 = 0, \; y_0 + 2ty_0 = 0, \; z_0 + 2tz_0 = 0 .$$

Check that $t = -1/2$ solves the system. So indeed, all normal lines to the sphere pass through the center. As an exercise, solve the same problem, but using the symmetric equations instead. You will see that the way we showed is easier.

5. Geometric notation. Parentheses are used for points, angular brackets $\langle , \rangle$ are used for vectors. Points in R^3 have 3 coordinates, vectors have 3 *components*. Gradient vectors are vectors. Adding a vector to a point results in another point, as shown in the parametric representation. So:

point + vector = point

which also means:

point - point = vector.

Vectors can be added to vectors, to give vectors. Points cannot be added to points.