**MATE 3063 problems**

1. Using the graphical method shown in class, find the following sets for the function $f(x) = x^2 - 2$:

$$\text{i)}\ f((-1,3)) \qquad \text{ii)}\ f^{-1}([-4,1])$$

2. * Same questions (i) and (ii), for the function $f(x) = \log|x+1|$.

3. Draw contour maps (families of level sets) for the following functions. Mark the level value corresponding to each curve.

$$\text{i)}\ f(x,y) = xy \qquad \text{ii)}\ f(x,y) = x^2 - y^2$$

$$\text{iii)}\ f(x,y) = \sqrt{y-x} \qquad \text{iv)}\ f(x,y) = x - y^2$$

4. Indicate for each of (i)–(iv) of the previous problem, whether there are values of the level for which the corrresponding set is degenerate - that is, does not look the same as other curves of the same family. We saw an example of this in one of the problems of quiz 0.

5. Using the method shown in class, find $f(A)$, where $A$ is the unit square centered at the origin, for each of:

$$\text{i)}\ f(x,y) = x - y \qquad \text{ii)}\ f(x,y) = x + 2y$$

6. * Find the range of the function $f(x,y) = \sqrt{9 - x^2 - y^2}$. (We found the domain of $f$; the range is the image of the domain. Find this image using the method shown in class).

7. * Find the domain of the function $f(x, y) = \log x + \log y$ and shade it in the xy-plane. Showing complete steps as in the exercises above, find the range of this function. Hint: $\log x + \log y = \log xy$.

8. * We showed in class that, if $B$ is the unit disc centred at the origin, then $[-\sqrt{2}, \sqrt{2}] \subset f(B)$, where $f(x,y) = x + y$. The goal of this problem is to show the reverse inclusion. The unit disc is parametrised by:

$$(x,y) = (r\cos\theta, r\sin\theta), \quad 0 \le r \le 1, -\infty < \theta < \infty \quad (1).$$

(Taking $0 \le \theta < 2\pi$ is another parametrisation, which is one-to-one, but does not simplify the steps).

Using the parametrisation (1), show that $x + y$ takes values only in the interval $[-\sqrt{2}, \sqrt{2}]$. Hint: since $x + y = r(\cos\theta + \sin\theta)$, the maximum and minimum of $x + y$ happen when $r = 1$. So all you have to do is to study the variation of the function $\cos\theta + \sin\theta$, $-\infty < \theta < \infty$.

Further hint (added March 20): finding extrema of functions of one variable is a skill you learnt in Calculus $n$, $n < 3$. For the function $g(\theta) = \sin\theta + \cos\theta$, you have also learnt in Precalculus that $g(\theta)$ can be put in the form $A\sin(\theta - \phi)$, where you can determine $A$ and $\phi$.

Use this method if you prefer.

9. Sketch the graphs of the functions

$$\text{i) } f(x,y) = \sqrt{x^2 + y^2} \qquad \text{ii) } f(x,y) = \frac{1}{\sqrt{x^2 + y^2}}$$

In general, if $g$ is a function of one variable, how is the graph of $f(x, y) = g(\sqrt{x^2 + y^2})$ obtained from the graph of $g$?

10. Find the radius of a square centered at $(0.5, 2)$ such that its image under the given function is contained in the given interval:

a) Function $f(x,y) = 2x$, interval $[0.7, 1.3]$

b) Function $f(x,y) = 2y - x$, interval $[3.4, 3.6]$.

11. * Find the radius of a square centered at $(0.8, \pi/2)$ such that its image under the given function is contained in the given interval:

a) Function $f(x,y) = \sin y$, interval $[0.7, 1.3]$

b) Function $g(x,y) = \cos(2y) + x$, interval $[-0.22, -0.18]$.

12. Find the limit if it exists, or show that it does not exist:

$$\text{i) } \lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2 + y^2}} \qquad \text{ii) } \lim_{(x,y)\to(0,0)} \frac{x^2 \sin^2 y}{x^2 + 2y^2}$$

13. Find the limit if it exists, or show that it does not exist:

$$\text{i) } \lim_{(x,y)\to(0,0)} \frac{x^2}{x^2 + y^2} \qquad \text{ii) } \lim_{(x,y)\to(0,0)} \frac{x^2 + y^2}{\sqrt{x^2 + y^2 + 1} - 1}.$$

14. We showed in class a graphical illustration of the definition of continuity of a function of one variable. Use this to illustrate:

a) Continuity of a function of your choice at $x = 0$

b) Lack of continuity of the function $y = \sin(1/x)$ at $x = 0$. Use colours.

15. Find the first partial derivatives:

$$\text{i) } z = \log(x + \sqrt{x^2 + y^2}) \qquad \text{ii) } u = xe^{-t}\sin(y) \qquad \text{iii) } u = x^{y/z}$$

16. Find the first partial derivatives:

$$\text{i) } f(t,x) = e^{\sin(t/x)} \qquad \text{ii) } f(x,y) = \int_x^y \cos(t^2)\, dt$$

$$\text{iii) } u = \sin(x_1 + 2x_2 + \ldots + nx_n) \qquad \text{iv) } z = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}$$

In (iii) and (iv), there are n variables named $x_1, x_2, \ldots x_n$.

17. * Use the definition of partial derivative as limit of difference quotient to find the partial derivatives with respect to $x$ and $y$ of $f(x, y) = \sqrt{2x - y}$.

18. Use implicit diferentiation to find $x_z$ and $x_y$:

$$\text{i) } xyz = \cos(x + y + z) \qquad \text{ii) } xy^2z^3 + x^3y^2z = x + y + z.$$

In each case, what are the points on the surface where it fails the x-axis test? (The surface passes the x-axis test at $x_0, y_0, z_0$ if, for a square of small enough radius centered at $(y_0, z_0)$ in the yz-plane, and any $(y_1, z_1)$ belonging to that square, the line $y = y_1, z = z_1$ intersects the surface at a single point. The surface $x = y^2 - z$ passes everywhere the x-axis test, and so does the surface given by any equation of the type $x = f(y, z)$.)

19. Find $z_x$ and $z_y$:

$$\text{i) } z = f(xy) \qquad \text{ii) } z = f(y/x)$$

20. Find the second partial derivatives:

$$\text{i) } f(x, y) = \log(x - 2y) \qquad \text{ii) } u = e^{-s}\cos(t)$$

21. (Corrected). If $f$, $g$ are twice differentiable functions of a single variable, and $c$ some constant, show that the function $u(t, x) = f(x - ct) + f(x + ct)$ solves the “wave equation”

$$u_{tt} - c^2u_{xx} = 0$$

22. You are told that a function $f$ has continuous second-order partial derivatives, and that $f_x(x, y) = x + 4y$, $f_y(x, y) = 3x + y^2$. Should you believe it?

23. * If $f(x, y) = x(x^2 + y^2)^{-3/2}\exp(\sin(x^2y))$, find $f_x(1, 0)$. Hint: at the given point, it is easier to use directly the definition of the partial derivative, as derivative of a certain function of a single variable.

24. Find the linearisation $L(x, y)$ of the function at the given point:

$$\text{i) } x\sqrt{2y}\text{, at } (1, 2) \qquad \text{ii) } \sin(2x + 3y) \text{ at } (-3,2).$$

25. * Using linear approximation (linearisation), estimate

$$\text{i) } \sqrt{4 - 1.1^2 - 2(0.92)^2} \qquad \text{ii) } \exp(x^2 - y^2)\text{, where } (x, y) = (-1.02, 1.01).$$

26. The radius of a right circular cone is increasing at a rate of 1.8 in/s while its height is decreasing at a a rate of 2.5 in/s. At what rate is the volume of the cone changing, when the radius is 120 in. and the height is 140 in.?

27. Exercise 39 of § 14.5 of Stewart. (You must refer to such exercises from the text by their number in this list, not in the book. This is number 27, not 39).

28. Exercise 45 of § 14.5 of Stewart.

29. * a)–b): exercises 47, 48 of § 14.5 of Stewart.

30. Find the maximum rate of change at the given point, and the direction where it occurs

i) $\cos(xy)$ at $(0, 1)$ ii) $\log(x^2 + y^2 + z^2)$ at $(1, 2, -1)$

31. * You are told that the function

$$f(x, y) = \frac{1}{x^2 + y^2}$$

has a rate of change exceeding four, in a certain direction, and at a point outside the unit disc. Should you believe it? The unit disc is given by the inequality $x^2 + y^2 \le 1$.

32. * a)–b): exercises 36, 38 of § 14.6 of the text. Use colours.

33. Find equations of the tangent plane and the normal line to the given surface at the given point. For the normal line, use parametric form.

i) $z + 1 = xe^y \cos(z)$, $(1, 0, 0)$ ii) $xe^{yz} = 1$, $(1, 0, 5)$

34. * Show that every plane that is tangent to the cone $y^2 + z^2 = x^2$ passes through the origin.

35. Exercise 63 of § 14.6 of the text.

36. Exercise 6 of § 15.1 of the text.

37. * Exercise 8 of § 15.1 of the text.

38. * Exercise 10 of § 15.1 of the text.

39. a)–b): exercises 13,14 of § 15.1 of the text. For (b), also find the volume, by analogy with the example done in class.

40. * Let R be the quadrangle with vertices $A = (0, 1)$, $B = (1, 0)$, $C = (2, 0.7)$, $D = (1, 1)$ (draw it). If $f(x, y) = -1$ on the triangle $ABD$ and $f(x, y) = 2$ on the triangle $BCD$, compute $\iint_R f(x, y)\, dA$.

41. * Let R be the square of vertices (0, 1), (1, 0), (2, 1), (1, 2). Show that

$$0 \le \iint_R \sin(\sqrt{x} + \sqrt{y})\, dA \le 2.$$

Hint: can $\sin(\sqrt{x} + \sqrt{y})$ take negative values over $R$?

42. * Let $H$ be the half-disc intersection of the region $y \ge 0$ and the disc centered at (0, 0) of radius 0.7. Show that

$$-\frac{\pi(0.7)^2}{2\sqrt{2}} \le \iint_H \sin(x + \sqrt{y})\, dA \le \frac{\pi(0.7)^2}{2}.$$

43. a)–c): exercises 7, 10, 13 of § 15.2.

44. a)–b): exercises 19, 22 of § 15.2.

45. * Exercise 27 of § 15.2. First, you must:

    a) Draw the intersection of the paraboloid with each of the four lateral planes: four plots, each in a different plane.

    b) Using the algebra of inequalities, show why the parabolic surface stays above the xy plane over the rectangle *R*.

46. Exercise 28 of § 15.2. Same plan as for the previous exercise.

In all the problems of § 15.3, sketch the region of integration when possible.

47. a)–b): exercises 2, 5 of § 15.3.

48. a)–c): exercises 8, 9, 13 of § 15.3.

49. a)–b): exercises 23, 28 of § 15.3.

50. Exercise 32 of § 15.3. Plot (shade) the projection of the solid in question on each of the coördinate planes, and show, using inequalities, that on the one side of the cylinder, one of the two given planes sits above the other, making the subtraction meaningful.

51. a)–b): exercises 39, 40 of § 15.3.

52. a)–b): exercises 46, 50 of § 15.3.

53. * a)–b): exercises 53, 54 of § 15.3. “Property 11” is a particular case (and the main one we have used) of monotonicity. See problems 40, 41, 42.

54. a)–b): exercises 11, 12 of § 15.4.

55. Exercise 18, § 15.4.

56. Exercise 22, § 15.4.

57. Exercise 33, § 15.4.

58.

    a) Find the surface area of the plane $2x + 5y + z = 10$ that lies inside the cylinder $x^2 + y^2 = 9$.

    b) Find the surface area of the helicoid with parametric equations

$$(x, y, z) = (u\cos v, u\sin v, v),\ \ 0 \le u \le 1,\ \ 0 \le v \le \pi.$$

59. (F11) One parametric representation of the elliptic region *R* in the xy-plane given by $x^2 + 2y^2 \le 9$ is in the form

$$x = ar\cos\theta, \quad y = br\sin\theta.$$

Find an explicit such parametrisation (i.e., find $a$, $b$), draw both $R$ and the corresponding region in the polar plane, and use the parametrisation to find the surface area of $R$.

60. a)–b): exercises 4, 6 § 15.6.

61. Exercise 13 § 15.6. Shade the projection of the solid on each of the coördinate planes.

62. Exercise 18 § 15.6. Shade the projection of the solid on each of the coördinate planes.

63. Exercise 19 § 15.6. Plot the tetrahedron.

64. a)–b): exercises 27, 28 § 15.6.

65. Exercise 33, § 15.6.

66. Exercise 34, § 15.6.

67. a)–b): exercises 6, 14 § 16.7.

68. a)–b): exercises 8, 10 § 16.7.

69. a)–c); exercises 20, 26, 27 § 16.7.

70. Exercise 3, 10 § 16.9.

71. a)–c): exercises 13, 14 § 16.9.