Some common corrections

1. The following symbols have different meanings:

   (a) The symbol “ $=$ ” means *equals,* and nothing else. It connects expressions, sets, quantities, or any objects of the same nature, but not statements. It is not an all-purpose connector, nor does it open a paragraph.

   (b) The symbol “ $\Rightarrow$ ” means *implies.* It connects sentences or logical statements; it does not connect values or expressions. For instance:

   $$\sin^2 x + \cos^2 x = 1 \;\Rightarrow\; 2\cos^2 x + 3\sin^2 x = 2 + \sin^2 x$$

   is correct, an equality being a particular kind of statement.

   $$\sin^2 x + \cos^2 x \Rightarrow 1$$

   does not make sense: “$\sin^2 x + \cos^2 x$” and “1” are not statements.

   (c) The symbol “ $\to$ ” means *tends to*, as in “has limit”. It cannot be substituted either to “ $\Rightarrow$ ” or “ $=$ ”.

2. Never write $0/0$, $a/0$, $1/\infty$, a/∞, ∞/∞, $0\times\infty$, $\ln(0)$, 0^0 or any other variant of this style.

3. Much confusion mars the use of the lim symbol. One way to misuse it is to equate a limit (which is a fixed value) with a function, as in

   $$\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^2 + yx^2} = \frac{y}{1+y}$$

   (nonsense: fixed value = function of y). Other times it is used with the wrong clause, such as $\lim_{(x,0)}$, $\lim_{(0,y)}$, or $\lim_{(x,y)\to(x,0)}$ (there is no such notation).
   To keep one variable fixed, or, more generally, to replace a two-variable limit by a single-variable limit, I have shown how to do it using parametrisation (for instance, $y = 0$, $x = t$; more generally, $x = x(t)$, $y = y(t)$). One can solve any problem about limits without using the lim symbol. Follow the style shown in class; in particular, instead of stating “limit of function equals …” state: “this function tends to this value as (x, y) tends to this point”. If the use of mathematical symbols enchants you, use $\to$ to indicate “tends to”.

4. Distinguish between the expression of a function and its value at a particular point. If I found that $g(x) = 1 + x$ and I need its value at $x = 1$, I don’t write

   $$g(x) = 1 + x = 2$$

   since “$g(x) = 1 + x$” is an identity (true for all x) but “$1 + x = 2$” certainly is not, but I write two separate statements:

   $$g(x) = 1 + x$$

$$g(1) = 2\,.$$

This is one more reason to name all functions, even if the data of the problem does not name them. If no name was assigned, it is still possible to distinguish the function and a particular value:

$$(1 + x)|_{x=1} = 2\,.$$

There are of course cases when it is appropriate to write "$1 + x = 2$": for example, as part of the sentence "Solve for x the equation $1 + x = 2$". But this is practically the only exception.

5. Existence of limit. We gave an example (ex 2) of a function which has the same limit along any line going through the origin (there are infinitely many), yet has no limit as $(x, y) \to (0, 0)$. The moral is: finding the same limit along certain paths, or as many paths as you want, does not mean the limit exists. Finding limits along paths (use parametrisation) only serves to show that there is no limit. To show that there is a limit, there are two ways we know: direct use of the definition (two-player game), or squeeze theorem (the use of polar coordinates, when possible, is equivalent to using the squeeze theorem).

6. If $f(x, y)$ is differentiable at (a, b), it is continuous at (a, b): refer to the definition of differentiable, and check that as $\Delta x, \Delta y \to 0, \Delta f \to 0$. The converse, unfortunately, is not true: the function $|x + y|$ is continuous everywhere, but fails to be differentiable at any point of the line $x + y = 0$ of the plane.
   A sufficiency criterion for differentiability is stated in Theorem 8 of §14.4. Make sure you can tell the difference between "sufficient" and "necessary", we will need this when we study local extrema.

7. Square and square root, are they tied at birth? The equations

$$x = y^2 \qquad (1)$$

$$y = \sqrt{x} \qquad (2)$$

   are not equivalent. (2) implies (1), but (1) does not imply (2). Solutions of (2) are solutions of (1), but solutions of (1) are not necessarily solutions of (2). (2) does not let y be negative, but in (1), y is as free to be negative as we are free to breathe.

   Or, think of it geometrically: in the xy-plane, (1) describes a parabola, but (2) describes half a parabola (plot them). They are not the same curve. In space, (1) describes a parabolic cylinder, symmetric about the xz plane, but (2) describes half of that surface, without the symmetry. When you encounter curves or surfaces with equation involving squares, leave them as given and don't spoil them by taking square root.

   Incidentally, do not use the fake identity "$\sqrt{y^2} = y$". The correct identity (which you will seldom need) is $\sqrt{y^2} = |y|$.

8. Square, square root, inequalities. The inequalities

$$a \le b \qquad (1)$$

$$a^2 \le b^2 \qquad (2)$$

are not equivalent in general. The different cases depend on the signs of a, b. So if, say $0 \le a \le b$, then (1) and (2) are equivalent. If $a \le b \le 0$, then $a^2 \ge b^2$. If $a \le 0 \le b$, then a^2 and b^2 can be in either order. To check, plot $y = x^2$, where a, b are values of x.

The region where $f(x, y) \le 0$, where f is some continuous function, will in general be determined by the curve of equation $f(x, y) = 0$. It will be on "one side" of the curve, with the caveat that for some degenerate curves, "one side" may be more than one connected region. The side in question is found by testing the sign of $f(x, y)$ at some points (in the favourable case, only one point) not on the curve $f(x, y) = 0$. Some examples:

(i) $x - y^2 + 1 = 0$ is a parabola. $x - y^2 + 1 \ge 0$ is the inside of that parabola, $x - y^2 + 1 \le 0$, the outside.

(ii) $h(x, y) = x^2 - 2y^2 + 4$. $h(x, y) = 0$ is a hyperbola with two branches. Testing at $(0, 0)$: $h(x, y) \ge 0$ is the connected region between the branches and containing the x-axis, $h(x, y) \le 0$ is the region with two separate parts, above and below each branch.

(iii) $h(x, y) = x^2 - 2y^2$. $h = 0$ is a degenerate hyperbola, with branches the lines $x - \sqrt{2}y = 0$, $x + \sqrt{2}y = 0$, intersecting at the origin. The region $h \le 0$ consists of a cone (two quarter planes) containing the y-axis. Describe the region $h \ge 0$.

(iv) $y^2 = 1$ is a degenerate conic, consisting of the two lines $y = \pm 1$. $y^2 \le 1$ is the strip inside the two lines, $y^2 \ge 1$ the region outside the strip. On no account should you replace $y^2 \le 1$ by "$y \le \pm 1$", which is meaningless (which of $+1$ or -1 is y smaller than?). Note also that "$\sqrt{y^2} \le \sqrt{1}$" reads "$|y| \le 1$", which is no improvement on $y^2 \le 1$.

Note that in all these examples, the region $f(x, y) \le 0$ has the same symmetries as the curve $f(x, y) = 0$. As for the square root function, you see that is is best to leave it at the door.

9. Distinguish between total and partial derivative. If $f(x, y)$ depends on t via $x(t), y(t)$, the rate of change of f with respect to t is denoted df/dt. The rate of change of f wrt x is $\partial f/\partial x$.