MATE 4000 assignment 1

1. Determine which of the following statements are true for all sets A, B, C, and D. If a double implication fails, determine whether one or the other of the possible implications holds. If an equality fails, determine whether one or the other of the possible inclusions holds.

   (a) $A \supset C$ and $B \supset C \Leftrightarrow (A \cup B) \supset C$

   (b) $A \supset C$ or $B \supset C \Leftrightarrow (A \cup B) \supset C$

   (c) $A \supset C$ and $B \supset C \Leftrightarrow (A \cap B) \supset C$

   (d) $A \backslash (A \backslash B) = B$

   (e) $A \cap (B \backslash C) = (A \cap B) \backslash (A \cap C)$

   (f) $A \subset C$ and $B \subset D \Rightarrow (A \times B) \subset (C \times D)$

   (g) Converse of (f)

   (h) Converse of (f), assuming A and B are nonempty

   (i) $A \times (B \backslash C) = (A \times B) \backslash (A \times C)$

   (j) $(A \times B) \backslash (C \times D) = (A \backslash C) \times (B \backslash D)$

2. Let $\mathcal{A}$ be a nonempty collection of sets. Which of the following statements is true:

   (a) $x \in \bigcup_{A \in \mathcal{A}} A \Rightarrow x \in A$ for every $A \in \mathcal{A}$

   (b) $x \in \bigcap_{A \in \mathcal{A}} A \Rightarrow x \in A$ for every $A \in \mathcal{A}$

3. Given sets A, B, and C, express each of the following sets in terms of A, B and C, using the symbols $\cup$, $\cap$ and $\backslash$:

   (a) $D = \{\, x \mid x \in A$ and $(x \in B$ or $x \in C) \,\}$

   (b) $E = \{\, x \mid (x \in A$ and $x \in B)$ or $x \in C \,\}$

   (c) $F = \{\, x \mid x \in A$ and $(x \in B \Rightarrow x \in C) \,\}$

4. Let R be the set of real numbers. For each of the following subsets of $R \times R$, determine whether it is equal to the cartesian product of two subsets of R:

   (a) $\{\, (x, y) \mid 0 < y \leq 1 \,\}$

   (b) $\{\, (x, y) \mid y > x \,\}$

(c) $\{ (x, y) \mid x$ is not an integer and y is an integer $\}$

5. Let $f : A \to B$ and let $A_i \subset A$ and $B_i \subset B$ for $i = 0$ and $i = 1$. Show that f^{-1} preserves inclusion, union, and difference of sets (for instance:

$$f^{-1}(B_1 \backslash B_0) = f^{-1}(B_1) \backslash f^{-1}(B_0)$$

and similarly for other expressions). Also, show that f preserves inclusion and union only:

(a) $A_0 \subset A_1 \Rightarrow f(A_0) \subset f(A_1)$

(b) $f(A_0 \cup A_1) = f(A_0) \cup f(A_1)$

(c) $f(A_0 \cap A_1) \subset f(A_0) \cap f(A_1)$

(d) $f(A_0 \backslash A_1) \supset f(A_0) \backslash f(A_1)$

6. Let $f : A \to B$ and $b : B \to C$.

(a) If $D \subset C$, show that $(gof)^{-1}(D) = f^{-1}(g^{-1}(D))$

(b) If f and g are injective, show that gof is injective.

(c) If gof is injective, what can you say about injectivity of f and g?

(d) If f and g are surjective, show that gof is surjective.

(e) If gof is surjective, what can you say about surjectivity of f and g?

(f) Summarise your answers to (b)–(e) in the form of a theorem.