Tips and common corrections

[1] Some comments about notation.

(a) The symbol " = " means *equals*, and nothing else. Points are lost on tests and quizzes for using it inappropriately.

(b) The symbol " $\Rightarrow$ " means *implies*. It connects sentences or logical statements; it does not connect values or expressions. For instance:

$$\sin^2 x + \cos^2 x = 1 \;\Rightarrow\; 2\cos^2 x + 3\sin^2 x = 2 + \sin^2 x\,.$$

The symbol " $\rightarrow$ " means *tends to*, and we will only use it to denote limits.

(c) The symbol " $\in$ " means *belongs to*. It relates elements to sets. The symbol " $\subset$ " means *is included in*. It allows to compare between sets. The two symbols are not interchangeable.

(d) Use of parentheses.

To multiply x by $-y$, we write $x(-y)$. "$x \cdot -y$" or "$x \times -y$" is incorrect, even if you reduce the size of the minus sign, raise it and stick it very close to y.
Also, $x - y$ and $x + (-y)$ are correct. "$x + -y$" is not, even if you reduce the size of the minus sign, raise it and stick it very close to y.
The derivative sign has precedence over the product. Therefore, $\dfrac{d}{dt} f(t)g(t)$ means one thing (namely, $g(t)\, df(t)/dt$) and $\dfrac{d}{dt}\left(f(t)g(t)\right)$ means another.

(e) We never use the integral symbol without the differential symbol. $\int f(x)$ is incorrect. If you are integrating with respect to x, you must write $\int f(x)\,dx$.

(f) The notations $\dfrac{dz}{dx}$, $\dfrac{\partial z}{\partial x}$ mean different things. The first is a total derivative (z is a function of x only, possibly through intermediate variables), whereas the second is a partial derivative.

[2] "$\sqrt{x^2} = x$" is wrong, since x may be negative. The correct formula is $\sqrt{x^2} = |\,x\,|$, which is more complete than "$\sqrt{x^2} = \pm x$". For the same reason, $y^{2/3} = x$ is equivalent to $y^2 = x^3$, (where y may be negative), but not to $y = x^{3/2}$, where y is never negative.

[3] On the real line, there is a total order ($\leq$) between numbers, which is compatible in some way with the algebraic operations, and which allows us to speak of negative and positive numbers. In particular, "$\sqrt{a}$" means: the positive solution x of $x^2 = a$. There is no such order defined on the complex numbers: we cannot write "$z \leq w$" unless both z and w are real numbers. Therefore:

(a) An inequality such as $|z| < 2$ (which makes sense, since $|z|$ is a real number), cannot be stated in any simpler way. In particular, "$-2 < z < 2$" is meaningless.

(b) The "square root function", which you find in advanced texts on complex variable, is not defined the way you would expect, and not in any form you can use in this chapter. So you are not allowed to use it; this means, for example, that $z^2 = w$ cannot be replaced by $z = \pm\sqrt{w}$. Also, the symbol $\sqrt{-1}$ is a quaint holdover, and it is preferable to use i. More generally and for the same reason, do not use $\sqrt[n]{\ }$ on complex numbers, n being any integer.

[4] The so-called "inverse trigonometric functions" ($\cos^{-1}$, $\tan^{-1}$, etc...) — inaptly named, since the trigonometric functions, being periodic, have no inverse — are only useful when used in connection with a calculator. Do not use them in this course. If you want to refer to an angle corresponding to the point (a, b) of the unit circle, and the angle is not any of those you know exactly, simply write: "let θ_0 be an angle solving $\cos\theta = a$, $\sin\theta = b$." Note that θ_0 is then only determined up to rational integer multiples of 2π.

[5] The following symbols have intrinsic conventional meaning in Mathematics. This means that you can use them anytime without having to provide the definition:

(i) $\mathbf{N} = \{0, 1, 2, \ldots\}$ is the set of *natural integers*.

(ii) $\mathbf{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ is the set of *rational integers*. These, unlike the natural integers, may be negative.

(iii) $\mathbf{R}$ is the set of real numbers, also called "the real line". $\mathbf{R} \supset \mathbf{Z} \supset \mathbf{N}$.

Small-case letters like m, n, and so on, have no intrinsic meaning, and must be defined in context. Therefore:

"The solutions of $\cos x = 0$ are $x = \pi/2 + n\pi$"

is incomplete, since you did not explain where n is chosen. But:

"The solutions of $\cos x = 0$ are $x = \pi/2 + n\pi$, $n \in \mathbf{Z}$"

or

"The solutions of $\cos x = 0$ are $x = \pi/2 + n\pi$, n rational integer"

or

"The solutions of $\cos x = 0$ are $x = n\pi/2$, n odd rational integer"

are all correct. The following notation is useful:

$$r = s \pmod q$$

(r congruent to s modulo q) means:

$$\text{for some } k \in \mathbf{Z}, \quad r = s + kq.$$

We illustrate how to use it below.

[6] The equation $z^n = w$. Principle:

(a) Two complex numbers in cartesian form are equal just in case their real and imaginary parts match: $x + iy = u + iv$ means $x = u, y = v$.

(b) Two complex numbers in polar form are equal just in case their modulus is the same, and their angles are congruent *modulo* 2π. So:

$$re^{i\theta} = \rho e^{i\phi} \text{ means } \quad r = \rho, \quad \theta = \phi \ (\text{mod} 2\pi).$$

To solve $z^n = w$, use polar form, not cartesian (why? well, try using cartesian form on the following example, and see how you succeed). We illustrate on: find the solutions z of

$$z^5 = 1.$$

First, express 1 in polar form: $1 = 1 \cdot e^{i0}$. (You can replace 0 by any angle equal to zero modulo 2π).
Then express the equation itself:

$$r^5 e^{i5\theta} = 1 \cdot e^{i0}$$

which, by (b), means:

$$r^5 = 1, \quad 5\theta = 0 \ (\text{mod} 2\pi)$$

which implies:

$$r = 1, \quad \theta = 0 \ (\text{mod} 2\pi/5)$$

The equation for θ says that, modulo 2π, the angle has one of the values:

$$0, \ \frac{2\pi}{5}, \ \frac{4\pi}{5}, \ \frac{6\pi}{5}, \ \frac{8\pi}{5}.$$

This corresponds to five points equally spaced over the unit circle. That there are five solutions is assured by the fundamental theorem of algebra, since $z^5 = 1$ is an equation of degree five.

[7] Finding disc of convergence of a complex power series using the ratio test.
The test examines whether the ratio of moduli of consecutive terms has a limit, and compares this limit with one. Three things to keep in mind:

(a) The symbol $|\cdot|$ must be used throughout.

(b) Refrain from expanding when you see a power.

(c) The limit is only taken at the last step, if it exists.

Worked example: find the disc of convergence of

$$\sum_{p=0}^{\infty} \frac{p+i}{p^2-i} \left(z+1-i\right)^p$$

Let w_p be the general term of the series.

$$\left|\frac{w_p}{w_{p-1}}\right| = \left|\frac{p+i}{p^2-i}\right| \left|\frac{(p-1)^2-i}{(p-1)+i}\right| \left|z+1-i\right|$$

$$= \left|\frac{p+i}{(p-1)+i}\right| \left|\frac{(p-1)^2-i}{p^2-i}\right| \left|z+1-i\right|$$

As $p \to \infty$, each of the first two factors has limit 1. This can be seen without expanding: for instance,

$$\left|\frac{(p-1)^2-i}{p^2-i}\right| = \left|\frac{\left(1-\frac{1}{p}\right)^2 - \frac{i}{p^2}}{1-\frac{i}{p^2}}\right|.$$

Therefore $\left|\frac{w_p}{w_{p-1}}\right|$ has limit $\left|z+1-i\right|$. The domain of convergence is given by

$$\left|z+1-i\right| < 1$$

and is the disc centered at $i-1$ and radius one.

[8] When solving exact differential equations, assume you found the potential function to be $F(x, y) = x^2 - 2y$. The family of solutions is then described, not by

$$\text{“}F(x, y) = x^2 - 2y + C\text{”}$$

(which implies that we are not sure what F exactly is), but rather, by

$$x^2 - 2y = C,$$

which describes a family of curves in the xy plane.

[9]

$$f(x) = g(x) \quad \text{and} \quad \int f(x)\,dx = \int g(x)\,dx$$

are not equivalent statements. For this reason, neither are

$$(1) \quad \frac{d\,(\mu(x)y)}{dx} = g(x) \qquad \text{and} \qquad (2) \quad \int \frac{d(\mu(x)y)}{dx}\,dx = \int g(x)\,dx\,.$$

(1) is what I call exact derivative form, not (2). When using the method of integrating factors, you are required to show step (1). As for (2), which is the next step, it is better to write

$$\mu(x)y = \int g(x)\,dx,$$

on the principle that, contrary to students' belief, the fewer integral symbols used, the better.