MATE 4009 Practice exercises

1. Complex numbers

   a) Convert from polar to cartesian:

   i) $\sqrt{2}(\cos(\pi/4) - i\sin(\pi/4))$ ii) $\cos\pi - i\sin\pi$.

   b) Convert from cartesian to polar:

   i) $1 - i\sqrt{3}$ ii) $-\sqrt{3} + i$ iii) $-4i$ iv) -1 v) $2 - 2i$ vi)* $-4\sqrt{3} - 4i$.

   Show that you cannot solve (vi) by making $\theta = \cos^{-1}(-\sqrt{3}/2)$ or $\theta = \sin^{-1}(-1/2)$ (hint: what are the respective ranges of the functions arccos, arcsin? what quadrants do they cover?) Show that making $\theta = \tan^{-1}(1/\sqrt{3})$ is not much better. Moral: do not try to use the inverse trigonometric functions, including arctan. Solve directly the system giving the values of $\cos\theta$ and $\sin\theta$.

2. Complex algebra

   a) Express in cartesian form:

   i) i^4 ii) $\dfrac{1}{1+i}$ iii) $(i + \sqrt{3})^2$ iv) $\left(\dfrac{1+i}{1-i}\right)^2$.

   b) Using one or both of $|z_1 z_2| = |z_1 z_2|$ and $|z| = \sqrt{\bar{z}z}$, find the modulus of:

   i) $\dfrac{2+3i}{1-i}$ ii) $\dfrac{z}{\bar{z}}$ iii) $(1+2i)^3$ iv) $\left(\dfrac{1+i}{1-i}\right)^5$.

3. Equations

   Solve for all possible values of the real numbers x, y:

   i) $x + iy = y + ix$ ii) $(x + iy)^3 = -1$ iii) $|x + iy| = y - ix$.

4. Plots

   Describe geometrically these sets in the complex plane. You may use the fact that $|z - w|$ is the distance between z and w in **C**.

   i) $|z - 1| = 1$ ii) $|z - 1| < 1$ iii) $z - \bar{z} = 4i$ iv) $\Re(z^2) = 4$ v) $z^2 = \bar{z}^2$.

5. Complex-valued series

   Test for convergence:

i) $\Sigma \frac{i^n}{n}$ ii) $\Sigma\left(\frac{1}{n^2}+\frac{i}{n}\right)$ iii) $\Sigma\frac{(3+2i)^n}{n!}$ iv) $\Sigma\left(\frac{1+i}{2-i}\right)^n$.

6. Complex power series

Find the disc of convergence:

i) $1-\frac{z^2}{3!}+\frac{z^4}{5!}-\ldots$ ii) $\sum_{n=1}^{\infty} n^2(3iz)^n$ iii) $\sum_{n=0}^{\infty}\frac{(n!)^2 z^n}{(3n)!}$ iv) $\sum_{n=0}^{\infty} n(n+1)(z-2i)^n$.

7. Euler's formula

a) Express in cartesian form. Hint: when useful, convert first to polar form:

i) $e^{i\pi}+e^{-i\pi}$ ii) $\frac{(i-\sqrt{3})^3}{1-i}$ iii) $(1-i)^8$ iv) $\left(\frac{1-i}{\sqrt{2}}\right)^{40}$.

b) Using the formula $|wz| = |w|\,|z|$ when appropriate, evaluate:

i) $|e^{\sqrt{3}-i}|$ ii) $|\,3e^{5i}\cdot 7e^{-2i}|$ iii) $\left|2e^{i\pi/6}\right|^2$ iv) $\left|\frac{1+i}{1-i}\right|$.

c) Solve the equation in each case, and show a figure displaying the solutions:

i) $z^4 = 1-i$ ii) $z^3 = -i$ iii) $z^6 = 1$.

8. Background

Exercises 1,3,4,6,9,13,15,16,17 of §1.1 of [NSS] (thereafter referred to as "text").

9. Direction fields

Exercises 1,3,7,12,13,16 pp. 22–24 of text. For exercise 16, answer the following questions:

a) Is there a solution curve of $dy/dx = x + 2y$ of the form $x + 2y = C$? For which value of C?

b) Can the solution $y(x)$ passing through $(x, y) = (4, -6)$ also take the value $y = -1$ for some $x \geq 2$? Explain.

10. Euler's method

Exercises 1,2,4,7 p. 28 of text. Use two-decimal digit arithmetic, use stepsize of 0.1, and take three steps only. Approximate f using linearisation. For exercise 7, replace the right-hand side by $1 + t\sin x$, and use the second-order approximation of f coming from a McLaurin series for two variables:

$$f(h,k) = f(0,0) + h\frac{\partial f}{\partial t}(0,0) + k\frac{\partial f}{\partial x}(0,0) + \frac{1}{2}\frac{\partial^2 f}{\partial t^2}(0,0)h^2 + \frac{\partial^2 f}{\partial t \partial x}(0,0)hk + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(0,0)k^2.$$

Compare the results obtained with those obtained by use of calculator, where f is evaluated using the good old keys.

11. Exact equations

    Exercises 1–8,10,16,17,21,24,28,29,32,33,34 pp. 65–67 of text.

12. Linear equations

    Exercises: see syllabus. All exercises from syllabus numbered 2–22, and at least three from those numbered 24–37.

13. Separable equations

    Follow the steps we outlined in class. Note that the summary in the box on p. 41 is incomplete, since it is missing the first step (find special solutions). Now read example 2, then the discussion on pp. 44–45, then go back to example 2 and answer the following: what is the restriction on C_1 in equation (6)? What is the restriction on K in equation (7)? Now solve exercises 1–6, 8, 11, 16, 17, 21, 24, 30, 31, 33 pp. 46–48 of text.

14. Substitutions

    Exercises 10, 12, 15, 17, 20, 45 pp. 79–80. Use the "shifted growth model" substitution to solve exercise 36 p. 48. As a step of this method, we assume known the formula giving the general solution of the growth model.

15. Mixing problems, population models

    Exercises 2,5,8,10,21,22, and two of 23–25 pp. 104–106. Where applicable, use the shifted growth substitution method rather than other methods.

16. Heating and cooling

    Exercises 2,6,7,9,12,16 pp. 113–114. Also, problem 5 of exam 1, Fall 08 (see "old tests"). In this problem, does a certain substitution apply?

17. Mass-spring oscillator

    All exercises of syllabus (p. 168).

    a) In the derivation of the mass-spring oscillator equation, assume that the equilibrium position is y_0 instead of $y = 0$, and modify equation (3) on p. 164 accordingly.

    b) Consider the mass-spring equation

    $$y'' + 6y + 9 = 0, \qquad y(0) = 1, \quad y'(0) = 0.$$

    What is the position of equilibrium? Find the amplitude of the oscillations.

18. Homogeneous linear equations

Exercises 1,11,12,15,17,18,23,26,27,29,32,35 (choose two of a–d),40,46. For linear dependence/independence, you may only use the definition given in class, even if there are only two functions. You may use neither the “definition” on p. 172, nor the lemma 1 on the same page.

19. Auxiliary equations with complex roots

    Choose exercises from the syllabus.

20. Superposition principle

    Exercises 2,3,7,19,21,23,26,29,38,40 pp. 201–202. For 40: take a look at §4.4 to find a particular solution. Also, 41, and one of 42–44. For a challenge: 48.