MATE 4009 Practice exercises

Complex numbers

1–7: 1,3,6,7,11,12,14 section 4 of Boas (Boas refers to the handout available at Magic Copy).

Complex algebra

8–18: 1,3,4,6,13,19,20,27,28,29,34 section 5 of Boas. For some of the last problems, the formula $|zw| = |z|\,|w|$ is more convenient to use than equation (5.1).

Equations and graphs

19–25: 38,39,43,44,45,47,50 section 5 of Boas.

26–30: 51,54,56,57,63 section 5 of Boas.

31: solve and describe geometrically the set $z = |z|$. Hint: what are re $(|z|)$ and im $(|z|)$?

Complex-valued series

32–36: 6,7,9,11,12 of section 6 of Boas.

Complex power series

37–43: 2,3,10,12,13,14,15 section 7 of Boas.

Additional exercises

44: (corrected). Solve the equation $z^5 = \sqrt{3} - i$ and plot the roots (solutions) on a certain circle.

45: Solve in the complex plane the equation $z^4 + 2z^3 - iz - 2i = 0$. Show the solutions on a single plot. Hint: one root of this equation is $z = -2$. Use this fact, and the fundamental theorem of algebra.

46: Outline the main steps that lead to the ratio test for convergence of a complex-valued series $\sum w_n$.

47: Consider a power series $\sum c_n z^n$ in the complex plane. If the domain of convergence of this series is a disc with finite radius, what can you say about the set of values of z where this series converges, but not absolutely? You may use the statement of the ratio test, and the following fact: "If the series $\sum c_n z^n$ converges for the value z_0, then it converges absolutely for any z such that $|z| < |z_0|$."

Background

48: Drag race. Alison and Kevin a participating in a drag race. Beginning from a standing start, they each drive with constant acceleration. Alison covers the last one-quarter of the distance in 3 seconds. Kevin covers the last one-third of the distance in 4 seconds. Who wins the race, and by how much time?

49: For each of these families of functions of x, find (and justify) whether it is a one-parameter or a two-parameter family:

i) $A\cos x + (B - A)\sin x$ ii) $x^3 + Ax^2 + \cos Ax^2/2 + Bx$ iii) $x^3 + \cos^2 Ax^2 + \sin^2 Ax^2 + Bx$

Direction fields

50: Plot the direction field (isoclines, direction markers, use colours preferably) of $dy/dx = x + 2y$. Find a straight-line solution. Sketch several solution curves. Can the solution $y(x)$ passing through $(x, y) = (4, -6)$ also take the value $y = -1$ for some $x \geq 2$? Explain.

51: Consider the differential equation

$$\frac{dp}{dt} = p(p-1)(2-p)$$

for the population p (in thousands) of a certain species at time t.

a) Sketch the direction field (first, sketch a suitable auxiliary graph).

b) If the initial population is 4000 (so $p(0) = 4$), what can you say about the limiting population $\lim_{t\to\infty} p(t)$?

c) If $p(0) = 1.7$, what is $\lim_{t\to\infty} p(t)$?

d) If $p(0) = 0.8$, what is $\lim_{t\to\infty} p(t)$?

e) Can a population of 900 ever increase to 1200?

Euler's method

52–55: exercises 1,2,4,7 of text. Use two-decimal digit arithmetic, use stepsize of 0.1, and take three steps only. Approximate f using linearisation.

Exact equations

56–71: exercises 1–8,9,16,17,21,24,28,29,32 of text.

Linear equations

72–95: all exercises from syllabus numbered 2–22, and at least three from those numbered 24–37.

Separable equations

Follow the steps we outlined in class. Note that the summary in the box on p. 39 is incomplete, since it is missing the first step (find special solutions).

96–108: exercises 1–6,8,11,16,17,24,31,33 pp. 43–44 of text.

Substitutions

109–112: exercises 17–20 p. 74.

113: use the "shifted growth model" substitution to solve exercise 36 p. 45. As a step of this method, we assume known the formula giving the general solution of the growth model.

114: in the example discussed in class (example 2 in the text), we found that the general solution in the x, z plane is given by

$$z^2 = 1 + De^{2x}, \quad D \in R.$$

Draw in the x, z plane several solutions corresponding to positive and negative values of D, marking the intercept with the x axis, and draw a similar plot in the x, y plane, indicating which solution curve in one plane corresponds to which in the other.

Mass-spring oscillator

115–120: § 4.1: 4,3,5,6,8,9 (5th edition), 3,4,5,6,7,9 (6th edition).

Homogeneous linear equations

121–127: § 4.2: 3,10,14,19,20,21,26 (5th edition). 4,9,14,19, 20,21,26 (6th edition).

Auxiliary equation with complex roots

128–139: § 4.3: 5th edition: 4, 6, 10, 9, 21, 22, 26, 28, 29, 35, 37, 38. 6th edition: 3, 5, 9, 10, 21, 22, 26, 28, 29, 35, 37, 38.

Principle of superposition and inhomogeneous equations

140–153: § 4.5: 5th edition: 1, 3, 6. 6th edition: 2, 4, 5. Both editions: 17, 19, 24, 25, 35, 36, 39, 40, 42, 46, 48.

Power series solutions

154–158: 4, 6, 8, 11, 13 of § 8.1.

159: we gave in class the example: Find the third Taylor polynomial of the solution of the initial-value problem

$$y' = \frac{1}{x + y + 1}, \qquad y(0) = 0.$$

Find the exact solution, and compare the third Taylor polynomial with that found in class.

160–168: 3, 6, 7, 13, 14, 23, 24, 35, 38 of § 8.2.

169–178: exercises 1, 4, 5, 12, 14, 16, 20, 22, 23, 32 of § 8.3.