Tips and common corrections

[1] Some comments about notation.

(a) The symbol “ = ” means *equals*, and nothing else. Points are lost on tests and quizzes for using it inappropriately.

(b) The symbol “ $\Rightarrow$ ” means *implies*. It connects sentences or logical statements; it does not connect values or expressions. For instance:

$$\sin^2 x + \cos^2 x = 1 \;\Rightarrow\; 2\cos^2 x + 3\sin^2 x = 2 + \sin^2 x\,.$$

The symbol “ $\rightarrow$ ” means *tends to*, and we will only use it to denote limits.

(c) Use of parentheses.

To multiply x by $-y$, we write $x(-y)$. “$x \cdot -y$” or “ $x \times -y$” is incorrect, even if you reduce the size of the minus sign, raise it and stick it very close to y.
Also, $x - y$ and $x + (-y)$ are correct. “ $x + -y$ ” is not, even if you reduce the size of the minus sign, raise it and stick it very close to y.
The derivative sign has precedence over the product. Therefore, $\frac{d}{dt} f(t)g(t)$ means one thing (namely, $g(t)\, df(t)/dt$) and $\frac{d}{dt}\left(f(t)g(t) \right)$ means another.

(d) We never use the integral symbol without the differential symbol. $\int f(x)$ is incorrect. If you are integrating with respect to x, you must write $\int f(x)\, dx$.

(e) The notations $\frac{dz}{dx}$, $\frac{\partial z}{\partial x}$ mean different things. The first is a total derivative (z is a function of x only, possibly through intermediate variables), whereas the second is a partial derivative.

[2] “$\sqrt{x^2} = x$ ” is wrong, since x may be negative. The correct formula is $\sqrt{x^2} = |\,x\,|$, which is more complete than “$\sqrt{x^2} = \pm x$”. For the same reason, $y^{2/3} = x$ is equivalent to $y^2 = x^3$, (where y may be negative), but not to $y = x^{3/2}$, where y is never negative.

[3] Polynomials are series, but not all series are polynomials.

$$1 + x + \frac{x^2}{2} + \frac{x^3}{6} \qquad \text{and}$$

$$1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} \qquad (1)$$

are respectively polynomials of degrees 3 and n, but

$$1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots \qquad \text{and}$$

$$1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots + \frac{x^n}{n!} + \ldots \qquad (2)$$

are bona fide infinite series, the second written less ambiguously than the first. The first occurence of three dots in (1) and (2) signifies that terms extend up to order n . The second occurence of $\ldots$ in (2) means that the series does not stop at order n, making n simply a dummy variable. (For those of you who learned computer programming, a dummy variable is the same as a local variable). As you see, three dots make a big difference.

[4] Neither

$$\tan\theta = b/a \qquad (1)$$

nor

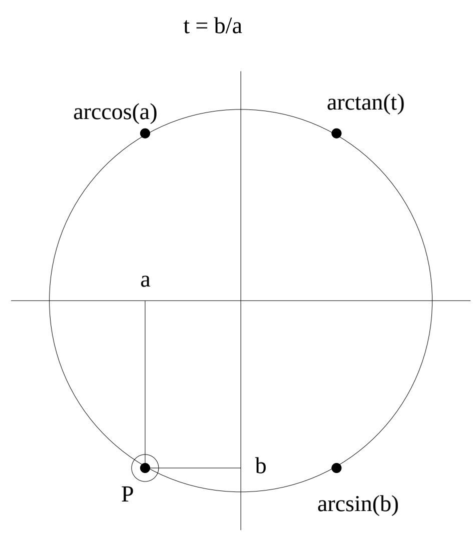
$$\theta = \tan^{-1} b/a \qquad (2)$$

are equivalent to the system

$$\cos\theta = a, \quad \sin\theta = b \qquad (3),$$

which you solve when you are putting complex numbers in polar form. We explained why: whereas solutions of (3) are determined up to multiples of 2π, (so: *one* point on the unit circle), those of (1) are determined up to multiples of π (so: *two* possible points on the unit circle). You may only use (2) if you are solving numerically for θ using calculator, and if, having found $\tan^{-1}(b/a)$, you use at least one of the equations (3) in order to determine the quadrant where θ resides.

Also, (1) and (2) are not equivalent to each other. Whereas (1) has infinitely many solutions residing in exactly two quadrants, (2) is a well-defined function of b/a, allowing θ to be a single value in the interval $(-\pi/2, \pi/2)$, and therefore, in quadrant one or four. See exercise 1 of the practice problems, added Jan 30. In the figure below, the solution of the system $\cos\theta = a$, $\sin\theta = b$ corresponds to point P. The other spurious solutions are each obtained by using some inverse trigonometric function.



[5] "Circle" and "disc" mean different things. The disc is a plane region. Its boundary, the circle, is a curve. The circle of radius one and center w in the complex plane has equation $|z - w| = 1$, and the disc of radius one and centre w has equation $|z - w| < 1$.

[6] On the real line, there is a total order ($\leq$) between numbers, which is compatible in some way with the algebraic operations, and which allows us to speak of negative and positive numbers. In particular, "$\sqrt{a}$" means: the positive solution x of $x^2 = a$. There is no such order defined on the complex numbers. Therefore:

(a) An inequality such as $|z| < 2$ (which makes sense, since $|z|$ is a real number), cannot be stated in any simpler way. In particular, "$-2 < z < 2$" is meaningless.

(b) The "square root function", which you find in advanced texts on complex variable, is not defined the way you would expect, and not in any form you can use in this course. So you are not allowed to use it; this means, for example, that $z^2 = w$ cannot be replaced by $z = \pm\sqrt{w}$. Also, the symbol $\sqrt{-1}$ is a quaint holdover, and it is preferable to use i.

Note: the above does not mean that we cannot use the minus sign with complex numbers. -1 is a complex number (it is even real), and $-z$ means $(-1)z$, which is the point symmetric to z about the origin in the complex plane.

[7]

$$f(x) = g(x) \quad \text{and} \quad \int f(x)\,dx = \int g(x)\,dx$$

are not equivalent statements. For this reason, neither are

$$(1)\quad \frac{d\,\big(\mu(x)y\big)}{dx} = g(x) \qquad \text{and} \qquad (2)\quad \int \frac{d(\mu(x)y)}{dx}\,dx = \int g(x)\,dx\,.$$

(1) is what I call exact derivative form, not (2). When using the method of integrating factors, you are required to show step (1) before showing step (2).

[8] A scalar expression is never comparable to a differential expression. The following are meaningless:

$$x + 3y = 4\,dx \qquad \text{equating scalar and differential}$$

$$x + 3y\,dy = 4\,dx \qquad \text{adding scalar and differential}$$

[9] When solving exact differential equations, assume you found the potential function to be $F(x, y) = x^2 - 2y$. The family of solutions is then described, not by

$$\text{"}F(x, y) = x^2 - 2y + C\text{"}$$

(which implies that we are not sure what F exactly is), but rather, by

$$x^2 - 2y = C,$$

which describes a family of curves in the *xy* plane.

[10] In word problems, I do not use any symbol without first defining it. So if I am going to use the symbol x or V, where x or V is not part of the problem statement, I must state: "let x be the amount of lead in kgs" or "let V be the amount of acid in mL".

In addition, the solution of word problems requires the use of verbal explanations throughout the solution, and this means complete sentences.

[11] (Style). In word problems, do not use units in front of every term of an equation. The following is incorrect:

$$\frac{dx}{dt} = 4L/\min(0.2kg/L) - (4L/\min)\frac{x}{100}.$$

Rather, write:

$$\frac{dx}{dt} = 0.8 - 4\frac{x}{100} \qquad (kg/\min)$$

The textbook does use the style I am asking you to avoid, in an explanatory fashion, in equations (2) and (3) on p. 97, (and again in example 2 of the following page), but foregoes it as soon as the mathematical equations are stated, starting from equation (4).

[12] The following symbols have intrinsic conventional meaning in Mathematics. This means that you can use them anytime without having to provide the definition:

(i) $\mathbf{N} = \{0, 1, 2, \ldots\}$ is the set of *natural integers.*

(ii) $\mathbf{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ is the set of *rational integers*. These, unlike the natural integers, may be negative.

(iii) $\mathbf{R}$ is the set of real numbers, also called "the real line". $\mathbf{R} \supset \mathbf{Z} \supset \mathbf{N}$.

Small-case letters like m, n, and so on, have no intrinsic meaning, and must be defined in context. Therefore:

"The solutions of $\cos x = 0$ are $x = \pi/2 + n\pi$"

is incorrect, since you did not explain where n is chosen. But:

"The solutions of $\cos x = 0$ are $x = \pi/2 + n\pi,\ n \in \mathbf{Z}$"

or

"The solutions of $\cos x = 0$ are $x = \pi/2 + n\pi$, n rational integer"

or

"The solutions of $\cos x = 0$ are $x = n\pi/2$, n odd rational integer" are all correct.