Comments on homework

Homework 1

In many (not all) of these problems, the solution includes solving a linear system as a step (or partially solving, if you are investigating existence, but not explicit form, of solutions). Always do this step using row (reduced or partial) echelon form, not setting up the system and eliminating, as you learnt in high-school.

A principle to keep in mind is that in terms of computations, if you want to compute, say, $A = B^{-1}C$, it is more efficient to use row echelon reduction on the augmented matrix $\begin{pmatrix} B & C \end{pmatrix}$ transforming it to $\begin{pmatrix} I & A \end{pmatrix}$, than to find B^{-1}, then multiply it by C.

Do not use the determinant to compute inverse: this method is very inefficient for large matrices, and the purpose of these exercises is to teach the right solution methods.

2. Transforming to (partial) row echelon form the matrix $\begin{pmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \mathbf{x}_3 \end{pmatrix}$ allows to answer at once parts (a), (b), (c). Note that the single form tells you that two certain vectors are l.i., and the three of them are l.d.

3. S is the span of three vectors. Transform the 4×3 matrix having these vectors as columns to echelon form, to show that they are linearly independent.

5. Two steps: show that some two of the vectors are linearly independent (which most of you did, often without explaining that you are using the definition), and also use an identity giving a nontrivial combination of the three being zero. Then the span has dimension two, for if $h \in \operatorname{span}(f, g)$, then $\operatorname{span}(f, g, h) = \operatorname{span}(f, g)$.

8. Let U be the matrix $\begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{pmatrix}$ and V be the matrix $\begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{pmatrix}$. The transition matrix from $[\mathbf{v}_1, \mathbf{v}_2]$ to $[\mathbf{u}_1, \mathbf{u}_2]$, call it P, has as column j the solution x of $Ux = \mathbf{v}_j$. In other words it solves the matrix equation

$$UP = V.$$

Therefore, is is found by transforming the augmented matrix $\begin{pmatrix} U & V \end{pmatrix}$ to the form $\begin{pmatrix} I & P \end{pmatrix}$.

9. Call B the matrix which has as columns the vectors of E. Then $[\mathbf{x}]_E$ solves the equation (in c) $Bc = \mathbf{x}$. Similarly for the other two coördinate vectors. Then the solution is obtained by reducing the 2×5 augmented matrix $\begin{pmatrix} B & \mathbf{x} & \dots \end{pmatrix}$ to the form $\begin{pmatrix} I & C \end{pmatrix}$.

11. Let V be the matrix of columns $\mathbf{v}_1, \mathbf{v}_2$. Since $V\begin{pmatrix} a \\ b \end{pmatrix} = a\mathbf{v}_1 + b\mathbf{v}_2$, you see that the matrix VS has columns $\mathbf{w}_1, \mathbf{w}_2$.

12. (Exercise 8 of text). Let U, V be matrices with columns the given ordered bases. Then there holds $US = V$. To solve for U using the reduction to echelon form, we need to change the order: the same equation is equivalent to $S^T U^T = V^T$. We can then use the same procedure as above (see pb 8): operate on the augmented matrix $\begin{pmatrix} S^T & V^T \end{pmatrix}$ to reduce to the form $\begin{pmatrix} I & U^T \end{pmatrix}$.

Homework 2

14. To show that a mapping L is not linear, it is often enough to show that $L(0) \neq 0$. In part (c), it is enough to find a matrix A and a scalar α such that $L(\alpha A) \neq \alpha L(A)$.

15. The kernel of a linear mapping defined on R^n is the same as the null space of its associated matrix.

17. A mapping $L : R^n \to R^m$ is onto if the column space of its matrix A has dimension m, since this is also the dimension of the range of L. In particular, if $m = n$, it is onto if and only if A is nonsingular.

18. A linear map L is one-to-one just in case its kernel is trivial ($\{0\}$). Indeed, $L(x) = L(y)$ is equivalent to $L(x - y) = 0$.