MATE 4052 set 9

Not to be handed in.

Exercises 8.1, 8.2, 8.3.

Let E be a Banach space, and denote by F the Banach space $\mathbf{L}(E, E)$.

(a) Show that for each n, the map $x \longmapsto x^n$ from F to itself is of class C^∞. Conclude that the mapping $x \longmapsto \exp(x) = \sum_{n\geq 0} (x^n/n!)$ is of class C^∞.

(b) Show that $x \longmapsto \exp(x)$ is a C^∞-diffeomorphism from a neighbourhood of 0 to a neighbourhood of 1_E, the inverse mapping being defined near 1_E by

$$y \longmapsto -\sum_{n\geq 1} \frac{(1_E - y)^n}{n}.$$

Let E_0 be the vector space of continuous functions from $[0, 1]$ to R, normed by $\|f\|_0 = \sup_{0\leq x\leq 1} |f(x)|$, and E_1 be the vector space of real-valued functions of class C^1 on $[0, 1]$ such that $f(0) = 0$, normed by $\|f\|_1 = \sup_{0\leq x\leq 1} |f'(x)|$. Show that $\varphi : E_1 \longmapsto E_0$ defined by $\varphi(f) = f' + f^2$ is a C^∞-diffeomorphism from a neighbourhood V of the origin in E_1 to a neighbourhood W of the origin in E_0 (consider $\varphi'(0)$). Compute the first and second derivatives of the inverse function $\psi : W \longmapsto V$.