MATE 4072 S14

Due 27/01

1. Problem 1 of § 13.2 of Boas.

2. Problem 7 of § 13.2 of Boas.

3. Problem 14 of § 13.2 of Boas.

4. Same as problem 3 of § 13.2, replacing cos$x$ by sin x.

5. In finding elementary solutions of the steady-state temperature in the semi-infinite rectangular plate, we assumed that the constant of eq. (2.5) (p. 622) is negative. Discuss what we obtain if we assume respectively that the constant is (a) positive or (b) zero.

6. (Final exam S13). Let S be the square given by $0 \le x \le 1,\ 0 \le y \le 1$. A function is required to be zero on the sides $x = 0, x = 1, y = 1$, and not identically zero on the side $y = 0$. Two candidates are:

$$\phi(x, y) = (A\cos(\lambda x) + B\sin(\lambda x)) \cdot (Ce^{-y} + De^{y}),$$

$$\psi(x, y) = (A\cos(\lambda y) + B\sin(\lambda y)) \cdot (Ce^{-x} + De^{x}).$$

   $A,\ B,\ C,\ D$ are constant numbers and λ is a positive constant. Which, if any, of the two functions are suitable, and are there conditions on the numbers A, B, C, D, λ? Your analysis must be rigorous and complete.

Due 3/02

7. Problem 3 of § 13.3.

8. Problem 6 of § 13.3.

9. Problem 7 of § 13.3. Follow the method we have used and ignore the long-winded "hints and comments", but observe the following: if the eigenfunctions are of the form $X(x) = \cos(\lambda x)$, which solutions λ of $X'(l) = 0$ correspond to non-zero eigenfunctions? To compute the Fourier coefficients of a cosine series, refer to equations (5.1) and (5.9) of chapter 7.

10. Complete the example started in class: slab of thickness l with the given initial temperature distribution, with the right face no longer held at 100° at time zero, but suddenly insulated. The eigenfunctions are those we found, of the form $\sin(\lambda_n x),\ n \ge 0$, for those λ_n solving $\cos(\lambda l) = 0$. Verify that they form an orthogonal system, then find the coefficients of the Fourier expansion of $u(0, x)$.
    Addendum: the initial distribution is $u_0(x) = 100x/l$.

Due 12/02

11. Problem 1 of § 13.4.

12. Problem 5 of § 13.4.

13. Problem 11 of § 13.4. Update: consider the string pinned at $x = 0$ and free at $x = l$, and the displacement specified by problem 3 of the text.

14. Refer to problem 7 of § 13.4, and take $l = c = 1$. Using the d'Alembert form of the solution of the wave equation $u_{tt} - u_{xx} = 0$, find $u(t, 0.5)$ at times $t = 0, 1, 2, 4$.

Due 24/02

15. Consider the displacement $u(t, r, \theta)$ of a pie-shaped membrane of radius a and angle $\pi/3$ which satisfies

$$u_{tt} - c^2 \Delta u = 0.$$

Determine the natural frequencies of oscillation if the boundary conditions are

$$u(t, r, 0) = 0, \ \ u(t, r, \pi/3) = 0, \ \ u(t, a, \theta) = 0.$$

16. Problem 3 of § 13.6.

Due 26/02

17. The fundamental frequency of the vibrating membrane is the lowest value of the family which we denoted by λ_{mn}. What is the effect on the fundamental frequency, of making the drum smaller, if tension and density (reflected in the c constant) are unchanged?

Due 3/03

18. If D is a bounded plane domain, and ∂D its boundary, Green's formula states that for any twice differentiable functions u and v,

$$\int_D v \Delta u dx + \int_D \nabla u \cdot \nabla v dx = \int_{\partial D} v (\nabla u) \cdot n d\sigma,$$

where n is the unit outward normal to D on ∂D. Consider elementary solutions of the vibrating membrane problem; the spatial factor satisfies the Helmholtz equation

$$\Delta F - CF = 0,$$

in addition to the homogeneous condition $F = 0$ on ∂D (where D is the disc of the membrane). We assumed in our analysis that $C < 0$. By choosing suitably u, v in Green's formula, show that when $C \geq 0$, the only solution of the Helmholtz equation is $F \equiv 0$. Note added: one way to show that a function is zero, is to show that the integral of its square is zero.

19. Sequel to problem 9: we assumed that the separation constant in

$$\frac{X''}{X} = C$$

is nonpositive, not necessarily strictly negative. Justify this assumption by considering, separately, the cases $C > 0$ and $C = 0$. Conclude that it is indeed the case that one of the eigenfunctions corresponds to $\lambda = 0$.

Due 10/03

20. Problem 1 of § 13.9.

21. Problem 4 of the same section. Replace the factor $\sin(kx)$ by $\sin(2\pi kx)$, and modify accordingly the exponential term.

22. Problem 5 of § 13.9.

Due 17/03. These problems are to hand in later, but feel free to solve many more on your own.

23. § 14.2: choose three from 1–21, omitting 18, 19 but including 21.

24. (a)–(c): problems 22–24 of § 14.2.

25. (a)-(b): choose two from 34–42, omitting 36.

26. Problem 45 of § 14.2.

27. (a)-(b): choose one from 54–57 and one from 58–63 of § 14.2

28. (a)-(b): 4, 5 of § 14.3.

29. (a): choose one of 8–10. (b): problem 16 of § 14.3.

Due 31/03

30. (a)-(b): chose one of 17–20, and one of 22–24 of § 14.3.

31.

(a) Integrate $\dfrac{1}{z^4 - 1}$ over the circle $|z + 3| = 1$, counterclockwise. Recall that the solutions of the equation $z^4 = 1$ all lie on a certain circle (which?)

(b) Evaluate $\displaystyle\int_\gamma \frac{\exp(z^3)}{(z - 1 - i)z^2}\,dz$ where γ consists of $|z| = 3$ counterclockwise, and $|z| = 1$ clockwise.

32.

(a) Show that $\int_\gamma (z - z_1)^{-1}(z - z_2)^{-1}\,dz = 0$ for any simple closed path γ enclosing the two distinct points $z_1,\ z_2$.

(b) Integrate $z^{-2}\tan(\pi z)$ around any ellipse of foci $\pm i$.

Due 28/04

33. Choose one of 3–8 of § 14.4.

34. Choose another of 3–8.

35. Choose one of 9–12 of § 14.4.

36. Choose another of 9–12.

Due 30/04

37. (a)-(b): chose two of exercises 1–9 of § 14.6.

38. (a)–(c): chose three of exercises 14–35 of § 14.6.

39. (a)–(c): chose three corresponding to those of the previous problem, from exercises 14’–35’.

40. $f(z) = \dfrac{e^{iz}}{z(z^2+1)^2}$ has a double pole at $z = i$. Putting $f(z) = \dfrac{g(z)}{(z-i)^2}$, find the residue of f at i by finding the coefficient of $z - i$ in the Taylor expansion of g in powers of $z - i$. Hint: let $z = i + t$, and find the coefficient of t in the McLaurin expansion of $h(t) = g(i + t)$. When multiplying or dividing series, you do not need to determine all terms to find the coefficients of low-order terms. Further hint: an expansion of $(w - t)^{-2}$ in powers of t is obtained by differentiating the expansion of $(w - t)^{-1}$ term by term.

41. Problem 37 of § 14.6.

Due 5/05

42. Using methods from § 14.7, evaluate $\displaystyle\int_0^{2\pi} \frac{\sin^2\theta}{5 - 4\cos\theta}\, d\theta$.

43. Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^4 + 16}$.

44. Evaluate $\displaystyle\int_0^{\infty} \frac{\cos 2x}{x^4 + 13x^2 + 9}\, dx$.

45. Problem 37 of § 14.7.

46. Problem 39 of § 14.7.

Due 7/05

47. (a)-(b): problems 5, 6 of § 15.1.

48. (a)-(b): problems 8, 10 of § 15.1.

49. Problem 14 of § 15.2.

50. Problem 15 of § 15.2.

51. Problem 19 of § 15.2.

52. (a)-(b): problems 11, 23 of § 15.3.

53. Problem 21 of § 15.3.

54. (a)-(b): problems 19, 22 of § 15.3.

55. (a)-(b): problems 16, 18 of § 15.3.

Due 12/05

56. (a)-(b): problems 7,8 of § 5.4.

57. Problem 10 of § 5.4.

58. Problem 21.

59. Problem 22 of § 5.4.

60. (a)-(b): problems 1, 7 of § 15.5.

61. (a)–(c): problems 8, 9, 12 of § 15.5.

62. (a)-(b): problems 2,3 of § 15.6.

63. Problem 8 of § 15.6.

64. Problem 16 of § 15.6.

Practice

65. An urn contains four red, seven green, and two white balls. You draw a ball at random, note its colour, and replace it. You repeat these steps four times. Let X be the number of red balls and Y be the number of green balls. Find:

    (a) $P(X \le 1)$.

    (b) $P(X + Y = 2)$.

66. Toss a fair coin ten times. Let X be the number of heads. Show, using an certain inequality, that the probability that X is within $\frac{10}{3}$ of its mean is at least 0.75.

67. A multiple-choice exam contains 50 questions. Each question has four choices. Find the expected number of correct answers if a student guesses the answers at random.

68. Problem 3 of § 15.8.

69. Some of pbs 11–20 of the same section.

70. Problem 21.

71. Problem 25 of § 15.8.

72. Problem 1 of § 15.10.

73. Problem 3 of § 15.10.

74. Problems 2, 4, 8, 9 of § 15.9.