MATE 5049 assignment 2

Sketch of solutions

Marks: 7 (12), 8 (12), 9 (6), 10 (4), 11 (14). Total: 48.

8. $f(y,z) = y + \sqrt{1+z^2}$ is strongly convex, as sum of a convex and a strongly convex function. The differential equation

$$\frac{d}{dx}\frac{y'}{\sqrt{1+y'^2}} = 1$$

yields upon direct integration $y'/\sqrt{1+y'^2} = x + b$, hence $y = -\sqrt{1-(x+b)^2} + c$.

(a) On $\mathcal{D}$: the boundary conditions give $1 - b^2 = (1+c)^2$, and also $b = (2-\sqrt{3})\cdot c$. Therefore:

$$(1 + (2-\sqrt{3})^2)\, c^2 + 2c = 0.$$

$c = 0$ corresponds to $b = 0$, and the other value $c = \dfrac{-2}{1+(2-\sqrt{3})^2}$ does not satisfy the original condition $c = -1 + \sqrt{1-b^2}$. So the solution is $y = -\sqrt{1-x^2}$.

(b) On $\mathcal{D}_1$: we refer again to the form $y = -\sqrt{1-(x+b)^2} + c$, and the boundary conditions are $y(0) = -1$ and $f_z(1/2) = 0$. The natural boundary condition implies $y'(1/2) = 0$, and since $y' = \dfrac{x+b}{\sqrt{1-(x+b)^2}}$, this means $b = -1/2$. Compute the corresponding value of c, and obtain $y = -\sqrt{1-\left(x-\frac{1}{2}\right)^2} - 1 + \dfrac{\sqrt{3}}{2}$.

11.

(a) $\tilde{f} = -y + \lambda\sqrt{1+y'^2}$. The differential equation to solve is

$$\frac{d}{dx}\frac{\lambda y'}{\sqrt{1+y'^2}} = -1$$

and upon integration, $\dfrac{\lambda y'}{\sqrt{1+y'^2}} = -(x+B)$. Further integration gives $y' = -\dfrac{x+B}{\sqrt{\lambda^2-(x+B)^2}}$, using the fact that $\lambda > 0$. Hence:

$$y = \sqrt{\lambda^2-(x+B)^2} + C\,.$$

Applying the endpoint conditions gives $B = 0, \quad C = -\sqrt{\lambda^2 - b^2}$, and

$$y = \sqrt{\lambda^2 - x^2} - \sqrt{\lambda^2 - b^2}.$$

As seen by rewriting this in the form $\left(y + \sqrt{\lambda^2 - b^2}\right)^2 + x^2 = \lambda^2$, this is the nonparametric equation of an arc of circle of radius λ, centre $\left(0, \ -\sqrt{\lambda^2 - b^2}\right)$ and passing through the points $(-b, 0)$ and $(b, 0)$. From the form of $y(x)$, we see that $\lambda \geq b$. Remains to see how the value of λ is related to the length l. From

$$l = \int_{-b}^{b} \sqrt{1 + y'^2}\, dx = \int_{-b}^{b} \frac{\lambda}{\sqrt{\lambda^2 - x^2}}\, dx,$$

we see that $l > 2b$, and that l is a strictly decreasing function of λ. By taking limits, there holds

$$\lim_{\lambda \to \infty} l(\lambda) = 2b, \qquad \lim_{\lambda \to b} l(\lambda) = \int_{-b}^{b} \frac{b}{\sqrt{b^2 - x^2}}\, dx = \pi b.$$

Monotonicity of $l(\lambda)$ implies that if $2b < l < \pi b$, there exists an unique radius λ solving the problem. The elegance of this argument is that you don't have to compute λ explicitly in terms of l.

(c) The angle β is given by $\beta = 2\arctan(b/\sqrt{b^2 - \lambda^2})$. Make $t = \tan(\beta/2)$, so that $\lambda^2 = b^2(1 + 1/t^2)$. Then the area is given by

$$\lambda^2 \frac{\beta}{2} - \frac{1}{2} 2b \sqrt{\lambda^2 - b^2} = \frac{b^2 \beta}{2\sin^2(\beta/2)} - b^2 \cot(\beta/2).$$

Its derivative with respect to β is $\dfrac{b^2}{2\sin^2(\beta/2)}\left(2 - \beta\cot(\beta/2)\right)$, which is nonnegative on $(0, \pi/2)$.