MATE 5049 assignment 7 (final version)

35. In a vector space X, an *affine variety* is the translate of some subspace S of X by some fixed element x_0: $V = x_0 + S$.

   a) Show that such x_0 belongs to V, and that S is determined by V alone (so that V is indifferently written as $x_1 + S$, where x_1 is any element of V, but S is always the same).

   b) Use (a) to conclude that, given a subset C of X, there is a smallest affine variety containing C (how is it obtained from $S(C - x_0)$, span of $C - x_0$?) We denote this variety by $v(C)$.

   c) The span of a set D, smallest subspace containing D, is also the set of all finite linear combinations of the form

$$\{ \sum \lambda_i x_i : x_i \in D \} .$$

   Use this fact to show that

$$v(C) = A = \{ \sum \lambda_i x_i : x_i \in C,\ \sum \lambda_i = 1 \}$$

   where all possible finite sums are allowed. Linear combinations with coefficients adding to one are called *affine combinations*.

36. It is asserted in Luenberger that $v([f, C]) = R \times v(C)$, where $[f, C]$ denotes the epigraph of f.

   a) Assume f takes finite values, and show that $R \times v(C)$ is an affine variety containing $[f, C]$ (thereby showing $v([f, C]) \subset R \times v(C)$.)

   b) To show the reverse inclusion, one has to show that any pair (r, x) where $x \in v(C)$ can be written as an affine combination of points in $[f, C]$. For this, one can use induction on the cardinality of n when writing $x = \sum_1^n \lambda_i x_i,\ x_i \in C,\ \sum \lambda_i = 1$. Show how the base of the induction works, by making n = 1. Hint: express (r, x) as an affine combination of the two points $(f(x), x)$ and $(f(x) + 1, x)$.

   For bonus, show how the general induction step works.

37.

a) Let $X = L^2[0, 1]$, and define $g(x) = \int_0^1 |x(t)|\,dt$. Find the conjugate functional of g. Hint: we have worked with piecewise C^1 functions, but not much in L^2. To find the domain of g^* (which is contained in $X = L^2$, since X is its own dual), assume first that x^* is continuous. The restriction you find on x^* carries over to L^2; check with me if in doubt.

b) Let $X = L^2[0, 1]$, and define $f(x) = \int_0^1 \left(\frac{1}{2}x^2(t) + |x(t)|\right) dt$ on X. Find the conjugate functional of f.

38. Let X be a Hilbert space, with norm $|\cdot|$. We want to show the "minimum norm" theorem: "Let x_0 be a point in X, and d be its distance from the subspace M. Then

$$d = \inf\{\, |u - x_0| : u \in M \,\} = \max\{\, \langle x^*, x_0 \rangle : |x^*| \le 1,\ x^* \in M^\perp \,\}\text{''}.$$

Follow these steps:

a) Show that the conjugate of $f(x) = |x - x_0|$ is $f^*(x^*) = \langle x^*, x_0 \rangle$. Hint: first, make $x_0 = 0$.

b) Prove the minimum norm theorem using directly the Fenchel duality theorem. Hint: make $C = X,\ D = M,\ g = 0$.

39. Problem 2 p. 270.

40. Problem 4 p. 271.