MATE 6540 assignment 1

1. A topological space is *normal* if it is separated (Hausforff), and if for any pair of disjoint closed sets F_i, $i = 1, 2$, there exists a pair of disjoint open sets U_i such that $F_i \subset U_i$. The following example is of a Hausdorff space which is not normal.
   Let $S = \{(x_1, x_2) : x_2 \geq 0\}$ (closed upper half-plane). We define a topology on S by its basis as follows: let R^1 be the boundary of S, and $S_+ = S \backslash R^1$ be the open half-plane (here, “open” and “closed” refer to the topology induced by R^2, which may be different from the one we will define). Let

   $$\mathcal{B}_1 = \{B_x(r) : x = (x_1, x_2) \in S_+,\ r < x_2\}$$

   and

   $$\mathcal{B}_2 = \{(B_x(r) \cap S_+) \cup \{x\},\ x \in R^1$$

   where the $B_x(r)$ are the open balls for the usual metric of the plane. Let $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$.

   a) Verify that $\mathcal{B}$ is the basis for a topology $\mathcal{T}$ on S.

   b) For this topology, verify that S is Hausdorff. (Therefore, points are closed).

   c) Show that $R^1 \backslash \{0\}$ is closed in S for this topology.

   d) Show that $R^1 \backslash \{0\}$ and the point 0 cannot be separated by open sets.

2. An open subset of R is the union of a sequence of pairwise disjoint open intervals.

3. Let X be a topological space and A a nonempty subset of X. A subset V is called a neighbourhood of A if there exists an open subset U of X such that $A \subset U \subset V$.

   a) The set of neighbourhoods of A is a filtre $\mathcal{F}$.

   b) Give a necessary and sufficient condition for the identity mapping of X into X to have a limit along $\mathcal{F}$, assuming X is separated.

   c) Let $X = \mathbf{R}$, $A = \mathbf{N}$. Show that there does not exist a sequence $V_1, V_2, V_3 \ldots$ of elements of $\mathcal{F}$ such that every element of $\mathcal{F}$ contains one of the V_i.

4. Let X, Y be topological spaces, f a mapping of X into Y. The following conditions are equivalent: (a) f is continuous and closed, (b) $f(\bar{A}) = \overline{f(A)}$ for every subset A of X.

Marks: 12 + 9 + 12 + 6.