MATE 6540 assignment 6

19. Exercise 1 of Chapter VII, p. 133 in Dixmier.

20. Exercise 2 of Chapter VII, p. 133 of Dixmier.

21. The Baire theorem (Hirsch-Lacombe, "Elements of functional analysis"). Let X be a metric space. Two players, Pierre and Paul, play the following "game of Choquet". Pierre chooses a nonempty open set U_1 in X, then Paul chooses a nonempty open set V_1 inside U_1, then Pierre chooses a nonempty open set U_2 inside V_1, and so on. At the end of the game, the two players have defined two decreasing sequences (U_n) and (V_n) of open sets such that

$$U_n \supset V_n \supset U_{n+1} \text{ for every } n \in N.$$

Note that $\bigcap_{n \in N} U_n = \bigcap_{n \in N} V_n$; we denote this set by U. Pierre wins if U is empty, otherwise Paul wins. We say that one of the players has a winning strategy if he has a method that allows him to win whatever his opponent does. Therefore, the two players cannot both have a winning strategy; *a priori*, it is possible that neither does.

a) Prove that if X has a nonempty open set O which is covered by a countable union of closed sets with empty interior, Pierre has a winning strategy. Hint: Pierre starts with $U_1 = O$, and responds to each choice of Paul's by using the given closed sets.

b) Prove that if X is complete, Paul has a winning strategy. Hint: if a decreasing sequence F_n of closed sets in X has diameter tending to zero, then $\bigcap_{n \in N} F_n \neq \varnothing$.

c) Let X be a complete metric space. Prove Baire's theorem: the union of a sequence of closed sets with empty interior has empty interior.

d) Verify that an equivalent way of stating Baire's theorem is: the intersection of a sequence of open dense sets in X is itself dense in X.

e) Baire's theorem is used in problem 18 of the previous assignment. Explain how.

Points: 9 + 6 + 14