MATE 6627 Midterm Examination October 16, 2006

Name:

You are entitled to take home two problems, one counting for twenty points, and one for fifteen; you must indicate which you choose. Try to complete the four others.

1. (20) Denote by l^1 the set of complex sequences $a = (a_n)$ such that the series $\sum |a_n|$ converges; $|a|_1 = \sum |a_n|$ is then a norm. Denote by l^∞ the set of bounded complex sequences, with the norm

$$|a|_\infty = \sup_{n \in N} |a_n|.$$

Also, denote by c_{00} the set of almost-zero sequences (those that have only finitely many nonzero terms), and by c_0 the subset of l^∞ consisting of sequences that tend to 0.

a) Prove that l^1, l^∞ are Banach spaces (you only need to show completeness).

b) Show that c_0 is closed in l^∞.

c) What is the closure in l^∞ of c_{00}?

d) What is the closure of l^1 in l^∞? (Use b) and c)).

2. (20)

a) Let X be a metric space and (f_n) a sequence in $C(X)$. Prove that, if $\{f_n\}_{n \in N}$ is equicontinuous at a point x of X, then for any sequence (x_n) of X that converges to x, the sequence $\left(f_n(x) - f_n(x_n)\right)$ converges to 0.

b) Set $f_n(x) = \sin nx$. Prove that $\{f_n\}_{n \in N}$ is not equicontinuous at any point x of R. *Hint.* Consider the sequence (x_n) defined by $x_n = x + \pi/(2n)$.

c) Let (Y, d) be a metric space and let A be a nonempty subset of Y. Let f be a Lipschitz function from A to R, with Lipschitz constant C. Set

$$g(y) = \inf_{x \in A} \left(f(x) + Cd(x, y)\right)$$

for all y in Y. Prove that g is a Lipschitz extension of f, also with constant C. Hint: first, show that g extends f. To show Lipschitz, use the intermediate step: given y, z in Y,

$$\forall \xi \in A, \exists x \in A : \ f(x) + Cd(x, y) \le f(\xi) + Cd(\xi, z) + Cd(y, z)$$

then take infima to obtain a bound on g(y) that depends on y, z.

3. (20) Let c_{00} be as in problem 1, endowed with the scalar product

$$(x|y) = \sum_{i \in N} x_i \bar{y}_i.$$

Let f be the linear form on c_{00} defined by

$$f(x) = \sum_{i \in N} \frac{x_i}{i + 1}.$$

a) Prove that f is continuous.

b) Set F = ker f. Prove that F is a closed vector subspace strictly contained in c_{00}.

c) Prove that $F^{\perp} = \{0\}$.

4. (15)

a) Consider the space $E = l^{\infty}$. Prove that the sequence (T_n) of E' defined by $T_n(x) = x_n$ has no pointwise convergent subsequence in E.

b) Assume that (x_n) and (y_n) are sequences contained in the unit ball of a scalar product space, and that $(x_n|y_n) \to 1$. Prove that $|x_n - y_n| \to 0$.

5. (15) Prove that every precompact metric space is separable.

6. (15) *A generalisation of Dini's lemma.* Consider a compact metric space X, and elements f and $\{f_n\}_{n\in N}$ of C(X). Assume that there exists a constant $C > 0$ such that for all p, q, in N,

$$|f_{p+q} - f| \leq C|f_p - f|$$

Prove that if the sequence (f_n) converges pointwise to f, it converges uniformly to f. Hint: given $\varepsilon > 0$, consider the sets $\omega_n = \{x : |f_n - f| < \varepsilon\}$